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Logic and Argument Analysis: An Introduction to Formal Logic and Philosophic Method (REVISED)

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LOGIC AND ARGUMENT ANALYSIS

An Introduction to Formal Logic and Philosophic Method

Companion Text to the Computer Tutorial Programs

ANALYTICS

Preston K. Covey, Jr.

Carnegie-Mellon University

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The project that produced this text and the computer tutorial package in logic and argument analysis, ANALYTICS, was funded by a grant from The Fund for the Improvement of Post-Secondary Education
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INTRODUCTION

A word of explanation is in order regarding the title of the set of computer-based tutorials, ANALYTICS, for which this text is a companion. The computer tutorials are designed primarily to exercise you in the use of basic tools and techniques of logical analysis. The science of formal logic and the systematic study of what makes reasoning good or bad were invented by Aristotle over twenty centuries ago. Modern logic has made considerable progress on Aristotle's monumental beginnings. But the nature of logical analysis as we pursue it today still owes a great deal to Aristotle's original inspiration and definition. Any history of philosophy or logic will attest to this. The following brief description of Aristotle's basic enterprise from W. T. Jones' History of Western Philosophy (Vol. I, p. 224) is fitting for our course of study:

Aristotle was the inventor of formal logic in the sense that he was the first person to draw up precise rules for distinguishing valid from invalid thinking. Suppose I know that all Greeks are mortal and that Aristotle is a Greek. It follows that Aristotle is mortal. . . . [Now], why would the conclusion that Aristotle is a man not follow from these premises? Of course there are premises from which the latter conclusion could be drawn—for instance, "All Greeks are men" and "Aristotle is a Greek." But even though it is true that Aristotle is a man, this proposition does not follow from the facts that he is a Greek and that all Greeks are mortal.

Thus, as Aristotle saw, we must distinguish between truth and validity. Truth is a characteristic of individual propositions: An individual proposition is true if it correctly classifies things and false if it does not. Thus "Aristotle is a Greek" is true, and "Aristotle is a Turk" is false. Validity is not a characteristic of individual propositions. It is the logical relation between premises . . . and the conclusion that follows from these premises. Thus, although the proposition "Aristotle is a man" is true, it follows validly from some premises but not from others.

The two chief questions Aristotle set himself to answer . . . were: (1) When we have true propositions, what are the rules of inference by which a conclusion can be [validly] drawn? (2) How can we know that the premises we start with are true?
In formal logic today we still distinguish (1) the logical FORM and VALIDITY of an argument from (2) the CONTENT or TRUTH of its premises and conclusion. We, like Aristotle, will be interested in both of the following questions:

1. What are the rules of inference by which conclusions can be validly drawn from given premises?

2. How can we determine whether the premises of an argument are true or sufficiently plausible to justify assent?

We will also be concerned with how these two questions are inevitably and usefully related: with how valid logical form is usefully related to the pursuit of truth (in philosophy, in particular).

This concern with the distinction and relation between the FORM and CONTENT of reasoning remains very much in the spirit of Aristotle's pioneering study of logic. This interaction is reflected throughout the computer programs in logic and argument analysis. This is why the package of programs is entitled ANALYTICS, reminiscent of the title of Aristotle's major treatises on logical analysis, the Prior and Posterior Analytics.

As you probably know and as you will see, you can't argue well about the truth of anything without logic; but logic, while necessary, is hardly sufficient for determining the truth in any dispute. The relation between formal logic and the pursuit of truth in ANALYTICS becomes especially conspicuous in the reconstruction and analysis of arguments.

We turn first, in Part One, to the matter of what makes arguments good or bad, valid or invalid, as a function of their LOGICAL FORM, the formal LOGICAL CONNECTION between premises and conclusions. In Part Two of this text, we will consider how, with the help of formal logic, we can assess the TRUTH or PLAUSIBILITY of the premises of philosophic arguments, especially the general principles that are crucial to arguments about values and fundamental to the governance of our lives and our society.
1.1. WHAT AN ARGUMENT IS: ARGUMENTS AS TOOLS OF ANALYSIS

What is an ARGUMENT? What often comes to mind is either an altercation between two people, or a line of assertion put forward by one person with the intent of convincing someone to believe something or persuading someone to do something. These are perfectly legitimate associations to call to mind. However, they are not exactly the sort of thing we will be studying when we analyze the logic of arguments.

We will not be concerned with arguments in the sense of altercations or verbal exchanges that take place between people. We will not be concerned with arguments as events in which we take part, but rather with arguments as sets of assertions, as artful constructions representing propositions people believe and reasons that can be given for those beliefs.

We may construct an argument or assemble a set of assertions in order to convince or persuade others to believe or do something. But arguments as logical constructions have other important purposes besides convincing or persuading.

Arguments, like theories, might be constructed for merely exploratory, analytical, or purely hypothetical purposes—to display what can be said for and against a certain position, or to explore the logical consequences of a position, or to reveal the 'hidden' assumptions of a position. The possible uses of logic and argument go well beyond attempts to convince or persuade—especially in areas of controversy (like philosophy or social policy), where, before we can convince anyone of anything, we need to get clear about what it is we believe, what the grounds and consequences of our beliefs are, what our other options are, what can be said for and against alternative views.

Arguments may be purely artificial—and hopefully artful—constructions whose sole purpose is to exhibit—as clearly as possible—the presumed LOGICAL CONNECTIONS among a set of beliefs or statements. Constructing arguments serves the purpose of making the presumed logical connections among statements or beliefs explicit and
clear. Within and consistent with this larger purpose, we may, of course, construct arguments in order to show that certain statements (called PREMISES) logically support other statements (called CONCLUSIONS) in order to convince other people to believe or act on those conclusions. But it will be useful to consider arguments both as sets of statements constructed for purposes of convincing or persuading and also as sets of statements constructed just for purposes of clearly exhibiting logical connections, for purposes of analytical inquiry into the grounds and consequences of our beliefs.

For example: proofs in mathematics can function as arguments in either sense. The proof of a formula, showing that it follows by valid steps from previously accepted axioms and proven theorems, may serve to convince us that the proposition itself should be accepted as a mathematical truth or theorem. Or, the proof that a contradiction follows from a set of formulae may serve simply to exhibit the fact that there is some inconsistency among those formulae.

The same goes for philosophy: we may construct arguments in order to marshall premises in support of conclusions; or we may construct arguments merely to exhibit the logical connections among propositions (to show, for example, that a set of statements is inconsistent). For either purpose, the pertinent logical tools are the same.

For our purposes, then, an ARGUMENT is a set of statements, some of which (the PREMISES) purportedly imply or "support" another (the CONCLUSION), such that, if the premises are true, we have some reason or evidence for accepting the conclusion: there is some purported EVIDENTIARY CONNECTION between the premises and conclusion.

Later we will see how argument construction is useful for purposes besides convincing or persuading people to believe or do things—for example, for making explicit the LOGICAL CONNECTION between some proposition and either the reasons that can be offered in its behalf or the consequences that follow from it.

1.2. WHAT MAKES AN ARGUMENT GOOD OR BAD?

The criteria by which something is judged good or bad are often relative—relative at least to the purpose(s) for which it is judged good
For example: to answer a seemingly straightforward question like "Is this a good knife?" one needs to know "Good for what?" A knife may be very good for purposes of gutting whales, and very bad for slicing vegetables; or vice versa. Some broken, rusty old knife may be good for nothing but arousing sentiment as a memento.

We discern value, goodness or badness, so far as we discern purposes for which things might be good or bad. This makes evaluation rational, factually verifiable, and intelligible, but not uncomplicated.

So also with arguments. The criteria according to which we judge them good or bad are relative to our purposes.

Even where we agree in point of purpose—say, that the purpose of an argument is to be convincing—a 'good-making' characteristic (being convincing) may be judged to vary according to a wide range of subjective standards: What I find convincing you may find too boring to pay any mind at all. Arguments that are very competent according to objective, logical criteria may still vary in goodness and badness (even in how convincing they are) according as they are boring, simple, complex, sophisticated, humorous, grammatical, colorful, elegant, misspelled, indiscreet, repetitive, obscene, racist, suspect, fanciful, sonorous, limpid, daring, startling, low-brow, dingy, scientific, contrary, or whatever—all depending on their context or the standards and interests of their audience.

Of all the myriad ways in which arguments may conceivably be good or bad, we will be concerned with only a prominent few that can be judged by relatively objective criteria. Specifically, we will be concerned with two factors of paramount importance when constructing or evaluating any argument:

1. The logical connection between premises and conclusion: The evidentiary connection by which premises purportedly support the conclusion.

2. The truth-value or plausibility of the premises.

In this text we will consider objective criteria for appraising arguments on both counts.

In Part One we will focus on the analysis of the logical connection
between the premises and conclusion of an argument: the EVIDENTIAL CONNECTION between premises and conclusion by virtue of which the premises are purported to lend evidence or support to the conclusion.

In Part Two we will learn how to test or assess the TRUTH-VALUE (i.e., the truth or falsity) and the PLAUSIBILITY of the premises of an argument—in particular, general normative principles, principles of right and wrong, principles that are crucial premises of philosophic arguments about social values, principles that are fundamental to the governance of society and our personal lives. Thus, the analysis of arguments will be a vehicle for analytical inquiry into the grounds and consequences of our social values. This is why arguments as logical constructions are useful as tools or vehicles for philosophic inquiry as well as tools of persuasion.

Some Important VIRTUES of Arguments

Of all the virtues an argument may have, we will be especially interested in the following FOUR and how to assess them:

1. **VALIDITY**

   This virtue has to do with the LOGICAL FORM of an argument, with the LOGICAL CONNECTION between its premises and conclusion, with whether the conclusion 'FOLLOWS' from/ is a LOGICAL CONSEQUENCE of the premises.

   How we can analyze this logical connection, represent the logical form of an argument, and assess or manifest its validity will be the subject of most of Part One.

2. **TRUE PREMISES**

3. **SOUNDNESS**

   This, like VALIDITY, is a technical term of logic.

   A SOUND argument both is VALID and has all TRUE PREMISES.
4. PLAUSIBLE PREMISES

Truth and plausibility are not necessarily the same.

What is true may not be demonstrably (provably) true or even plausible in a given context of observational evidence and belief: It was and is true that the earth revolved around the sun, but this was not a plausible explanation of the observed phenomena in the context of the orthodox theology and cosmology before Copernicus demonstrated the converse.

And what is plausible within a given context of limited evidence and belief may not be true: It was (and may still be) plausible for certain islanders to believe that the earth is flat, that the horizon marks the edge, and that whoever sails that far will fall off the edge—but this is, as we know, false. So, what is true may not (yet) be demonstrably true or even plausible. And what is plausible may turn out to be false and even demonstrably so.

Since what is true is not always demonstrably or obviously true, nor always even plausible within our limited frames of reference, AN IDEAL ARGUMENT, for purposes of convincing people of its conclusion, would be logically VALID and would have premises that were either DEMONSTRABLY TRUE or else SUFFICIENTLY PLAUSIBLE to command assent within present frameworks of knowledge and belief.

Such ideal arguments are often hard to come by, especially in philosophy and disputes about values (though even here they are not impossible to produce).

For purposes of analytical inquiry, validity and prima facie plausible premises will often suffice. Our interest in this course will be in analyzing our present frameworks of knowledge and belief, especially our beliefs about social values, the plausibility of the grounds and logical consequences of our most basic social values. For this analytical purpose, we will be especially interested in assessing two minimal virtues of arguments:

- The VALIDITY of arguments (Part One)
1.3. TEST YOUR ANALYTICAL JUDGMENT ABOUT ARGUMENTS: EXERCISES

The following exercises are for fun and illustration. You will learn specific tools and techniques for analyzing arguments so you can easily answer questions like the following. For now, just see what you think. Make a note of your answers so you can compare them with what you learn later in the course.

1.3.1. JUST WHAT IS THE ARGUMENT HERE?

In the following items, some inference is made, some conclusion is drawn or implied. Some premises from which the conclusion is supposed to follow are either stated explicitly or tacitly assumed. In each case:

(a) Construct an argument to represent the reasoning involved: state the (explicit or tacit) conclusion that is advanced and the (explicit or tacit) premises from which it is supposed to follow.

(b) Consider: How do you know or decide what the premises or conclusion of an argument are when they are not explicitly stated?

(c) Consider: Are the premises of the argument plausible? What does this mean? How do you decide?

(d) Consider: Does the conclusion follow logically from the premises? What does this mean? How do you decide?

1. Two current American coins add up to 30 cents, yet one of them is not a nickel. Therefore, one of the coins is a quarter.

2. Jack is standing on the street beside a car with its hood up. Jill happens by. Jack says, with some consternation, "I guess I should hitch a ride home." Jill says, "But there's a garage just around the corner." What's the argument here?

3. The sign outside the restaurant reads: "Cheap food is not good. Good food is not cheap." What's the argument here?

4. He: "So, you think that a fetus has a right to life then?"
She: "I don't want to take a stand on whether a fetus has a right to life— that's philosophy. But I'll go so far as to say that unless it turns out that a fetus does not have a right to life, abortion is wrong."

He: "You can't duck the right-to-life issue like that. You do think abortion is wrong, don't you?"

She: "Yes—of that much I'm convinced."

He: "Well, if you hold that unless a fetus has no right to life abortion is wrong and you also think that abortion is wrong, then, whether you want to cop to it or not, you are logically committed to the proposition that a fetus does have a right to life."

5. The confounded case of Scott Free. Scott Free is to be put to death. He is told that his manner of death is to be decided as follows: Scott is to make a statement. If the statement is true, then he will be shot. If the statement is false, then he will be hanged. On hearing this, Scott spontaneously exclaims: "I'll be hanged!" The judge in the case ponders which mode of execution is now indicated. Then the judge lets Scott go free, he says, on the basis of inescapable logic. What could his argument be?

6. It is said that in some heavenly massage parlor there is a wonderful and busy masseuse who will massage all if only those who will not massage themselves. (We'll call her Trixie because this feat is quite a trick.) Now, the question is: Who massages Trixie? (Note: There is one and only one indisputably and logically correct conclusion to be drawn on this matter. What is it? What's the argument for it? Compare problem (5) in the next section, 1.3.2.)
1.3.2. DOES THE CONCLUSION 'FOLLOW'? IS THE ARGUMENT VALID?

Besides answering the above question about the following arguments, ask yourself, think about:

(a) Is the argument SOUND? What does this mean? How do you decide?
(b) Are the premises of the argument PLAUSIBLE? What does this mean? How do you decide?
(c) What good is it if the premises of the argument are plausible or even true but the conclusion does not follow logically from the premises?

1. Everyone is afraid of Dracula. Dracula is afraid only of God. Therefore, Dracula is God. Valid or Invalid?

2. God is, by definition, all-good and all-powerful. If God existed, there would be no evil in the world. But evil abounds. Therefore, God does not exist. Valid or Invalid?

3. If you're not in Pennsylvania, you're not in Pittsburgh. But you are in Pennsylvania. So, you are in Pittsburgh. Valid or Invalid?

4. If you're illiterate, you're not reading this. You are illiterate. So, you're not reading this. Valid or Invalid?

5. There is a god (we'll call him/her God) who created all and only those things that did not create themselves. Therefore, God created himself even though God did not create himself, and, moreover, there is no god (God) who created all and only those things that did not create themselves. Valid or Invalid?

6. Every flame I've encountered so far has been hot enough to burn me. Therefore, all flames are likely to burn me. Valid or Invalid?
ANSWERS: Argument (1) is valid; the conclusion that Dracula is God follows necessarily from the premises. But surely this conclusion is false: so what's wrong with the premises? How do we know it's false, by the way?

Argument (2) is also valid. Suppose you disagree with the conclusion: since the argument is valid, how can you fault the argument? Even if you don't disagree with the conclusion, how can you criticize the argument on grounds other than invalidity?

Argument (3) is invalid, even though the premises and conclusion are all quite true. Note: Arguments that are rife with truth can nonetheless be logically faulty. How can this be? We'll soon see.

Argument (4) is absolutely valid, even though its second premise and its conclusion are (happily for you) quite false. Remember: the validity of an argument has nothing to do with the truth or falsity of its premises. Hard to figure? Hang on! You'll see why soon enough.

It would be reasonable to think that the conclusion of Argument (5) does not follow, that such a thing could not reasonably follow from its premise. But the fact is, Argument (5) is technically and provably valid. What is confusing perhaps is the paradox that a negative conclusion like 'There is no god who . . .' could logically follow from a premise stating that there is such a god. This same paradox is involved in the Trixie problem (6) in the last section (1.3.1). If you think about the logical meaning of the premise, the paradox can be sorted out: the crux of the matter is the logical force of the phrase 'all and only.' Now, either God creates himself or he does not. Suppose he does. Then it follows from the premise that he does not: remember, the premise states that God creates only those who do not create themselves. Suppose then that God does not create himself. Then it follows from the premise that he does create himself: the premise states that God creates all those that do not create themselves. Thus: Either God creates himself or he doesn't. If he does, then he doesn't. And if he doesn't, then he does. The premise contains a hidden contradiction. Anything follows from a contradiction—this is why contradictions, especially 'hidden' ones, are so invidious to straight thinking. The premise describes a logically contradictory entity, a logical impossibility: that's why it follows that there is no such entity or god. Likewise, there can be no such masseuse as Trixie in problem (6), Section 1.3.2: if Trixie is a logical impossibility, then Trixie does not, can not possibly exist. Therefore, no one massages Trixie. (Poor Trixie!) You will soon learn to analyze and avoid such 'hidden' contradictions easily enough and to show them up for what they are.
Argument (6) is invalid even though its premise seems very good evidence all by itself for believing its conclusion. In fact, it would seem unreasonable if not insane for a person not to accept the conclusion of argument (6) on the basis of its premise. This is to say that some arguments that are not valid can be perfectly reasonable. But too many arguments that seem reasonable, that have the ring of truth about them, but are invalid can still be pernicious. That's why we're going to be very careful to make sure that the arguments we deal with are, above all, valid and provably so: while it may not always do so, INVALIDITY can lead us from truth into falsehood—and it is this above all that we shall need to avoid, especially when arguing about social values.

By the way, it's easy to make argument (6) valid—do you see how? Just add another premise—do you see what it is? The missing premise is some version of what is often called 'the principle of induction.' We assume this sort of principle implicitly in many of our arguments. Making argument (6) valid simply requires us to make this tacitly assumed principle explicit:

If all things of a kind (e.g., flames) that we've examined so far have had a certain property (e.g., being hot enough to burn us),

Then all things of that kind are likely to have that property

With the addition of some such general principle as a premise, argument (6) becomes manifestly valid. But is it sound? Is the above premise (always) true? Is it a plausible premise? What's the difference?
1.4. VALIDITY, INVALIDITY, AND LOGICAL FORM

An ARGUMENT, for present purposes, is a set of statements, some of which (the PREMISES) purportedly support or imply some other statement (the CONCLUSION).

An argument purportedly provides some sort and degree of CONDITIONAL WARRANT for its conclusion: If its premises are evidently true or credible, then one has some reason to accept the conclusion. Some EVIDENTIARY CONNECTION is posited between the premises and conclusion: This means that such credibility as the premises possess is somehow passed along or 'lent' to the conclusion. Just how do credible premises 'lend' credibility to a conclusion? What kinds of evidentiary or logical connections are there?

One very special kind of evidentiary connection that can obtain between the premises and conclusion of an argument is called, variously: deductive VALIDITY, logical IMPLICATION, logical CONSEQUENCE.

An argument that is deductively VALID provides the strongest possible conditional warrant for its conclusion: If the premises are true, the conclusion is not just metaphorically 'lent' some 'support'; it is absolutely GUARANTEED to be true.

A useful converse relation also obtains: If a set of statements (say, a theory) LOGICALLY IMPLIES a false or otherwise unacceptable consequence, then at least one of the statements must be false or likewise unacceptable.

This CONNECTION between a set of statements (say, the premises of an argument) and some LOGICAL CONSEQUENCE (say, the conclusion of the argument) has nothing to do with the content or actual truth or falsity of the statements.

DEDUCTIVE VALIDITY is accountable rather to LOGICAL FORM, to certain skeletal or structural features of the statements in question.

By analogy: whether the human body stands or falls depends in large
part on its skeleton, as well as on its musculature; whether a bridge stands or falls depends in large part on its structural design, as well as on what it's made of. Likewise, whether an argument stands or falls, whether it supports its conclusion or not, depends in large part on its LOGICAL FORM, its skeletal structure, as well as on the truth or credibility of its premises. Examples follow.
The following argument (A) is VALID, and this is by virtue of its having a certain skeleton or LOGICAL FORM, for example (A'), depicted to its right.

(A) (1) If you are illiterate (A') (1') If I, not R
you are not reading this

(2) You are illiterate (2') I

Therefore, (3) you are NOT reading this  
(3') Not R

The fact that statements (2) and (3) are false does not affect the VALIDITY of the argument: If (2) as well as (1) were true, (3) would have to be true. (A) obviously is not seriously intended as an argument in the sense of an attempt to convince you of its conclusion. Whether we regard it as a serious or interesting argument does not change the LOGICAL CONNECTION between statements (1) and (2) and statement (3): (1) and (2) together LOGICALLY IMPLY (3). Moreover, what's of interest about this connection is that any statements having the logical forms (1') and (2') would together logically IMPLY a statement of the form (3').

In the following argument, (B), the premises and conclusion all happen to be true:

(B) (1) If you are illiterate (B') (1') If I, not R
you are NOT reading

But, (4) you are NOT illiterate (4') Not I

So, (5) you are reading (5') R

Close your eyes and the conclusion, statement (5), is false, while the premises remain true. Hence, this argument is INVALID. But not just because of any accident or fact about the world that momentarily renders the conclusion false while the premises are true. It is invalid because the LOGICAL FORM of the argument (B') fails to GUARANTEE a true conclusion, given true premises. We can know and prove this about the argument FORM (B') irrespective of anything we may know about the particular statements asserted in argument (B).

An argument skeleton of the form (B') fails to guarantee the truth of its conclusion, given true premises, if any argument of that form can have...
true premises BUT a false conclusion. Knowing nothing about you or the truth of statements (4) and (5), I know that argument (B) is invalid so far as its logical form, (B'), is the same in relevant respects as the following argument's, (C'):

\[
\begin{align*}
(C) & \quad (6) \text{ If I'm on the moon} \\
& \quad \text{I'm not on Venus} \quad \text{(True)} \\
& \quad (7) \text{ I'm not on the moon} \quad \text{(True)} \\
& \quad \text{Therefore, (8) I'm on Venus (False!)} \\
(C') & \quad (6') \text{ If M, not V} \\
& \quad (7') \text{ Not M} \\
& \quad (8') \text{ V}
\end{align*}
\]

Demonstrate the fact as you will, any argument whose relevant FORM is the same as (B') or (C') is INVALID so far as it is possible for an argument of that form to have true premises and a false conclusion: An INVALID ARGUMENT FORM can lead us from truth into falsehood. A VALID argument form cannot.

1.5. FORMAL SYMBOLIC LOGIC: WHY SYMBOLIZE ARGUMENTS?

Deductive logic is FORMAL insofar as it typically attributes validity/invalidity to LOGICAL FORM: it seeks to discover rules governing the use of those logical elements of our language that make arguments valid or invalid.

For example: the crucial elements of logical form singled out in arguments (A)-(C) were the sentential connective 'IF' and the negation term 'NOT.' Connectives like IF are crucial parts of the skeletons of arguments. The validity or invalidity of arguments (A)-(C) can be accounted for by the way the statements of the argument were constructed and combined using skeletal parts like 'IF' and 'NOT.' To make the skeleton or form of these arguments stand out clearly, it was convenient to symbolize (let single letters stand in for) the component statements that make up the arguments.

Formal logic is typically SYMBOLIC so far as it is convenient (for purposes, say, of easy pattern recognition and formal manipulation) to depict the statements and crucial logical elements of natural language (like 'if,' 'unless,' 'not') in some standard notation. It is often convenient to reduce the logical form and import of the variety of logical expressions found in ordinary language (e.g., 'if,' 'only if,' 'unless') to some standard symbolic form for purposes of easily construing the validity or invalidity of arguments. It can be very useful to examine the pure
logical form and import of statements, quite apart from knowing their truth or falsity—especially when the matter at hand is controversial, the truth of the matter is elusive, and we are not sure what to believe.
For example: Suppose a person is wondering whether a human fetus can be shown to have a right to life. She's not at all sure what to believe on this issue. But she does think that, unless a fetus has no right to life, abortion is wrong. In any case, she can't help feeling that abortion is just not right. A quarrelsome friend then claims that she's effectively committed to a position on the right-to-life issue after all, and had better face up to it. That is, he claims that propositions (9) and (10) logically commit her to (12), as follows:

\[(D) \quad (9) \text{ UNLESS it's the case that fetuses have no right to life, abortion is NOT right} \]

\[(9') \text{ Unless not R not A} \]

\[(10) \text{ Abortion isn't right} \]

\[(10') \text{ Not A} \]

So, (11) it's NOT the case that fetuses have no right to life--

\[(11') \text{ Not not R} \]

\[(12) \text{ fetuses do have a right to life} \]

\[(12') \text{ R} \]

Is it true that she is logically committed to believe (12) if she believes (9) and (10)? Do (9) and (10) logically imply (12)? Is (D) a valid argument? How can we tell?

Suppose her friend, while trying to argue for abortion and raise doubts in her mind about (10) by capitalizing on her doubts about (12), holds that abortion is not wrong unless fetuses have a right to life. But, he must confess, he thinks fetuses do have a right to life. She presses the point that he must, then, logically, concede that abortion is wrong, on the following deduction:

\[(E) \quad (13) \text{ UNLESS fetuses have a right to life, abortion is NOT wrong} \]

\[(E') \quad (13') \text{ Unless R, not W} \]

But (12) fetuses do have a right to life

\[(12') \text{ R} \]

So, (14) abortion is wrong

\[(14') \text{ W} \]

Is she right? Does (14) follow logically from (12) and (13)? This may be unclear; the logic of the matter may get lost in the verbiage. This is not uncommon. This is why we try to simplify or clarify the logical form of an argument by reducing it to a standard symbolic schema, by stripping away the verbiage and focusing on the crucial logical connections.
The foregoing hypothetical dispute is not about the TRUTH OR FALSITY of beliefs (about the rights of fetuses or the rights and wrongs of abortion). It is rather about the LOGICAL CONNECTIONS among the propositions in question. The dispute hangs in part on some LOGICAL CONNECTION, (9) or (13), that each disputant posited between the rights of fetuses and the rights and wrongs of abortion.

In fact, each party is incorrect about what the other is logically committed to concede: Arguments of the form (D') and (E• ) are, on one account, clearly INVALID. This may or may not be clear from the 'sound' or logical 'ring' of the arguments as given in ordinary language. The crux of the matter here is how we interpret the precise logical force of the ordinary conditionalizing connective 'UNLESS.' Symbolic logic can legislate the dispute and make the issue more transparent as follows.

Conditionals of the form (9') 'Unless not R, not A' have the same LOGICAL FORCE, the same LOGICAL MEANING as statements of the form 'A only if not R,' 'If A, not R' and 'If R, not A.' Why this should be so will require some study and justification, but the point can be illustrated by the following deductive sequence of LOGICALLY EQUIVALENT statements, any of which 'follows logically' from any other:

\[
(15) \text{ IF it's raining out, it's not dry out} \\
(16) \text{ It's raining out \textbf{ONLY} IF it's not dry out} \\
(17) \text{ It's NOT raining out \textbf{UNLESS} it's not dry out} \\
(18) \text{ \textbf{UNLESS} it's NOT dry out, it's not raining out} \\
(19) \text{ IF it's NOT the case that it's not dry out, it's not raining out} \\
(20) \text{ IF it's dry out, it's NOT raining out} \\
(15') \text{ IF R, not D} \\
(16') \text{ R \textbf{ONLY} IF not D} \\
(17') \text{ Not R \textbf{unless} not D} \\
(18') \text{ Unless not D, not R} \\
(19') \text{ If not not D, not R} \\
(20') \text{ If D, not R}
\]

Whatever actual sentences the sentence symbols 'R,' 'A,' 'D' stand for makes no difference to the logical force of these equivalent conditional statements, to the logical relation posited between the sentences connected by 'if,' 'only if' or 'unless.' At bottom, then, statement (9) may be seen to have the equivalent logical force of statements (4), (6) and statements(15)-(20).
It is convenient to symbolize the logical force of these diverse but logically equivalent connections in a standard way, with a single symbol, say, an arrow '⇒'. The logical form of arguments (B) and (D) may then be readily represented as, at bottom, the same:

\[
\begin{align*}
(B') & \quad \text{If I, not R} \\
& \quad \text{Not I} \\
& \quad \quad \quad \quad R \\
(B'') & \quad \text{I} \Rightarrow \neg \text{R} \\
& \quad \text{Not I} \\
& \quad \quad \quad \quad \neg \text{R} \\
(D') & \quad \text{Not A unless not F} \\
& \quad \text{Not A} \\
& \quad \quad \quad \quad F \\
(D'') & \quad \text{A} \Rightarrow \neg \text{F} \\
& \quad \text{Not A} \\
& \quad \quad \quad \quad \neg \text{F}
\end{align*}
\]

Arguments of the form (B') are not valid. Neither, then, is any argument of the form (D'), since the underlying logical form of (B') and (D') are equivalent, as represented by (B'') and (D''), above. That arguments of the forms (B') or (D') are invalid is readily seen from the following example, which has the same logical form:

You're not in New York unless you're not in France (True)
You're not in New York (True)
Therefore, you're in France (False!)

Can you symbolize the logical form of this argument to show that it has the same logical form as (B') and (D')?
Arguments of the following form are also **INVALID**:

(E') Not \( W \) unless \( F \)  
\[
\begin{array}{c|c}
F & F \\
\hline
W & W \\
\end{array}
\]

(E'') \( W \Rightarrow F \)

(E') is **INVALID** because an argument with the same underlying logical form (E'') can have true premises but a false conclusion, can lead us from truth into falsehood, as follows:

If you're a whale, you're a mammal \((\text{True})\)

You are a mammal \((\text{True})\)

So, you're a whale \((\text{False!})\)

Or, equivalently:

You're NOT a whale UNLESS you're a mammal \((\text{True})\)

But you are a mammal \((\text{True})\)

So, you're a whale \((\text{False!})\)

Deductive logic, formal and symbolic, is concerned with discovering and demonstrating various sorts of **logical form** and **logical connectedness**, such as define the **validity/invalidity** of arguments, the relations of **logical implication** or **consequence**, **logical equivalence** and familiar derivative properties such as logical **consistency/inconsistency**. Judgments about these sorts of formal logical relations play an important role in everyday reasoning and a crucial role in philosophic argument.

These logical connections are conveniently defined and studied with the aid of symbolic notation. The advantage of formal symbolic logic is analogous to that of an x-ray device: it allows us to depict and scan the supporting skeleton of an argument, and to isolate distinctively structural flaws, apart from the often obscuring verbal flesh and musculature. As you become practiced in depicting the logical form of an argument symbolically, you will develop a kind of 'x-ray vision' into the structural strengths and weakness in the skeletons of arguments—and you will not be confounded by
logical disputes like those over arguments (D) and (E) above.
2.1. SENTENTIAL LOGIC

Deductive logic studies those logical terms or skeletal elements of language (like 'if,' 'not') that are crucial to determining the validity or invalidity of arguments. You now know that whether an argument is valid or not depends on its LOGICAL FORM. But how do we determine the logical form of an argument?

There are many LOGICAL ELEMENTS of language that could be crucial to the logical form of a sentence or argument. We are going to study, for starters, the most basic ones, the most basic building blocks and connective tissue of arguments. These are logical terms like 'not,' 'and,' 'or,' 'if' by which we connect or negate sentences. At the most basic level, arguments are built up by putting simple sentences together by means of SENTENTIAL CONNECTIVES. The way in which sentences are connected (using words like 'and,' 'or,' 'if') or the way sentences are negated (using 'not'), when these sentences are constructed into arguments, determines the validity/invalidity of these arguments. You've seen examples of this phenomenon of language already.

Argument (A) below is VALID. But argument (B-1) is not. The logical form of (B-1) is (B'), depicted to its right. The ARGUMENT FORM (B') is INVALID because it's possible for an argument with this form to have true premises and a false conclusion: This is shown by argument (B-2), which has the same form as (B-1). Look closely at ARGUMENT FORM (A') and ARGUMENT FORM (B') to be sure you see how the LOGICAL FORMS of arguments (A) and (B-1)/(B-2) are different.
The logical form of argument (A) is different from the logical form of arguments (B-1)/(B-2) in one crucial respect: premise (2) of argument (A) negates the 'THEN'-clause of premise (1), and (A) concludes with the negation of the 'IF'-clause; whereas premises (2) of arguments (B-1)/(B-2) negate the 'IF'-clauses of their respective premises (1), and (B-1)/(B-2) conclude with the negation of the 'THEN'-clause.

Differences in how sentences are combined into argument patterns using 'IF-THEN-' and 'NOT' can make all the difference as to the validity or invalidity of arguments.

Note: The fact that premise (2) of argument (A) is false makes no difference to the VALIDITY of the ARGUMENT FORM, (A'): If (2) as well as (1) were true, if you were indeed not in Pennsylvania, then you would not be in Pittsburgh. The ARGUMENT FORM is VALID because IF the premises were true, the conclusion would be guaranteed to be true.

Likewise, the fact that the premises and conclusion of argument (B-1) are all true does not make the argument valid: the ARGUMENT FORM (B') is INVALID because it fails to GUARANTEE that EVERY argument of that form with true premises will have a true conclusion. Argument (B-2) has the form
(B'): (B-2) has TRUE premises but a FALSE conclusion. INVALIDITY can lead us from truth into falsehood—this is why we want to be able to discern and avoid it.

You have now seen examples of how the validity of an argument can depend on the way sentences are combined using SENTENTIAL CONNECTIVES like 'IF' and 'NOT.' SENTENTIAL LOGIC determines the rules that govern the validity or invalidity of arguments so far as validity depends on how sentences are combined using the SENTENTIAL CONNECTIVES.

Terms like 'not,' 'never,' 'it is not the case that' are not really used to connect sentences so much as to negate them; but negation expressions like 'not' are crucial to the patterns in which sentences are combined into arguments. So, for convenience, we will refer to negation expressions like 'not' as sentential connectives.

The CONNECTIVES studied by SENTENTIAL LOGIC represent five basic types of LOGICAL OPERATION that we can perform on sentences:

1. **NEGATION**, by means of terms like 'NOT':

   Given any arbitrary sentence, say
   
   You are reading
   
   we can form its negation thus:
   
   You are NOT reading
   
   It is not the case that you are reading

2. **CONJUNCTION**, by means of terms like 'AND':

   Given any two arbitrary sentences, say,
   
   You are reading
   You are bored
   
   we can conjoin them in a conjunction:
   
   You are reading AND you are bored.
   
   We can, of course, negate and conjoin sentences:
   
   You are reading AND you are NOT bored.
3. **DISJUNCTION**, by means of terms like 'OR' we form disjunctions:
   You are reading OR you are bored.

   You are reading OR you are NOT reading.

4. **CONDITIONALIZATION**, by means of expressions like 'IF' or 'IF-THEN': We can form conditionals like
   
   (a) IF you are reading, THEN you are bored.

   (b) IF you are bored, THEN you are reading.

   (c) IF you are NOT bored, THEN you are reading.

   (d) IF you are reading, THEN you are NOT bored.

   Depending on how we combine certain sentences (like 'You are reading,' 'You are bored!) into more complex sentences (like (a) or (b) using a connective like 'IF,') the meanings of the more complex sentences that result are different. For example, the meaning of sentence (a) is different from that of sentence (b); likewise for sentences (c) and (d). Notice that these differences in meaning are accountable to differences in **LOGICAL FORM**: The component sentences ('You are reading,' 'You are bored') are the same in (a) and (b); but the 'IF'-clause in sentence (a) is the 'THEN'-clause in sentence (b), and vice versa. (We will study what these differences mean in sections 2.4 - 2.5 on the sentential connectives and their logical force.)

5. **BICONDITIONALIZATION**, by means of 'IF AND ONLY IF'

   We form biconditionals like

   (e) You are bored IF AND ONLY IF you are reading.

   Notice that a BICONDITIONAL SENTENCE like (e) above, is, in effect, a **CONJUNCTION** of two CONDITIONAL sentences (f) and (g), as follows:

   (f) You are bored IF you are reading

   AND

   (g) You are bored ONLY IF you are reading.

   The logical force or meaning of the BICONDITIONAL is clearly different from the meaning of either CONDITIONAL even though the component sentences of each are the same. The logical force or meaning of (f) is also different from that of (g). The logical
force of (f) is the same as (a) above. And the logical force of (g) is the same as (b) above. How and why this is the case will be explained in section 2.4. But you might want to try to reason it out for yourself with the following examples, having the same logical forms as examples (a), (f), (b) and (g), respectively.

(a') IF it's raining, THEN streets are wet.

(f') The streets are wet IF it's raining.

(b') IF the streets are wet, THEN it's raining.

(g') The streets are wet ONLY IF it's raining.

Note that (a') and (f') are true; whereas, (b') and (g') are false. Thus, the way in which complex sentences like (a'), (b'), (f'), (g') are constructed out of simpler sentences (like 'It's raining,' 'The streets are wet') using SENTENTIAL CONNECTIVES like 'IF' or 'ONLY IF' can make a difference to whether the resulting complex sentences are true or false. Notice that (h) has the same LOGICAL FORCE (the same LOGICAL MEANING) as (a'), and, of course, both are true:

(a') IF it's raining, THEN the streets are wet

(h) It's raining ONLY IF the streets are wet.

Because sentences like (a') and (h) have the same LOGICAL MEANING, it is convenient to be able to symbolize their logical form in the same way. Sentences that look different in ordinary language can have the same logical force and the same underlying logical form. So it's useful to be able to depict this fact by symbolizing them in the same way. For example:

(a') IF it's raining, THEN streets are wet. (a'') R \Rightarrow W
(h) It's raining ONLY IF the streets are wet. (h') R \Rightarrow W
(b') IF the streets are wet, THEN it's raining.(b'') W \Rightarrow R
(g') The streets ae wet ONLY IF it's raining. (g'') W \Rightarrow R

Symbolization exhibits the fact that the logical force and underlying logical form of sentences (a') and (h) are the same.
As are those of \((b')\) and \((g')\).

2.2. SYMBOLIZING SENTENTIAL LOGICAL FORM

For purposes of depicting the LOGICAL FORM of sentences in sentential logic (i.e., depicting the crucial LOGICAL CONNECTIONS between the component sentences combined by SENTENTIAL CONNECTIVES to form more complex sentences), it will be convenient to resort to the following conventions.

We will define an ATOMIC SENTENCE as a sentence in which no sentential connectives occur as logical operators. Thus, the following are all atomic sentences:

- It's raining.
- You are reading.
- 'Not' is a sentential connective.
- All the streets are wet.

We will define a MOLECULAR SENTENCE as a sentence in which at least one sentential connective occurs as a logical operator. Thus, the following are molecular sentences:
It's NOT raining.

You are bored IF you are reading.

'Not' occurs in this sentence twice AND it is NOT a logical operator the first time it occurs.

IF the word 'not' is merely mentioned in a sentence AND it is NOT used as a sentential connective to negate anything, THEN 'not' does not occur in that sentence as a logical operator.

The sentential connectives 'not,' 'and,' 'but,' 'or,' 'if,' 'only if,' etc. are logical operators with which we construct molecular sentences out of atomic sentences AND they are logical operators which are studied in sentential logic BUT they are NOT the only logical operators we will study SINCE there are other logical terms (like 'all,' 'none,' 'some') that are important, which are studied in what's called quantificational logic.

Sentential logic AND quantificational logic are both concerned with the deductive validity of arguments BUT they are NOT both concerned with the same logical operators, SINCE sentential logic studies the logical terms that operate as sentential connectives AND quantificational logic studies logical terms that operate as quantifiers.

From these examples you can see that there are a variety of terms that OPERATE as SENTENTIAL CONNECTIVES besides 'not,' 'and,' 'or,' and 'if.' 'But,' 'only if,' 'since' and a host of others operate as sentential connectives as well. Nonetheless, there are still only FIVE basic LOGICAL OPERATIONS performed on sentences that are important in sentential logic. These are the five ways in which MOLECULAR sentences can be constructed out of ATOMIC sentences: NEGATION, CONJUNCTION, DISJUNCTION, CONDITIONALIZATION, BICONDITIONALIZATION.

When a sentential connective is used to perform one of these logical operations, it is used as a LOGICAL OPERATOR. When it is merely MENTIONED (between quotation marks) it is not used as a logical operator.

The use of any one of the vast variety of connective expressions in ordinary language can be reduced to one of the five basic operations studied in sentential logic. It is the rules governing these LOGICAL OPERATIONS that it is important to understand in sentential logic: The ways in which MOLECULAR sentences are constructed by these operations and then combined into ARGUMENTS is what determines the VALIDITY or INVALIDITY of those arguments in sentential logic.
In sentential logic, the most basic units of CONTENT, the most basic elements of arguments are ATOMIC SENTENCES. What determines the VALIDITY of arguments is how the premises and conclusion of arguments are constructed out of ATOMIC SENTENCES by means of any of the five kinds of SENTENTIAL CONNECTIVE. SENTENTIAL CONNECTIVES are the logical mortar used to build MOLECULAR SENTENCES out of ATOMIC SENTENCES. The sentential LOGICAL FORM of a sentence is determined by the way its ATOMIC component sentences are combined using sentential connectives. The logical form of an ARGUMENT is determined by the logical form of the SENTENCES that are its premises and conclusion.

In sentential logic, we can represent the LOGICAL FORM of a sentence as follows. We will let capital letters like P, Q, R, etc. stand for SENTENCES. These letters are called SENTENTIAL VARIABLES. (Their function is similar to that of the variables x, y, z, etc. in algebra, which stand for numbers). We let certain symbols stand for the five logical operations of negation, conjunction, disjunction, conditionalization, and biconditionalization. These symbols are called LOGICAL OPERATORS: They represent the logical operations that we perform on sentences when we negate or connect them by means of any of the five kinds of SENTENTIAL CONNECTIVES. (The function of these symbols is similar to that of mathematical symbols like '+,' '-', '·,' etc., which represent various mathematical operations that we perform on numbers.) The symbols we will use for the five basic logical operations that we perform on sentences in sentential logic are:

1. - (the minus sign) for NEGATION
2. & (the ampersand) for CONJUNCTION
3. v (a 'v' for the Latin 'vel' for 'or') for DISJUNCTION
4. => (an arrow) for CONDITIONALIZATION
5. <=> (a two-way arrow) for BICONDITIONALIZATION

Different sentential connections in English (e.g., 'and,' 'but') are represented by the same LOGICAL OPERATOR (e.g., '&') when they represent the same LOGICAL OPERATION (e.g., conjunction). With a single symbol to represent each of the basic logical operations we can conveniently represent the logical force of various connective expressions used in ordinary language. For example:
(a) It's raining AND it's sunny.  \( (a') \) R & S

(a') It's raining BUT it's sunny. \( (a') \) R & S

(b) IF it's raining, THEN it's NOT sunny. \( (b') \) R \( \rightarrow \) -S

(b') It's raining ONLY IF it's NOT sunny. \( (b') \) R \( \rightarrow \) -S

The above symbolizations readily represent the fact that the same logical operations are being performed in sentences (a) and (a'); likewise in sentences (b) and (b'); despite the fact that different conjunctive or conditionalizing expressions are used in each pair of examples. The situation here is again similar to that in mathematics where different expressions in ordinary language are used to denote one and the same mathematical operation; for example: 'Two plus two makes four' and 'Two and two equals four' are both rendered as '2 + 2 = 4.'

In symbolically depicting the logical form of sentences (and the premises and conclusions of arguments) we will follow these conventions:

1. Assign a sentential variable to each and every distinct atomic sentence. E.g., the sentence 'It's not raining' should be depicted as '-P,' where P stands for the atomic sentence 'It's raining' (even though there's nothing logically incorrect with letting sentential variables stand for molecular sentences like 'It's not raining."

This assures that all of the relevant sentential logical structure of any sentence or argument is explicitly depicted.

2. Be sure to assign the same sentential variable to the same atomic sentence wherever that sentence occurs in a given argument. E.g., the argument 'It's raining only if the streets are wet. The streets aren't wet. Therefore, it's not raining' should be depicted as follows, where P stands for 'It's raining' and Q stands for 'The streets are wet':

\[
\begin{align*}
(1) & \quad P \rightarrow Q \\
(2) & \quad -Q \\
(3) & \quad -P
\end{align*}
\]

3. Separate the conclusion of an argument from the premises or
previous lines of an argument by a line (as in the example above) to indicate which sentence form represents the conclusion.

4. Number the lines representing the premises and conclusion of an argument, as above, for ease of reference.

When we depict the logical form of a sentence (as above), the formal or symbolized representation is called a SENTENCE FORM. Thus: \((a')\) is a sentence form representing the logical form of sentence \((a)\); \((a'')\) is the completed and standardized symbolization of that sentence form, where we let \(P\) stand for 'It's raining' and \(Q\) stand for 'The streets are wet'; we have, for example, the following:

\[
\begin{align*}
(a) & \quad \text{The streets are wet IF it's raining.} \\
(a') & \quad Q \text{ IF } P \\
(a'') & \quad P \Rightarrow Q
\end{align*}
\]

Examples \((a')\) and \((a'')\) represent two 'levels' of sentence form, partially symbolized and fully symbolized, respectively. Any number of sentences can have the sentence \((a')\) symbolized by \((a'')\). A SENTENCE FORM in sentential logic is just a schema or formula that can represent the logical form of any number of sentences, which, like the above, represent the same logical operation.

We give an INTERPRETATION of a sentence form when we assign actual SENTENCES to the SENTENTIAL VARIABLES. For example, we can give an interpretation of the sentence form

\[
\text{If P, then Q} \quad \text{(equivalent to \((a')\), above)}
\]

by letting \(P\) stand for 'The streets are wet' and letting \(Q\) stand for 'It's raining.' In this case, we have given an interpretation of the sentence form that is false. SENTENCE FORMS themselves are neither true nor false. Any sentence may be substituted for any sentential variable. Until some substitution or interpretation is given, the sentence form by itself refers to no proposition that we can discern to be true or false.

There are six kinds of sentence forms in sentential logic. Any sentence has one of the following basic forms. Any sentence is either
An ATOMIC sentence (containing no operative connective), represented by a single sentential variable: $P$

or else it is a MOLECULAR sentence of one of the following forms:

(2) A NEGATION: $-P$

(3) A CONJUNCTION: $P \& Q$

(4) A DISJUNCTION: $P \lor Q$

(5) A CONDITIONAL: $P \Rightarrow Q$

(6) A BICONDITIONAL: $P \Leftrightarrow Q$

Molecular sentences can contain more than one operative connective:

(b) If you study hard and pay attention, you will pass the exam.

Sentence (b) has the form: (b') If $P$ and $Q$, $R$.

It is symbolized: (b'') $(P \& Q) \Rightarrow R$

Sentence (b) is, at bottom, a CONDITIONAL whose 'IF'-clause is a CONJUNCTION. Notice that in the English sentence itself and in its schematization (b'), this is intuitively clear. In the symbolization of (b) we use PARENTHESES to show the same effect. Were we to symbolize (b) without parentheses, as follows:

$$P \& Q \Rightarrow R$$

this SENTENCE FORM would be AMBIGUOUS and might be read in either of two ways:

1. As the CONJUNCTION of an atomic sentence ($P$) and a conditional ($Q \Rightarrow R$):

   $$P \& (Q \Rightarrow R)$$

2. Or as a CONDITIONAL whose 'IF'-clause is a conjunction:
(P & Q) \Rightarrow R

In symbolizing sentences we will sometimes need to use parentheses to indicate unambiguously which of two or more connectives is the MAJOR CONNECTIVE of the sentence. A few examples will serve to show how parentheses are used to disambiguate sentence forms and make clear which of the basic five molecular forms a sentence has:

P \& Q \lor R \text{ might be read as either of the following: }

(P \& Q) \lor R: \text{ the DISJUNCTION of a conjunction } (P \& Q) \text{ with } R

P \& (Q \lor R): \text{ the CONJUNCTION of } P \text{ and a disjunction } (Q \lor R)

P \& Q \lor R \Rightarrow S \text{ might be read as any of the following: }

P \& (Q \lor (R \Rightarrow S)): \text{ a CONJUNCTION}

(P \& Q) \lor (R \Rightarrow S): \text{ a DISJUNCTION}

((P \& Q) \lor R) \Rightarrow S: \text{ a CONDITIONAL, with a disjunctive IF-clause}

(P \& (Q \lor R)) \Rightarrow S: \text{ a CONDITIONAL, with a conjunctive IF-clause}

Notice that PARENTHESES must be BALANCED; that is, counting out from the innermost parenthesized unit, the number of left-handed parentheses '(' must be equal to the number of right-handed parentheses ')'; for example:

\[
\begin{array}{c}
(P \& Q) \lor R \\
* & * \\
1 & 1 \\
\end{array}
= 1 \text{ right, 1 left}
\]

\[
\begin{array}{c}
P \& (Q \lor (R \Rightarrow S)) \\
* & * & * \\
2 & 1 & 1 & 2 \\
\end{array}
= 2 \text{ right, 2 left}
\]

Note that the sentence form

\[-P \& Q\]

is always to be read as the CONJUNCTION of the negation \((-P)\)
with \( Q \), rather than as the negation of a conjunction:

\[- (P \& Q)\]

All molecular sentence forms which contain two or more connectives of the same kind, for example

\[P \& Q \& R\]
\[P \lor Q \lor R\]
\[P \rightarrow Q \rightarrow R\]

must also be disambiguated to indicate which of the connectives is the major one, thus:

\[(P \& Q) \& R\] or \[P \& (Q \& R)\]
\[(P \lor Q) \lor R\] or \[P \lor (Q \lor R)\]
\[(P \rightarrow Q) \rightarrow R\] or \[P \rightarrow (Q \rightarrow R)\]

(You will see what difference these different groupings make later in the computer programs. Logically speaking, which grouping is chosen is a matter of indifference.)

Unambiguous sentential formulae (sentence forms) that employ only the legal symbols and contain only balanced parentheses are called WELL FORMED FORMULAE or WFF's. When symbolically depicting the logical form of molecular sentences:

1. Correctly identify the MAJOR CONNECTIVE that determines the basic form of the sentence (and parse the sentence accordingly with parentheses where needed)

2. Assign sentential variables to each atomic sentence.

3. Depict all the operative connectives in the sentence by the appropriate symbols.

4. Be sure that your symbolized formula is WELL FORMED (with parentheses BALANCED and only the legal symbols for connectives)

The procedure for depicting the LOGICAL FORM of sentences symbolically will become second-nature after a bit of practice with the computer lessons on the sentential connectives (in the SENT program) and the computer tutorials on symbolization (in the SYMBOL programs).
The purpose of learning to symbolize sentences and sentential connectives is to be able better to perceive the logical form of arguments and to determine their validity or invalidity. In particular, you will learn better to understand the logical force and meaning of the variety of conditionalizing expressions ('if,' 'only if,' 'unless,' etc.) that play a crucial role in the construction of arguments.

The next section begins to apply the concept of logical form to the assessment of validity.
2.3. **CAN WE PROVE VALIDITY OR INVALIDITY?**

You know already that the validity or invalidity of an argument depends on the argument's **logical form**. To review examples of this fact and the way in which logical form is determined and depicted in **sentential logic**, see sections 2.1 and 2.2. What follows are definitions and a review of the question of how we can **prove** whether or not an argument is valid.

The **allegation** that an argument is valid or invalid is a claim about the argument's **LOGICAL FORM**. To be precise, when talking about validity or invalidity, we are, at root, talking about **ARGUMENT FORMS**—it is the **LOGICAL FORM** attributed to an argument that is valid or invalid.

For any given **ARGUMENT FORM**, there are an infinite number of possible arguments that have that form. So, an allegation that an argument form is valid or invalid is, in effect, a claim about every possible argument that could have that form: an allegation that an argument form is valid or invalid is a statement about the **infinite** number of possible arguments that have that form.

If an **ARGUMENT FORM** is valid, then **EVERY** of the possible infinity of arguments that have that form is valid. If an **ARGUMENT FORM** is invalid, then **EVERY** of the possible infinity of arguments whose relevant form is the same is likewise invalid. Do you see why a statement about validity, a statement about an **ARGUMENT FORM**, is a statement about an **INFINITE** number of possible arguments which have that form?

Consider the now familiar **ARGUMENT FORM** (called **Modus Ponens**):

(MP) (1) If P, then Q
     P \rightarrow Q

(2) P
     P

(3) Therefore, Q
     Q
Other arguments that have this same form are:

(MP-1) (1) If you're in New York City, then you're in New York
(2) You're in New York City
(3) So, you're in New York

(MP-2) (1) If Nork is in Tolway, then Nork is in Orslik
(2) Nork is in Tolway
(3) So, Nork is in Orslik

(MP-3) (1) If barrigroves are mimsy, landaus sweat
(2) Barrigroves are mimsy
(3) So, landaus sweat

The argument below (MP-4) also has in effect the same relevant form as (MP), because, although it contains a piece of logical structure ('NOT') that is not present in the other examples, the basic form of the argument consists in (1) a conditional (2) whose antecedent is affirmed and (3) whose consequent is then concluded. We traditionally call this argument form Modus Ponens (MP), a Latin term that is short for affirming the antecedent (2) of a conditional statement (1) in order to be able to affirm its consequent (3):

(MP-4) (1) If you are illiterate, you are not reading
(2) You are illiterate
(3) So, you are not reading

By giving an infinite number of interpretations to the sentential variables P and Q in the argument form (MP), we could produce an infinite number of arguments with that form. The number of possible arguments having any given logical form is clearly infinite: between the meaningful and nonsensical sentences that we could substitute for any sentential variables in any argument form, the possibilities are surely infinite.
The fact that for any ARGUMENT FORM there are an infinite number of possible arguments having that form may be seen to pose a problem for determining whether any ARGUMENT FORM is valid. Consider the following various definitions of validity:

An ARGUMENT \((A)\) is VALID if, and only if:

- NO argument with the same ARGUMENT FORM \((A')\) has true premises but a false conclusion. Or, equivalently:
- its ARGUMENT FORM \((A')\) is valid

An ARGUMENT FORM \((A')\) is valid if, and only if:

- NO argument with the same form \((A')\) has true premises but a false conclusion; or
- Every argument with the same form \((A')\) that has true premises also has a true conclusion.

Notice that the allegation that either an argument or an argument form is valid is a statement about ALL POSSIBLE arguments having the same form. Recall that the number of possible arguments having any given form is potentially INFINITE.

This would seem to pose a problem for knowing or PROVING that any argument or argument form is indeed valid: How, after all, can we possibly examine EVERY POSSIBLE argument of a given form to see whether the ones with true premises also have true conclusions? This seems an endless, inconclusive and impossible task, given that the possibilities we would have to examine are infinite. How can we ever know that any argument or argument form is valid? Let's first consider one way to show that an argument FORM is invalid.
The 'COUNTER-EXAMPLE' TECHNIQUE

For Showing Arguments Forms to Be Invalid

The allegation that an argument is valid is tantamount to a universal generalization (like 'All swans are white'), to the effect: All arguments of this form that have true premises also have a true conclusion. We know that to refute a universal generalization about all things of a kind (e.g., all swans, all arguments with a certain logical form), all we need to do is find one case in point (e.g., a swan) that is an exception to the generalization (e.g., a case of a black swan, a swan that is not white).

So, to try to refute the allegation that an argument is valid, to show that an argument is not valid, we have the same procedure open to us: The allegation that an argument is valid is a generalization about ALL arguments with the same logical form; the allegation is that EVERY argument with the same logical form that has true premises also has a true conclusion. To refute this allegation, to show that an argument or ARGUMENT FORM is INVALID, we need to find an argument with the SAME LOGICAL FORM that has obviously true premises but an obviously false conclusion.
For example, the following argument form (DA) is INVALID. This may not be obvious from argument (DA-1), whose premises and conclusion are all true (invalid arguments can ring with truth and, so, deceive). That the argument form (DA) is invalid is obvious from the second example (DA-2), which shows that an argument with the logical form (DA) can have TRUE premises but a FALSE conclusion: AN INVALID ARGUMENT FORM CAN LEAD US FROM TRUTH INTO FALSEHOOD. Consider:

\[
\text{(DA)} \quad (1) \text{ If } P, \text{ then } Q \quad \text{(DA')} \quad P \Rightarrow Q
\]
\[
(2) \text{ Not } P \quad -P
\]
\[
\hline
(3) \text{ Therefore, not } Q \quad -Q
\]

\[
\text{(DA-1)} \quad (1) \text{ If you're in Moscow, you're in Russia}
\]
\[
(2) \text{ You're NOT in Moscow}
\]
\[
\hline
(3) \text{ Therefore you're NOT in Russia}
\]

\[
\text{(DA-2)} \quad (1) \text{ If you're in Alaska, you're in North America (True)}
\]
\[
(2) \text{ You're NOT in Alaska (True)}
\]
\[
\hline
(3) \text{ Therefore you're NOT in North America (False!)}
\]

This INVALID argument form is called 'the fallacy of denying the antecedent' because on the basis of denying the antecedent (2) of a conditional statement (1), the conclusion of the argument (3) then denies the consequent of the conditional. This is logically fallacious: (3) does not follow from (1) and (2), as example (DA-2) clearly shows.
A similarly INVALID argument form is 'the fallacy of affirming the consequent' where, on the basis of (2) affirming the consequent of (1) a conditional, the argument then invalidly (3) affirms the antecedent:

\[(AC) \quad (1) \text{If } P, \text{ then } Q \quad (AC') \quad P \rightarrow Q\]
\[(2) \quad Q \quad (3) \text{Therefore, } P \]

\[(AC-1) \quad (1) \text{If you're on the moon, you're in our solar system (True)} \]
\[(2) \text{You're in our solar system (True)} \]
\[(3) \text{Therefore, you're on the moon (False!)} \]

While example (AC-1) clearly proves that ARGUMENT FORM (AC) [and any argument whose relevant logical form is the same as (AC)] is INVALID, invalid arguments of this form may deceive us with the ring of truth:

\[(AC-2) \quad (1) \text{If you're on earth, you're in our solar system} \]
\[(2) \text{You're in our solar system} \]
\[(3) \text{Therefore, you're on earth} \]

The LOGICAL FORM of (AC-2) is INVALID [as proven by (AC-1)], even though its premises and conclusion all happen to be true: true though they all be, the conclusion (3) does not follow from premises (1) and (2).

The technique we just used to prove that a given ARGUMENT FORM is invalid is called the 'COUNTER-EXAMPLE TECHNIQUE.' This technique is familiar to you in the case of refuting (finding exceptions to) generalizations about all things of a certain kind. The generalization 'All swans are white' is disproven by the counter-evidence, the counter-instance or counter-example of a black swan. The generalization
that all philosophers smoke pipes could be refuted by a counter-example: a case of a philosopher who did not smoke a pipe.

Likewise, the allegation or presumption that an argument's form is valid can be defeated by finding a counter-example to the proposition 'Every argument with the same form that has true premises also has a true conclusion':

A counter-example would be an argument with the same form that had true premises but a false conclusion—like (DA-2) and (AC-1) above.

Remember: The proposition that an argument or argument form is valid is a universal generalization to the effect that every argument with the same form that has true premises also has a true conclusion. A counter-example that would defeat the claim of validity would be any argument of the same form that had obviously true premises but an obviously false conclusion.

(Why, for purposes of proving the invalidity of an argument form, is it important that the counter-example have obviously true premises and an obviously false conclusion? Be sure you can answer this question, or discuss it with your instructor.)

So, as you see, we have a technique for proving that invalid argument forms are in fact invalid. But does this technique bring us any closer to being able to prove that any argument form is valid? Not a bit.

Disproving a universal generalization is a good deal easier than proving one. To prove a universal generalization to be true, in effect, requires us to examine every case in the universe to which the generalization applies.

In the case of 'All swans are white' we would need to examine ALL swans to know that they all are white. Failure to find a counter-instance does not prove that a universal generalization is true; it only allows some probability that the generalization is true and only proves that we have failed to find any counter-evidence. In fact, the proposition 'All swans are white' is false, although it took quite some time for the western world to discover this fact (by finding counter-examples, black swans, in Australia).
The failure to find a counter-example to prove that an argument form is invalid does not prove that the argument form is valid. It only proves that we have failed to find otherwise. This failure may be accountable to a failure to search thoroughly enough, a failure in imagination, or the complexity of the argument form, or the fact that the argument form is after all valid!

Is the following argument form invalid?

P or not Q
Q or not R
R only if (J or K)
K only if M
M or P

Therefore, P

How long would you care to spend looking (in your imagination) for a counter-example to try to prove it invalid? Care to try?

It would be nice to have a straightforward DECISION PROCEDURE for determining the invalidity of argument forms. The counter-example technique can require a lot of work and imagination. Moreover, it can not provide us with any proof that an argument is VALID.

To prove that an argument or argument form is VALID, we would have to examine, in effect, EVERY argument of that form to see whether ALL the arguments with that form that had true premises also had true conclusions, such that NO argument of that form could lead from truth to falsehood.

SENTENTIAL LOGIC provides a technique for doing just that, a DECISION PROCEDURE for, in effect, examining ALL the possibilities in order to determine whether any argument of a certain form can possibly have true premises and a false conclusion. So far as the validity or invalidity of arguments depends on how their premises and conclusions are constructed by means of SENTENTIAL CONNECTIVES, an analysis of the sentential connectives can provide a straightforward PROOF of validity or invalidity within this domain of logic.
To see how this proof technique works, we look next in sections 2.4 - 2.5 at what's called the truth-functional analysis of the five basic logical operations that we can perform on atomic sentences when we combine them into more complex propositions and arguments by use of sentential connectives. Do you remember operations are? (See section 2.1)
2.4. SENTENTIAL CONNECTIVES AND TRUTH-FUNCTIONALITY

You already, at least implicitly, know about truth-functionality from your commonsense understanding of the sentential connectives 'not' and 'and' (and their cognates, like 'hardly' and 'but,' etc.).

A sentential connective is **TRUTH-FUNCTIONAL** so far as the truth-value of any molecular sentence formed with it is a unique function of the truth-values of its component or atomic sentences.

For example, you know that the truth-value (truth or falsity) of the NEGATION of a sentence P is a unique function of the truth-value of P: If P is true, then its NEGATION (-P) is false. If P is false, then its NEGATION (-P) is true.

You know that the truth-value of a CONJUNCTION of two sentences (P & Q) is a unique function of the truth-values of its component atomic sentences P and Q:

(P & Q) says, in effect:

Both P AND Q are true; or

P is true AND Q is true.

So where both conjuncts are true, their conjunction is true; otherwise, the conjunction is false. This truth-functional rule for conjunction (or any truth-functional operation) can be represented schematically in what's called a **TRUTH-TABLE**. We take it that any sentence P is either true or else it's false--it has one of two possible truth-values. In tabular form, where 'T' stands for 'true' and 'F' for 'false':

<table>
<thead>
<tr>
<th>P</th>
<th>T</th>
<th>P is either true,</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F</td>
<td>or else it is false</td>
</tr>
</tbody>
</table>
Any two sentences P and Q will then have four possible combinations of truth-values; in tabular form, as follows:

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>(2)</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>(3)</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>(4)</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

where both are true
where one is true and the other false
where the one is false and the other true
where both are false

The truth-functional rule for conjunction counts as conjunction (P & Q) true where both conjuncts are true [represented by line (1) below]; otherwise [where one or more conjuncts are false—lines (2)-(4) below], a conjunction is counted as false. This is, of course, nothing more than our common truth-functional sense. In tabular form:

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>P &amp; Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>(2)</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>(3)</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>(4)</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Where both conjuncts true, the conjunction as a whole is true;
otherwise, it is false.

On the basis of the truth-functional analysis of the sentential connectives, we can prove the validity or invalidity of argument forms whose crucial logical components are sentential connectives.

For lessons on the truth-functional analysis of each of the basic sentential connectives, you are referred to the SENT program in ANALYTICS. For practice in analyzing the truth-functional meaning of the basic connectives, you are referred to the TRUTH program in analytics. For lessons on how the truth-functional analysis of the sentential connectives is applied to proving validity and logical equivalence, you are referred to the VALIDITY and EQUIV programs.
2.4.1. NON-TRUTH-FUNCTIONAL INTERPRETATIONS OF CONNECTIVES

Truth-functional analysis does not capture everything that we may mean to say by use of sentential connectives, even in the case of the simple conjunction term 'and.'

Connectives like 'and,' which can be analyzed truth-functionally, can also bear other dimensions of meaning. For example, 'and' may carry a temporal or causal meaning, where it's not merely asserted by a conjunction (P & Q) that both are true, but where it's also asserted that the state of affairs described by P occurred before the state of affairs described by Q. For example, the following conjunctions can mean different things, and (1) can be true whereas (2) can be false, even though both conjuncts might be true in each case:

1. Sally got married and Sally got pregnant
2. Sally got pregnant and Sally got married

The order in which Sally did these things can be important to whether we count these conjunctions true or false: the conjunction term 'and' can be intended to convey not only that Sally did both things, but also that she did one before or because of the other. We can make these additional meanings of 'and' explicit as follows:

(1') Sally got married and then Sally got pregnant
(2') Sally got pregnant and because of this Sally got married

Note also that conjunctive terms like 'but,' 'however,' 'nevertheless' carry a certain sense of reservation and contrast not carried by the simple 'and'—a sense, like the temporal or causal sense of 'and,' that goes beyond the purely truth-functional analysis of CONJUNCTION. When we represent conjunction by the truth-functional operator '&,' we are representing the most basic sense of any conjunction of two sentences (P & Q), the bottom line, as it were, namely: BOTH conjuncts are asserted as true—never mind whatever other contrasts or (causal or temporal) relationships are imputed between the states of affairs that P and Q describe.
Conditional connectives perhaps best illustrate the variety of function that connectives can ordinarily enjoy in everyday usage. For example, statements of the form 'If P then Q' often refer to temporal or causal relations between states of affairs:

If there's lightning then there'll be thunder (afterwards)

When there's lightning, there's thunder

Where there's smoke, there's fire

If there's smoke, there's fire (as its cause)

These dimensions of meaning of ordinary conditional statements are clearly important in everyday life. In truth-functional logic we put them aside in order to concentrate on the 'bottom-line' interpretation of the various sentential connectives, namely, the truth-functional interpretation that allows us to account for the validity or invalidity of the most basic of argument forms.
2.4.2. CONJUNCTIONS AND DISJUNCTIONS

The standard conjunctive expression is 'AND.'

Its 'bottom-line,' truth-functional interpretation is represented by the truth-table for '&'

The conjunction of any two sentences, P and Q, symbolized 'P & Q,' asserts, at bottom: Both P AND Q are true

Other conjunctive expressions, which can give different colorations to conjunctions in ordinary language, also assert, as a matter of their 'bottom line' truth-functional interpretation, that both their conjuncts are true. For example,

I'm tired; however I will help you

asserts, however begrudgingly, that

It's true that I'm tired AND it's true that I will help you
A conjunction of P and Q, truth-functionally interpreted, differs in what it asserts as a matter of its 'bottom-line' (that is, in what it asserts about the truth-values of its component sentences) from the disjunction of P with Q:

An Inclusive Disjunction of P with Q, symbolized P v Q, asserts: either one disjunct or the other is true--at least one is true--or both are true. An inclusive disjunction does not insist that both its components are true (unlike a conjunction of P and Q, which asserts, in effect, that both P and Q are true).

An EXCLUSIVE DISJUNCTION of P with Q, is really a conjunction of the inclusive disjunction (P v Q) and the negation -(P & Q), symbolized:

(P v Q) & -(P & Q)

[P or Q but not both P and Q]

An EXCLUSIVE DISJUNCTION asserts, in effect: at least one disjunct is true, but not both of them. (Unlike an inclusive disjunction, an exclusive disjunction excludes the possibility of both disjuncts' being true. While a conjunction insists that both P and Q are true, an exclusive disjunction insists that it is not the case that both are true). A typical case of the exclusive use of 'OR' would be:

You can have your cake, or you can eat it

[but you can't both have and eat it]

When we wish to avoid the ambiguity of 'or' in ordinary language, we can simply spell out what we mean by adding BUT NOT BOTH, as in:

You can either have a good time tonight or pass the exam tomorrow, but you can't do both.
There are a variety of conjunctive expressions in ordinary language that have the same 'bottom line' truth-functional force as 'AND' in sentences of the form:

P and Q, symbolized: P & Q

The following are all conjunctions with the same logical force as 'P and Q' and all of them would be symbolized 'P & Q':

P but Q
P, however Q
Although P, Q
P; nevertheless Q
While P, Q
P; Q
P, yet Q
P, albeit Q
Whereas P, Q
P, even if Q

Notice the last example: P, even if Q. This conjunctive expression contains an 'if' and to that extent looks like a conditional expression. But the following example of this form

I'm damn well going to finish the race even if I am tuckered out.

is clearly a conjunction by which I'm asserting both that I am tuckered out but that I'm going to finish the race nonetheless. This statement is not a conditional: it is not asserting that my being tuckered out is a condition (sufficient or otherwise) of my finishing the race; it does not mean that if I am tuckered out, then I will finish the race; nor that I am not tuckered out unless I'm going to finish the race. (The fact that I am now tuckered out is presumably in no way conditional upon either my determination to finish the race or the fact that I will finish it.)
The word 'if' by itself can serve as a conjunctive rather than as a conditionalizing expression, as in:

It's desirable, if costly, to get the roof repaired

This statement, surely or probably, does not mean:

It's desirable to repair the roof IF it's costly to do so

Surely I don't mean to say that the costliness of the repair is a condition of its desirability. (If anything, the costliness by itself probably makes the repair undesirable. Who finds costliness desirable?) The statement is commonsensically interpreted as a conjunction:

It's desirable, although costly, to get the roof repaired

Just as some conditional-like expressions are sometimes conjunctive in their function, some conjunctive-like expressions can be conditional in their function. For example:

'Spare the rod and spoil the child'

presumably means:

If you spare the rod, then you will spoil the child

Another example from your text: You were given the following case of an invalid argument:

(1) If you are illiterate, you are not reading

But (2) You are not illiterate

Therefore (3) You are reading
Then it was observed:

Close your eyes and the conclusion (3) is false while the premises (1) and (2) remain true.

This statement could, and probably should, be logically parsed as a conjunction whose left conjunct is a conditional, as follows:

If/when you close your eyes, then the conclusion is false, and, even though you close your eyes, the premises remain true.

Here your closing your eyes is a sufficient condition for making the conclusion false: the falsity of the conclusion is conditional in some way on your closing your eyes (or otherwise paying it no notice). Here's another example:

Take a bad attitude and you'll surely fail

This statement probably means:

If you have a bad attitude, then you'll fail

not that both propositions are already true, i.e. that:

You (already) have a bad attitude and you will fail

As a piece of advice, the statement can be taken to contend that there is some conditional connection between your attitude and your success or failure, and that failure will follow upon your taking a certain attitude (to which you may not have yet succumbed). Far from asserting that you do in fact have a bad attitude, the statement may be meant to warn you against taking one.
2.4.3. CONDITIONALS AND 'UNLESS' EXPRESSIONS

The standard conditional expression is 'IF' or 'IF--THEN--'.

Its truth-functional interpretation is represented by the truth-table for the arrow '=>'

A standard conditional of the form 'IF P, then Q,' truth-functionally construed and symbolized 'P => Q' asserts, variously and in effect:

If P is true, then Q is true

P's being true is sufficient for Q's being true

P is true only if Q is true

Q's being true is necessary for P's being true.

P is not true unless Q is true

NOTE that certain other expressions in ordinary language can assume conditional force, equivalent to 'if.' Sentences of the following forms can all have the same truth-functional or logical force, symbolized P => Q:

If P, Q

Q, provided that P

When P, Q

Q, in case P

Q, in the event that P
Notice that 'IF—THEN---,' '—ONLY IF---' and 'NOT—UNLESS---' expressions can be used to assert the same truth-functional or logical relation between the antecedent and consequent of a conditional:

<table>
<thead>
<tr>
<th>ANTECEDENT</th>
<th>=&gt;</th>
<th>CONSEQUENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>=&gt;</td>
<td>C</td>
</tr>
<tr>
<td>IF A</td>
<td>THEN</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>ONLY IF</td>
<td>C</td>
</tr>
<tr>
<td>NOT A</td>
<td>UNLESS</td>
<td>C</td>
</tr>
</tbody>
</table>

For now think about the translational key as a step-by-step translation of UNLESS expressions into a standard ONLY-IF conditional expression. Be sure to reason this procedure through with examples (e.g., 'It's NOT raining UNLESS it's wet': 'It's raining ONLY IF it's wet'), so that you understand intuitively the implicit negation operation involved, as follows:

- **STEP (1):** Think of the 'unless'-clause as the consequent, like the 'then'- or 'only if'- clause in conditionals: so, 'unless' is symbolized by the arrow '=>.' REFER TO THE SCHEMA PROVIDED ABOVE

- **STEP (2):** The antecedent of the 'unless' statement (the clause preceding 'unless') is negated.

- **STEP (3):** In 'not-unless' expressions, this is tantamount to eliminating the 'not' in 'not-unless' by double negation ('Not not P only if Q' is equivalent to 'P only if Q').
Notice that the translation of the 'IF--THEN---' expression into an equivalent 'UNLESS' expression entails the introduction of a NEGATION. Certain negation operations are implicit in conditionals. Consider the following series of logically equivalent expressions:

1. If it's raining, then it's wet: \( R \Rightarrow W \)
2. It's raining only if it's wet: \( R \Rightarrow W \)
3. If it's not wet, then it's not raining: \( \neg W \Rightarrow \neg R \)

To assert a statement of the form:

1. If \( P \) is true, then \( Q \) is true 
or, equivalently:
2'. \( P \) is true only if \( Q \) is true

is to assert (implicity):

3'. If \( Q \) is not true, then \( P \) is not true.

Given the logical equivalence between conditionals and disjunctions stated by the Implication Rule (see Bertie rule sheet and CHAPTER 4), we can translate 'UNLESS' expressions either into a standard CONDITIONAL or into a standard DISJUNCTION.

If you can understand the transformation of 'unless' expressions into only if conditionals, you can then see why 'unless' expressions can also be treated as disjunctions (and you can prove their truth-functional equivalence with a truth-table).

Treating 'UNLESS' as if it were 'OR' raises the question of whether 'unless statements, like 'OR statements, might not be ambiguous. But therein lies another tale. For now, you will be safe translating 'UNLESS statements into equivalent 'ONLY IF' conditionals on the model given above.
2.4.4. BICONDITIONALS: NECESSARY AND SUFFICIENT CONDITIONS

The standard biconditional expression is really conjunctive:

'--IF AND ONLY IF--.'

Its truth-functional meaning is represented by the truth-table for the double arrow '<=>'

The expression 'just in case' is conventionally used in philosophical literature as an abbreviation for 'in case and just in case': 'P JUST IN CASE Q' means the same as 'P IF AND ONLY IF Q.'

A biconditional P if and only if Q is at bottom a conjunction of two conditionals: P if Q and P only if Q. A biconditional of the form P if and only if Q asserts, in effect: P is a necessary and sufficient condition for Q (and Q is a necessary and sufficient condition for P). That is: P is true when, and only when, Q is true; Q is true when, and only when, P is true. The biconditional asserts, in short: P and Q have the same truth-value—i.e., P is true whenever Q is true and false whenever Q is false. The biconditional is itself true when in fact P and Q have the same truth-value (i.e., when in fact P and Q are either both true or both false). The biconditional is false when P and Q have different truth values (i.e., when one is true and the other is false). The truth-functional rule for biconditionals is summarized in the truth-table in the TRUTH program (see the list of commands in TRUTH or type HELP) and it is also summarized in section 2.5.5 below.

Notice that the biconditional statement:

It's raining if and only if it's wet

is a much stronger statement than the conditionals:

It's raining only if it's wet.

If it's raining, then it's wet.

The above conditionals are true, whereas the above biconditional is false. Can you explain why in this case the biconditional is false? (Analyze it as the CONJUNCTION of two conditionals.)
Biconditionals are useful where we want explicitly to lay down both NECESSARY and jointly SUFFICIENT conditions for some proposition's being true. For example, the principle

Something has a right to life IF AND ONLY IF ... 

would assert (in place of the elipsis'...') the CONDITIONS

on which something would be granted a right to life, each and every one of which purportedly must be satisfied for something to have a right to life. Have you any notion of what those necessary and jointly sufficient conditions conceivably might be? (You might try constructing such a principle.) The following principle is arguably false. Can you explain why?

Something has a right to life if, and only if, it is alive.

Do we wish to say that viruses have such a right? Would potentially fatal viruses not then have a right to life if merely being alive were a SUFFICIENT condition for having this right? (On the other hand, being alive seems a plausible NECESSARY condition for having a right to life. So, one might argue that the biconditional is false since one of its conjuncts is false, namely, the conditional:

Something has a right to life \textbf{if} it is alive

Something has a right to life \textbf{only if} it is alive
2.4.5. 'UNLESS' AGAIN, UNLESS YOU DON'T WANT IT

We would like to have a reliable rule of thumb for interpreting 'UNLESS' expressions — unless, of course, we don't care to be clear about exactly what to expect when people say things like the following:

UNLESS you fail the final, you will pass the course

What might this mean? Here are four possibilities:

1. You will NOT pass the course ONLY IF you fail the exam:   \(-P \Rightarrow F\)
2. IF you fail the final, THEN you will NOT pass the course: \(F \Rightarrow -P\)
3. You will NOT pass IF AND ONLY IF you fail the final: \(-P \iff F\)
4. Either you will fail the final or you'll pass the course: \(F \lor P\)

Presumably, the conditions—necessary or sufficient—for passing the course are more clearly laid out in the conditional sentences 1-3. Because 'unless' statements so often imply that certain conditions are necessary or sufficient for some state of affairs to come about, we will translate them into some expressly conditional form. Notice that by the implication and transposition rules the disjunctive sentence 4 is equivalent to the conditional sentence 1, as follows:

\(F \lor P \iff \neg F \Rightarrow P \iff \neg P \Rightarrow F\)

Translating 'unless' as 'or' is easy, and makes for ease in symbolization, because no 'implicit' negations are involved as in the cases of sentences 1 through 3. But, since (a) translating 'unless' as 'or' does not often result in a clear statement of the conditional import of the 'unless' expression and since (b) not all 'unless' expressions are suitably rendered in the particular form of sentence 1 (logically equivalent to sentence 4), we will not translate 'unless' as 'or.' We will worry instead about which conditional form best captures the conditional import of 'unless' expressions.

Now, a good rule of thumb is to translate 'unless' expressions into the weaker conditional form of sentence 1 above, as follows:
(i) 'UNLESS' is replaced by 'ONLY IF' -- and thus

the 'UNLESS' clause becomes the CONSEQUENT of a conditional

(ii) The other clause is NEGATED

and becomes the ANTECEDENT of this conditional

THUS:

UNLESS you fail the final, you will pass the course

ONLY IF you fail the final, will you NOT pass the course

You will NOT pass the course ONLY IF you fail the final: \(-P \Rightarrow F\)

We say this is a 'weaker' interpretation of the conditional import of the 'unless' expression than sentence 2 above, because sentence 1 posits failing the final only as a necessary condition for not passing the course, and thus allows the possibility that you will still pass the course even in the event that you fail the final. Consider:

1. You will NOT pass the course ONLY IF you fail the final: \(-P \Rightarrow F\)
2. You will NOT pass the course IF you fail the final: \(F \Rightarrow -P\)

Suppose you were to fail the final: it would follow from 2 that you would not pass the course. But this would not follow from 1: here you still have a chance to pass the course anyway.

In general, given a statement of the form:

\[ P \text{ unless } Q \]

it can be crucial to know how to take its conditional import:

1. \textbf{Not } \textbf{P } \textbf{only if } \textbf{Q}

   \textit{posits } \textit{Q} \textit{ as a merely necessary condition for the negation of } \textit{P}
2. If $Q$, then not $P$

posits $Q$ as a sufficient condition for the negation of $P$

Consider, for example, where someone says to you:

$I'll$ kill you UNLESS you hand over your money

It would be nice to know that the conditional import of this statement were the 'stronger':

**IF** you hand over your money, **I will NOT** kill you

rather than the 'weaker':

**I will NOT** kill you **ONLY IF** you hand over your money

It's always nice to be sure of what one's getting for one's money, especially when one's life is at stake.

Granted, interpreting 'unless' expressions is not likely to be a life-and-death matter. But how are we to know which interpretation to draw in any case?

We adopt the general rule of thumb that 'unless' be rendered in the weaker sense illustrated above if only because it the safer interpretation in many or most pragmatic contexts. If it is not obvious in any given context what the conditional import of 'unless' is meant to be, we can ask for clarification: 'Do you mean that **IF** I hand over my money, then you WON'T kill me?'

Where there is no one to ask our rule of thumb may be safe more often than not -- unless there are clear independent reasons for departing from it. Sometimes a little reflection will show that 'unless' is better rendered one way rather than the other. Consider the following cases:

(A) It's not raining unless the streets are wet

(B) It's raining unless the streets are dry
Consider now the alternative interpretations:

A-1. It's NOT not raining ONLY IF the streets are wet

   It's raining ONLY IF the streets are wet: \( R \Rightarrow W \)

A-2. IF the streets are wet, then it's NOT not raining

   IF the streets are wet, then it's raining \( W \Rightarrow R \)

B-1. It's NOT raining ONLY IF the streets are dry

   IF it's NOT raining then the streets are dry \( -R \Rightarrow D \)

B-2. IF the streets are dry, then it's NOT raining \( D \Rightarrow -R \)

In these cases a little reflection will show that, if we want to interpret 'unless' in a way that will be sure to be in accord with what is true, our rule of thumb does not apply to sentence B, although it just as clearly does apply to sentence A: in general, it's true that it's raining ONLY IF the streets are wet; but it is not always true that IF it's NOT raining, then the streets are dry (because it may be that the streets are wet from a recent rain, or dew, or a street cleaner and that it is not presently raining).

Thus: the 'weaker' conditional interpretation of sentence A seems decidedly more plausible than the stronger; but the stronger interpretation of sentence B seems just as decidedly more plausible than the weaker.

You will encounter similar cases where the stronger interpretation of 'unless' seems called for in the context of what you know to be the case. In such cases you should be guided by your commonsense, linguistic intuitions and reflection. It's at least useful to be aware of the three alternative interpretations of 'unless' for purposes of this kind of critical reflection on the conditional import of whatever is said.

We can now amend our rule of thumb for interpreting 'unless' expressions:
Render sentences of the form 'P unless Q' as 'Not P only if Q'
UNLESS a stronger interpretation is arguably indicated.

Is this a good rule of thumb?

How do you think we should interpret the 'unless' in the statement of the rule?

Can you think of any cases where 'unless' should arguably be rendered using 'IF AND ONLY IF.' How would you write the rule for translating 'unless' into 'if and only if'? How would you translate a sentence of the form 'P unless Q' into a purportedly equivalent sentence form using 'if and only if'?

A good rule of thumb is to translate 'unless' expressions using 'only if' (refer to the schema in section 2.4.3). Why?
2.5. SUMMARY: THE TRUTH-FUNCTIONAL CONNECTIVES

2.5.1. NEGATION: 'NOT'

The TRUTH-VALUE (truth or falsity) of a DENIAL or NEGATION, for example,

You are NOT reading

depends upon, is a direct FUNCTION of the truth-value of its COMPONENT sentence

You are reading

Where its COMPONENT sentence is TRUE, as in the example above, the NEGATION is FALSE, and conversely. Commonsensically:

To DENY a TRUE statement is, in effect, to make a FALSE statement.

And, conversely, to DENY a FALSE statement is, in effect, to make a TRUE statement.

The logical form of a negation

\[ \neg R \]

is symbolized

\[ \neg R \]

where '\( \neg \)' stands for NEGATION and 'R' stands for 'You are reading'

Then:

It is NOT the case that you are NOT reading

(a NEGATION of a negation) is symbolized:

\[ \neg \neg R \]
The general rule by which we decide the truth-value of the NEGATION of any arbitrary sentence P is:

Where P is TRUE, NOT P is FALSE; and

Where P is FALSE, NOT P is TRUE.

This rule states the TRUTH-CONDITIONS of NEGATION: the conditions under which a negation (Not P) is TRUE (i.e., where P is FALSE) and the conditions under which a negation (Not P) is FALSE (i.e., where P is TRUE).

The TRUTH-CONDITIONS for NEGATION -- for any TRUTH-FUNCTIONAL SENTENTIAL CONNECTIVE -- can be displayed briefly in a TRUTH-TABLE.

NEGATION is considered TRUTH-FUNCTIONAL because the truth-value of a negation (Not P) varies as a direct FUNCTION of the truth-value of its COMPONENT sentence (P), as the following truth-table will show.

Let 'T'='true' and 'F'='false'. Any sentence P has one of TWO POSSIBLE truth-values: P is either TRUE or else P is FALSE:

<table>
<thead>
<tr>
<th>P</th>
<th>Not P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

So, a truth-table summarizes ALL POSSIBLE truth conditions for NEGATION and it shows how NEGATION is TRUTH-FUNCTIONAL, that is, how the truth-value of a negation (-P) varies as a direct FUNCTION of the truth-value of its COMPONENT (P):

<table>
<thead>
<tr>
<th>P</th>
<th>-P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

This truth-table is a summary of the rule for deciding the truth-value of negations.
2.5.2. **CONJUNCTION: 'AND'**

The logic of conjunction concerns the **TRUTH-FUNCTIONALITY** of 'AND', that is, it concerns how the truth-value of the **CONJUNCTION** of any two sentences

\[ P \text{ AND } Q \]

varies as a direct **FUNCTION** of the truth-values of its **COMPONENTS**

\[ P \]

\[ Q \]

Suppose we know that the following is true about any arbitrary sentence \( P \),

It is not the case that \( P \)

which, for convenience, we abbreviate as

\( \text{Not } P \)

and symbolize as

\( -P \)

where, equivalently, we could say that

\( P \text{ is false} \)

What then do we know, what can we infer or deduce about a statement of the following form (where \( Q \) stands for any arbitrary sentence other than \( P \))?  

\[ P \text{ and } Q \]

Is it true or false?

What if we **KNOW** that \( Q \) is true? What would be the truth-value of the conjunction

\[ P \text{ and } Q \]

where, as we've already supposed, \( P \text{ is false} \)

**GIVEN** that at least one conjunct (\( P \)) is false, any conjunction of the form

\[ P \text{ and } Q \]
MUST be false. This is commonsense.

Only ONE conjunct's being true is not sufficient to count the conjunction as a whole true.

A conjunction of any two arbitrary sentences

\[ P \text{ and } Q \]

asserts, in effect, the following:

'P' is true AND 'Q' is true

or, more emphatically,

BOTH 'P' AND 'Q' are true.

A conjunction of any two arbitrary sentences

\[ P \text{ and } Q \]

is counted as TRUE, commonsensically:

If, and ONLY if, BOTH component sentences (CONJUNCTS) are TRUE.

A conjunction is FALSE:

If AT LEAST ONE component sentence (CONJUNCT) is FALSE.

The rule for deciding the truth-value of any CONJUNCTION is, simply

The conjunction is true ONLY on condition that BOTH conjuncts are true; otherwise, it is false.

We can summarize the rule for deciding the truth-value of the conjunction of any two arbitrary sentences

\[ P \text{ and } Q \]

in a TRUTH-TABLE.

The truth-table for conjunction displays the TRUTH-CONDITIONS for conjunction: the conditions under which a conjunction is counted as true or false.

A truth-table summarizes ALL POSSIBLE COMBINATIONS of truth-values for the component sentences of a conjunction, as follows.

The POSSIBLE COMBINATIONS of truth-values for any TWO sentences, \( P \) and \( Q \), are FOUR, as follows:
P : Q

---!---

(1) T : T where both are TRUE
(2) T : F where one is TRUE and the other is FALSE
(3) F : T where one is FALSE and the other is TRUE
(4) F : F where both are FALSE

The conjunctive 'AND' is considered a TRUTH-FUNCTIONAL connective because the truth-value of any CONJUNCTION is a direct FUNCTION of the truth-value of its COMPONENT sentences.

A truth-table shows how the truth-value of a conjunction varies as a direct function of the truth-values of its components, according to our simple rule:

P : Q : P and Q

---!---!---------

T ! T ! T Where BOTH conjuncts are TRUE, the
! ! conjunction is true. Otherwise,
T ! F ! F where AT LEAST ONE
F ! T ! F conjunct is FALSE, the
F ! F ! F conjunction is false.

The rule states -- and the above table displays -- the TRUTH-
FUNCTIONALITY of the sentential connective 'AND'.

You should be aware that there are several CONJUNCTION terms in English besides 'AND'. The following are all equivalent:

P and Q
P but Q
Although P, Q
Q, however P
P, nevertheless Q

All are symbolized

P & Q

and assert, as their 'bottom line' logical 'cash value':

P is true and Q is true.
2.5.3. **DISJUNCTION: 'OR'**

The use of the disjunctive connectives

\[ \text{OR} \]

\[ \text{EITHER } _{\text{OR}} _{\text{OR}} \]

in ordinary language is AMBIGUOUS.

That is, a sentence of the form

\[ P \text{ or } Q \]

may mean either of the two following different things:

1. Either \( P \) is true, or \( Q \) is true --
   AT LEAST ONE is true --
   but NOT BOTH.

   This is called the EXCLUSIVE sense of 'or'.

2. Either \( P \) is true, or \( Q \) is true --
   AT LEAST ONE is true --
   OR BOTH are true.

   This is called the INCLUSIVE sense of 'or'.

We let the letter

\[ v \]

stand for

\[ \text{or} \]

in the INCLUSIVE sense.

We will understand disjunctions of the form

\[ P \text{ or } Q \]

in the INCLUSIVE sense, symbolized

\[ P \text{ v } Q \]

and meaning

Either \( P \) or \( Q \) OR BOTH.

When an EXCLUSIVE disjunction is intended (where the possibility
of both alternatives' being true is meant to be excluded) we will SPELL OUT this exclusive sense explicitly, as follows:

\[ P \lor Q \text{ [in the EXCLUSIVE sense]} \]

means

Either \( P \) or \( Q \) BUT NOT BOTH.

So, we can spell out the sense of

EXCLUSIVE 'OR'

in terms of a combination of

INCLUSIVE 'OR'

CONJUNCTION ('and'/ 'but')

NEGATION ('not')

An EXCLUSIVE disjunction of any two arbitrary sentences \( P \) and \( Q \), once spelled out fully, may be analyzed as

A CONJUNCTION of

an INCLUSIVE DISJUNCTION and the NEGATION of a conjunction:

\[ (P \lor Q) \land \neg(P \land Q) \]

Either \( P \) or \( Q \) BUT Not both \( P \) and \( Q \)

A truth-table will show us under which of all the possible truth conditions inclusive and exclusive disjunction differ in truth-value.

Recall: We analyze the EXCLUSIVE DISJUNCTION of \( P \) and \( Q \) as the

CONJUNCTION: \( (P \lor Q) \land \neg(P \land Q) \)

\( P \) or \( Q \) but not both \( P \) and \( Q \)

consisting of the

INCLUSIVE DISJUNCTION: \( P \lor Q \)

and the

NEGATION: \( \neg(P \land Q) \)

denying the conjunction: \( P \land Q \)

Let's see how the truth-value of
the INCLUSIVE \( P \lor Q \)

and

the EXCLUSIVE \( (P \lor Q) \land \neg(P \land Q) \)

vary as a function of the truth-values of their component sentences.

\[
\begin{array}{cccccc}
\text{Incl.} & \text{Disj.} & \text{Conj.} & \text{Negation} & \text{Exclusive Disj.} \\
\hline
P \lor Q & P \lor Q & P \land Q & \neg(P \land Q) & (P \lor Q) \land \neg(P \land Q) \\
\hline
(1) & T & T & T & F & F \\
(2) & T \lor F & T \lor F & T & F & T \\
(3) & F \lor T & T \lor F & F & T & T \\
(4) & F \lor F & F \lor F & F & F & F \\
\end{array}
\]

Lines (2), (3) and (4) represent the fact that INCLUSIVE and EXCLUSIVE disjunctions are true if AT LEAST ONE disjunct is true and false if both disjuncts are false.

Line (1) shows how INCLUSIVE and EXCLUSIVE disjunctions differ: INCLUSIVE allows both alternatives to be true. EXCLUSIVE disjunction excludes the possibility of both alternatives' being true.

AGAIN: We will interpret disjunctions like 'P or Q' in the INCLUSIVE sense and count them as true whenever AT LEAST ONE disjunct is true and also when both disjuncts are true.

We symbolize the inclusive disjunction of P with Q as: \( P \lor Q \)

When we wish to posit an EXCLUSIVE disjunction between two statements where we wish to say that AT LEAST ONE but NOT BOTH are true, we will simply spell this intention or interpretation out in the following explicit formulation: \( P \lor Q \) but not both P and Q, which we symbolize: \( (P \lor Q) \land \neg(P \land Q) \) using the more elementary logical operations of inclusive disjunction ['\lor'], conjunction ['\land'], and negation ['\neg'].
Conditionals are expressed variously as follows.

All the conditional expressions below are symbolized: \( P \Rightarrow Q \)

- If \( P \) then \( Q \) (\( P \) is a sufficient condition for \( Q \))
- \( Q \), if \( P \)
- \( P \) only if \( Q \) (\( Q \) is a necessary condition for \( P \))
- Not \( P \) unless \( Q \)

In the standard form conditional:

If \( P \) then \( Q \)

we call the 'if _' clause the ANTECEDENT.

We call the 'then _' clause the CONSEQUENT.

We will symbolize an 'if _ then _' conditional as follows

\[ \text{Antecedent} \Rightarrow \text{Consequent} \]

The ANTECEDENT is the 'if _' clause. Notice that 'then' is not always stated:

\( P \) if \( Q \)

is the logical equivalent of

If \( Q \) then \( P \)

or

If \( Q \), \( P \).

which would be symbolized as follows:

\( Q \Rightarrow P \)

All conditionals are symbolized in this format

\[ \text{ANTECEDENT} \Rightarrow \text{CONSEQUENT} \]

regardless of the order in which the antecedent and consequent occur in English. In '\( P \) if \( Q \)', \( Q \) is the antecedent and '\( P \)' is the consequent.
For any conditional of the form:

If P then Q

symbolized:

\[ P \rightarrow Q \]

there are four possible combinations of truth-values for its
ANTECEDENT (in this case P) and CONSEQUENT (in this case Q):

\[ P \rightarrow Q \]

where BOTH antecedent and consequent are TRUE
(1) \( T \rightarrow T \)

where the ANTECEDENT is TRUE but the CONSEQUENT FALSE
(2) \( T \rightarrow F \)

where the ANTECEDENT is FALSE and the CONSEQUENT TRUE
(3) \( F \rightarrow T \)

where BOTH antecedent and consequent are FALSE
(4) \( F \rightarrow F \)

But what in each case, (1)-(4), is the truth-value of the CONDITIONAL
\[ P \rightarrow Q \]?

Let's take it line by line.

Consider. A conditional of the form

\[ P \rightarrow Q \]

asserts, in effect

IF P is true, THEN Q is true.

When P IS true AND Q is also true, how do we count the conditional?

Well, P is true in this case.

So, the conditional 'speaks' truly if the consequent is also true.

In this case, the consequent, Q, IS also true. So the conditional
is true in this case:

\[ P \rightarrow Q \]

(1) \( T \rightarrow T \)

Suppose now P IS TRUE but Q is FALSE.

What is the truth-value of the following conditional?

If P, then Q

It is FALSE, of course. Consider, again, what a conditional asserts:
The conditional

If P then Q

asserts that

Q is true when P is true.

So, where P IS true, but Q fails to be true,

the conditional assertion is assuredly false, as shown in line (2) below:

P : Q : P => Q

---!!--!------
(1) T : T : T
(2) T : F : F

Consider the following concrete examples:

If I, Charlie Conditional, promise you

If Carter wins re-election, I'll eat my hat

and

It's true that Carter wins, but I do NOT eat my hat

then is my promise to you not false?

If I say

If you get an A on the exam, you'll get an A in the course

and

You A the exam but you do NOT get an A in the course

would you say I had told you a falsehood or gave you a false hope?

I rather imagine.

Thus far, the rule for conditionals is pretty much common sense.

In truth-tabular form:

P : Q : P => Q

---!!--!------
(1) T : T : T
(2) T : F : F
But what of the two other possibilities?

\[ P \land Q \land P \rightarrow Q \]

---!----!---------

(3) \( F \land T \) ! How are we to account the truth-value
(4) \( F \land F \) of the conditional in these cases?

Consider, again, the analogy of a conditional promise.

I, Charlie Conditional, promise

If you A the exam, you'll A the course.

Suppose: (if it's possible)

You fail to A the exam but you get an A in the course anyway

Would you say I made a FALSE claim or promise?

I claimed that

You'd get a A in the course

IF you get an A on the exam

NOT that

You wouldn't get on A in the course anyway, even if you FAILED to get an A on the exam.

In fact, I made NO CLAIM WHATSOEVER about what would be the case in the event that you failed to A the exam; I did not claim that failure to A the exam would result in failure to A the course.

In the following case:

\[ P \land Q \land P \rightarrow Q \]

---!----!---------

(3) \( F \land T \) ! How shall we count the conditional?

It would be odd to count the conditional as false just because \( P \) is false.

Consider:

(i) The conditional really makes no claim about the case where \( P \) is false. It claims rather that WHERE \( P \) IS TRUE, \( Q \) is true.

(ii) The conditional does not claim that \( Q \) is false when \( P \) is false or that \( Q \) is true ONLY if \( P \) is true.
So, the conditional is NOT FALSE in this case.

Where I claim:

If you fail the course, I'll eat my hat

and

You do NOT fail the course

and yet

I eat my hat anyway

we could hardly say I made a FALSE claim.

I never said I would NOT eat my hat if you did NOT fail the course.
I made NO CLAIM at all about what I would do if you did NOT fail the course.

So, my conditional claim

If you fail the course, I'll eat my hat

is NOT FALSE when the antecedent is false (You do NOT fail the exam)
and the consequent is nonetheless true (I eat my hat).

A conditional, which in effect asserts

**IF P is true, Q is true**

is NOT a FALSE claim where P is false. Because it makes NO CLAIM WHATSOEVER about the case where P is false, it makes no FALSE claim where P is false.

By parity of reasoning, how do we decide the following case?

Suppose I claim:

If you pass the exam, you pass the course.

Suppose, also:

You do NOT pass the exam

and:

You do NOT pass the course.

Can I be accused of a FALSE claim, of lying to you?

I did not claim you WOULD pass the course even if you failed the exam; indeed, I said nothing at all about the case where you fail the exam.

So, it seems wrong to count my merely conditional 'IF-FY' claim as false.
In sum, conditionals with FALSE ANTECEDENTS are always counted as TRUE because, and in the sense that, they are NOT FALSE claims. For this and other reasons the rule for conditionals is as follows:

\[
P \land Q \land P =\rightarrow Q
\]

---!----!-------

\[
\begin{array}{ccc}
T & T & T \\
* & T & F \\
F & T & F \\
F & F & T \\
\end{array}
\]

* A conditional is FALSE IF, and ONLY IF, it has a TRUE ANTECEDENT and a FALSE CONSEQUENT; otherwise, it is counted as true.

REMEMBER:

If P, then Q
P only if Q
Q, if P
Not P unless Q

are all symbolized: \( P \rightarrow Q \)

Make note of this! They are all equivalent expressions. You need to master the translation of these expressions into STANDARD FORM.

For a discussion of how to translate 'UNLESS' expressions into a standard conditional form, see section 2.4.3 above.
A biconditional of the form
\[ P \text{ if and only if } Q \]
and symbolized
\[ P \leftrightarrow Q \]
asserts, in effect, the following:
- \( P \) is true if but only if \( Q \) is true
- \( Q \) is true if but only if \( P \) is true

In other words, the biconditional asserts, in effect, that
- \( P \) has the same truth-value as \( Q \): \( P \) is true IF \( Q \) is true
  - AND
  - \( P \) is true ONLY IF \( Q \) is true (so, \( P \) is false if \( Q \) is false)

So, the truth-functional rule for biconditionals is simple:
- A biconditional of the form \( P \leftrightarrow Q \) is true
  - Whenever both \( P \) and \( Q \) have THE SAME TRUTH-VALUE
    - (whenever both are true or BOTH are false)
- A biconditional \( P \leftrightarrow Q \) is false
  - Whenever \( P \) and \( Q \) have different truth-values
    - (whenever one is true but the other is false)

This truth-functional rule for biconditionals is summarized in the
following truth-table:

<table>
<thead>
<tr>
<th>P ! Q</th>
<th>P &lt;=&gt; Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T ! T</td>
<td>T</td>
</tr>
<tr>
<td>T ! F</td>
<td>F</td>
</tr>
<tr>
<td>F ! T</td>
<td>F</td>
</tr>
<tr>
<td>F ! F</td>
<td>T</td>
</tr>
</tbody>
</table>

NOTICE that where both P and Q are false (and so have the same truth-value) the biconditional is counted as TRUE [on line (4) of the table above].

This is because a biconditional asserts that P is true if but only if Q is true (that P is false if Q is false): It does NOT assert that either P or Q is actually true -- only that IF one is true THEN the other is also true.
3.1. PROVING VALIDITY IN SENTENTIAL LOGIC

You know that an argument form is valid if, and only if, it is impossible (e.g., impossible according to the truth-functional meanings of the sentential connectives) for any argument having that form to have true premises but a false conclusion.

Argument forms can be proven valid (or invalid) within the domain of truth-functional sentential logic by means of truth-tables. Truth-tables allow us to examine all possible combinations of truth-values among the premises and conclusion of an argument form (truth-functionally construed), so that we can conclusively determine whether there is any case (any assignment of truth-values to the component atomic sentences of the argument form) in which the conclusion can be false while all the premises are true.

The truth-functional test of validity comes to this: If there is any interpretation, any assignment of truth-values to the atomic sentential variables of an argument form that renders the conclusion false and all the premises true, then the argument form in question is invalid. If there is no case, no possible assignment of truth-values to the component atomic sentences of the argument form in which all the premises are true but the conclusion false, then the argument form is valid.

3.2. THE TRUTH-TABULAR PROOF OF VALIDITY

To show that an argument form is invalid as depicted in truth-functional sentential logic, we show that we can assign truth-values so as to make the conclusion false and all the premises true. If there is no possible truth-value assignment that both makes the conclusion false and all the premises true, then the argument form is valid: any argument having that form is valid because it is not possible that any argument of that form have all true premises and yet a false conclusion.
We wish to see if, by virtue of an argument's truth-functional logical form, it is possible for all the premises of the argument to be true while its conclusion is false. Once an argument's logical form is depicted in sentential logic we can survey all possible interpretations of that argument form by surveying all possible truth-value assignments to the atomic components of its premises and conclusion. Since we wish to know whether one particular pattern of truth-values is possible, we can narrow our search: we consider only those truth-value assignments which render the conclusion of the argument false.

By use of a truth-table we can summarize all the conditions under which the premises and conclusion of an argument are rendered either true or false. Consider: any sentence \( P \) is either true or false. For any two sentences \( P \) and \( Q \) there are four possible combinations of truth-values, namely, where:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

(We know this fact about any two sentences without knowing anything about their content, their actual truth or falsity or the state of the world. Why is this so?) For any three sentences \( P \), \( Q \), and \( R \) there are eight possible combinations of truth-values:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

For two sentences: \( 2^2 = 4 \).

For three sentences: \( 2^3 = 8 \), etc.
Consider the argument forms below. We wish to know, for each one, whether it is valid or not. We would then know whether any argument having one of these forms was valid or invalid within the framework of truth-functional sentential logic.

(A) \( P \rightarrow Q \)
\[ \begin{array}{c|c|c}
P & Q & \rightarrow \\
T & T & \checkmark \\
T & F & \\
F & T & \\
F & F & \checkmark \\
\end{array} \]

(B) \( P \rightarrow Q \)
\[ \begin{array}{c|c|c}
Q & P & \\
T & \checkmark & \\
F & \\
\end{array} \]

(C) \( P \lor Q \)
\[ \begin{array}{c|c|c}
-P & Q & \\
\checkmark & \checkmark & \\
\checkmark & F & \\
\end{array} \]

(D) \( P \lor Q \)
\[ \begin{array}{c|c|c|c|c}
P & Q & P & \checkmark & F & \checkmark \\
T & T & \checkmark & \checkmark & & \\
T & F & \checkmark & & \\
F & T & & \checkmark & \\
F & F & & & \\
\end{array} \]

We wish to know whether there are any conditions under which the premises of an argument having one of these forms could be true while its conclusion is false. If so, the argument form--and any argument having that form--is invalid. If there are no such conditions, then the argument form--and any argument having that form--is valid, such that, if the premises are true, the conclusion cannot possibly be false. A truth-table will serve to survey all possible conditions under which the premises or conclusion of any argument having the above forms are true or false. For example, for argument form (A) we have:

Form (A)

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>Conclusion</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* We will concentrate on all and only those conditions, those possible truth-value assignments to the constituent sentences \( P \) and \( Q \), which render the conclusion false (those lines marked by asterisks '*'). All the possible combinations of truth-value assignments to the components \( P \) and \( Q \) under which the conclusion would be false are represented by lines (2) and (4) of the truth-table above. The question then is: Is it possible for all of the premises of an argument of form (a) to be true under any of the truth-value assignments that render the conclusion false? Clearly not. No more than one of the premises can be true under the assignments which render the conclusion false. The argument form is valid: no argument of this form can have a false conclusion and true premises. Remember: A
VALID argument form will never lead us from truth into falsehood.

We could as well have concentrated on all and only those truth-value assignments which rendered all the premises true and then inspected the truth-table to see whether under any of those conditions the conclusion was rendered false. Those conditions are represented by lines (1) and (3) in the truth-table for argument (A) above. By filling out those lines, you can see for yourself that there are no conditions under which the conclusion is rendered false while all the premises are true. No argument of this form can possibly have all true premises and a false conclusion. And so for argument form (C): wherever the conclusion is false, at least one premise must also be false.*

<table>
<thead>
<tr>
<th>! Conclusion !</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td>P Q ! Q ! P v Q ! -P</td>
<td></td>
</tr>
</tbody>
</table>

(1) T ! T ! T ! T ! F
*(2) T ! F ! *F ! T ! F*
(3) F ! T ! T ! T ! T
*(4) F ! F ! *F ! F* ! T

By contrast, arguments form (B) and (D) are invalid: it is possible for the conclusion of an argument of either form to be false while its premises are all true. This is readily shown by a truth-table survey of all the possible combinations of truth-value assignments to the component variables of the argument form. If under some combination of assignments the conclusion is rendered false while the premises are true, the argument form—and any argument having that form—is invalid as an argument of that form fails to guarantee the truth of its conclusion given the truth of all its premises. An invalid argument form is unreliable because it can lead us from true premises to a false conclusion. Proof of the invalidity of argument forms (B) and (D) by the truth-table method is left to you as an exercise.

The nice thing about the truth-table method is that it is conclusive, exhaustive, and purely mechanical. Of course, you could also use the counter-example method (discussed in the last chapter) to prove argument forms (B) and (D) invalid. But whereas the counter-example method for proving invalidity requires ingenuity and imagination, the truth-table method does not: the latter is an infallible, purely mechanical decision procedure. And, whereas the counter-example method cannot be used to prove an argument form valid, the truth-table method provides an unerring decision-procedure for determining validity in sentential logic.
3.3. RULES OF INFERENCE AND VALID ARGUMENT FORMS

The sections that follow show you how to use truth-functional sentential logic and the technique of truth-table analysis to PROVE that certain argument forms are VALID.

VALID ARGUMENT FORMS, in effect, provide us with RULES OF VALID INFERENCE—rules for constructing and evaluating arguments.

An inference rule allows us to draw CONCLUSIONS from given PREMISES by a series of valid INTERMEDIATE STEPS.

A VALID DERIVATION or VALID DEDUCTIVE ARGUMENT can be either a simple derivation of a single conclusion directly from given premises or a longer series of steps, where each step is either a confessed PREMISE or an INTERMEDIATE STEP or CONCLUSION validly derived from previous steps.

The ARGUMENT FORMS whose VALIDITY we will prove are the INFERENCE RULES that you may use in constructing valid derivations in the BERTIE, RECON and ARGUE programs, which will check each step in your derivation for validity according to these rules.

We have a systematic decision procedure (the truth-tabular method) for PROVING that a given inference rule represents a VALID ARGUMENT FORM. But we also want to understand intuitively WHAT THE RULES MEAN. To reason with the rules in ordinary life, we need to understand, in commonsense terms, what the rules allow.

I will give you an intuitive analysis and commonsense reading of the rules I show you how to prove. I leave it to you to be sure that you understand clearly what the rules say. This just takes some thought and practice in using them. You will gain this practice in the BERTIE, RECON, and ARGUE programs.
3.4. THE CONJUNCTION RULE

The conjunction rule says:

GIVEN any two sentences, premises, or previous lines in a derivation:

(1) P
(2) Q

we may logically derive their conjunction:

P & Q

In schematic form:

\[
\begin{array}{c|c}
P & P \\
Q & Q \\
\hline
P & Q & Q & P
\end{array}
\]

The above ARGUMENT FORMS are VALID, which means, in an argument having this form, that

The conclusion will not be false IF the premises are true.

Remember: The claim that an argument form is VALID is not a claim about the truth or falsity of the PREMISES alone. Nor is it a claim about the truth or falsity of the CONCLUSION alone.

The claim that an ARGUMENT FORM is valid is a claim about the LOGICAL CONNECTION between the premises and conclusion of ANY and EVERY argument of that form.

That an argument form is VALID means: for ANY and EVERY argument of that form,
IF the premises are true, THEN the conclusions MUST be true.

NO argument with a valid form can possibly have true premises and a false conclusion, though its premises and conclusion may have any other combinations of truth-values.

By the simple rule for the truth-functionality of the sentential connective 'AND,' we can readily see that, in an argument or derivation of the form:

\[
\begin{align*}
& P \\
& Q \\
\hline
P & \& Q
\end{align*}
\]

IF: P is true and Q is true

THEN: the conjunction of P and Q must be true.

When the premises 'P' and 'Q' are both true, the conclusion 'P & Q' cannot be false. WHY?

Obviously, because a CONJUNCTION is true just in case BOTH conjuncts are true. That is: a conjunction is true IF both conjuncts are true, and a conjunction is false ONLY IF at least one conjunct is false. So, an argument whose conclusion is the conjunction of its premises cannot have true premises and a false conclusion.

We can prove any ARGUMENT FORM valid by the rules of sentential logic if we can prove that NO argument with that form--NOT ONE--could have true premises and a false conclusion.

How can we survey EVERY argument of a form in order to show that NO argument with that form has true premises and yet a false conclusion?

A TRUTH-TABLE allows us systematically to examine ALL POSSIBLE COMBINATIONS of truth-values among the component sentences which form the argument.
Consider arguments of the form

\[ P \quad Q \]

\[ P \land Q \] where the conclusion is the conjunction of the premises

Granted this argument form is rather trivial for everyday purposes, it's easy with such a simple example to show how we can PROVE the following:

The argument form is VALID.

That is, NO argument of this form can have true premises and a false conclusion.

That is, there is NO TRUTH-VALUE ASSIGNMENT to the component sentences of any argument of this form that renders the premises true and the conclusion false.

The following simple truth-table test will show that

\[ P \quad Q \]

\[ P \land Q \] is a valid argument form.

<table>
<thead>
<tr>
<th>CONJUNCTION:</th>
<th>Conclusion</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td>P \land Q</td>
<td>P \land Q</td>
<td>P \qquad Q</td>
</tr>
<tr>
<td>(1) T \land T</td>
<td>T</td>
<td>T \qquad T</td>
</tr>
<tr>
<td><em>(2) T \land F</em></td>
<td>F</td>
<td>T \qquad F*</td>
</tr>
<tr>
<td><em>(3) F \land T</em></td>
<td>F</td>
<td>F* \qquad T</td>
</tr>
<tr>
<td><em>(4) F \land F</em></td>
<td>F</td>
<td>F* \qquad F*</td>
</tr>
</tbody>
</table>

* In no case where the conclusion is false are all the premises true.
KEEP IN MIND: In this rule schema

\[ \begin{align*}
\text{P} \\
\text{Q} \\
\hline
\text{P \& Q}
\end{align*} \]

as in all the other rule schemas, complex sentence forms can stand in for the simple sentential variables, \((P, Q)\).

The rule says:

Given ANY two sentences whatever
you may derive their conjunction.

Thus, the following deductions are all valid by the CONJUNCTION rule

\[ \begin{align*}
\text{R} & \quad (R \leftrightarrow \lnot S) \\
(F \& S) & \quad L \& M \\
\hline
\text{(F \& S) \& R} & \quad (L \& M) \& (R \leftrightarrow \lnot S) \\
\end{align*} \]
3.5. THE SIMPLIFICATION RULE

The schemata or argument forms representing this rule are SIMPR and SIMPL, for deriving the right and left conjuncts, respectively:

SIMPR: $P \& Q$

\[\frac{\text{--}}{Q}\]

SIMPL: $P \& Q$

\[\frac{\text{--}}{P}\]

What the rule says, in commonsense terms, is this:

Given the conjunction of any two sentences: $P \& Q$

You may derive one of the conjuncts, say: $P$

You can readily see why this is a VALID argument form. ASSUMING that BOTH $P$ and $Q$ are true, then certainly any one of them MUST be true.

An argument or deductive step with this form CANNOT have a true premise or line (1) and a false conclusion or intermediate step (2) because a CONJUNCTION can be true ONLY IF both conjuncts are true.

\[\text{(1)} \quad P \& Q \quad \frac{\text{--}}{P}\]

We can easily PROVE that this deductive schema is VALID, that step (2) LOGICALLY FOLLOWS FROM step (1), by showing that it is IMPOSSIBLE by the rules of truth-functional logic that step (1) be TRUE and step (2) FALSE.
That is, we can show that there is NO combination of TRUTH-VALUE ASSIGNMENTS to the component SENTENTIAL VARIABLES of the argument schema that both makes step (1) true and renders step (2) false:

<table>
<thead>
<tr>
<th></th>
<th>Conclusion</th>
<th>Premise</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Q</td>
<td>P</td>
</tr>
<tr>
<td>(1)</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td><em>(3)</em></td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td><em>(4)</em></td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

* Notice that we need to look only at those cases or truth-value assignments (lines of the truth-table) that render the conclusion of the argument form FALSE. WHY?

We want to show that in NO CASE where the conclusion P is false, can the premised conjunction of P and Q be true.

Whenever testing the truth-functional validity of an argument form we need look only at those cases (truth-value assignments) where the conclusion is false: to see if, in any case, the premise(s) or line(s) from which it is derived can be true.

Make sure you understand why this is a sufficient test of validity for any argument form!
Given the conjunction of any two sentences, you may derive either conjunct.

The rule allows only VALID deductions because:

IF a conjunction is true, either conjunct **MUST** be true.

I leave it to you as an exercise to **PROVE** for your own satisfaction, by the truth-table method illustrated above, that any and all of the following deductive schemas (argument forms) are VALID, that any deductions of their form are warranted by the rule of SIMPLIFICATION.

**NOTE:** The complexity of either conjunct in the following conjunctions does not affect the validity of the deduction. Be sure you understand why.

\[
\begin{align*}
-\neg P \land Q & \quad -\neg P \land (Q \lor R) & -\neg P \land (Q \lor R) \\
\quad -\neg P & \quad (Q \lor R) & \quad -\neg P \\
(S \iff R) \land (P \lor \neg S) & \quad (S \iff R) \land (P \lor \neg S) \\
\quad S \iff R & \quad P \lor \neg S \\
-\neg P \land Q & \quad -(S \lor P) \land R & -(Q \Rightarrow L) \land \neg S \\
\quad Q & \quad -(S \lor P) & \quad -\neg S
\end{align*}
\]
3.6. THE ADDITION RULE

This rule is quite simple: the argument form representing the rule is

\[ P \quad \vdash \quad P \lor Q \]

This is what the rule says:

Given any sentence whatever, for example: \( P \)

You may deduce the disjunction of that with ANY SENTENCE WHATSOEVER

IF you already have \( P \)

Then you can disjoin \( P \) with ANY SENTENCE WHATSOEVER (say, \( Q \)):

The deductive step from (1) \( P \)

\[ \quad \vdash \quad P \lor Q \]

is valid because there's NO WAY that (1) '\( P \)' can be true and
(2) '\( P \lor Q \)' false.

A disjunction is true if at least one disjunct is true.

So, in a deduction of the form: (1) \( P \)

\[ \quad \text{Therefore:} \quad (2) \ P \lor Q \]

If \( P \) is true, \( P \lor Q \) MUST be true.

Therefore, the argument form: \( P \)

\[ \quad \vdash \quad P \lor Q \quad \text{IS VALID.} \]
Alternatively: a disjunction is false only if BOTH disjuncts are false

So, in a deduction of the form: (1) P

Therefore: (2) P V Q

If 'P v Q' is false, both 'P' and 'Q' are false; so premise 'P' is false

ANY argument of this form with a false conclusion must have a false premise: NO argument of this form can have a false conclusion and a true premise.

The validity of the ADDITION argument schema:

\[
\begin{array}{ccc}
P & \rightarrow & P V Q \\
\end{array}
\]

is obvious from the following truth-table. The truth-table surveys ALL POSSIBLE combinations of truth-values among the component variables of the argument form.

<table>
<thead>
<tr>
<th>ADDITION</th>
<th>Conclusion</th>
<th>Premise</th>
</tr>
</thead>
<tbody>
<tr>
<td>P ! Q</td>
<td>P V Q</td>
<td>P</td>
</tr>
<tr>
<td>(1) T ! T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2) T ! F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) F ! T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*(4) F ! F</td>
<td>F</td>
<td>F*</td>
</tr>
</tbody>
</table>

*NOTE: In that case where the conclusion is false, the premise must also be false. ANY argument of this form with a true premise must have a true conclusion.
Consider what the rule means:

Given ANY SENTENCE WHATEVER,
You may disjoin it with ANY SENTENCE WHATEVER

So, all of the following correspond to the valid form of the ADDITION rule schema:

\[
\begin{array}{cccc}
P & P \lor Q & F \rightarrow S & K \\
\hline
P \lor P & (P \lor Q) \lor R & (F \rightarrow S) \lor L & K \lor (S \leftrightarrow M)
\end{array}
\]

All of the above have this formal feature in common:

Given some sentence (of whatever form),
That sentence is disjoined with some other sentence (of ANY form)
The schemas or argument forms representing this rule are (DSR), where the right disjunct is detached.

\[
P \lor Q
\]

\[-P
\]

\[Q\]

or, equivalently, (DSL) where the left disjunct is detached:

\[
P \lor Q
\]

\[-Q
\]

\[P\]

The rule says, simply and commonsensically:

GIVEN the disjunction of any two sentences \( P \lor Q \) and the denial/negation of ONE disjunct: \(-P\)

You may then deduce the OTHER disjunct: \( Q \)

The schema or logical form of this rule is VALID: valid because no deduction or argument of this form with true premises can possibly have a false conclusion. This we can PROVE.
Consider what the rule says, in ordinary intuitive terms:

(1) EITHER the one disjunct (P) OR the other disjunct (Q) is true — ONE OR THE OTHER is true
(2) Now, it's NOT the one disjunct (P) that's true.
(3) So, it MUST be the other (Q) that's true.

Given what we know about the truth-functionality of disjunction, we know the following:

ASSUMING that EITHER P is true OR Q is true
and that it's NOT P that's true

THEN IT LOGICALLY FOLLOWS that Q is true.

Be sure you understand the common intuitive sense of this rule! While we can reason out the intuitive sense of the DISJUNCTIVE SYLLOGISM rule, we can also PROVE that NO deduction or argument with the above form can possibly have true premises and a false conclusion:

<table>
<thead>
<tr>
<th>Conclusion</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Q</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F*</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F*</td>
</tr>
</tbody>
</table>

* Note that we need go no further; we already have one false premise in line (4), so we already know, in this case, that we cannot have ALL the premises true when the conclusion is false.

All of the truth-value assignments that render the conclusion (Q) false are marked by the asterisk '*':: IN NO CASE where the conclusion is false are all the premises true.
THE STRATEGY FOR TESTING THE VALIDITY OF AN ARGUMENT FORM is as follows:

1. Construct a truth-table for the premises and conclusion of the argument form in question.

2. Start with the CONCLUSION: find those truth-value assignments (lines of the truth-table) which render the conclusion false.

3. Then proceed, premise by premise, to check whether any of the truth-value assignments that renders the conclusion false also allows ALL of the premises to be true.

IF there is NO truth-value assignment to the component sentential variables of the argument form that renders the conclusion false and all of the premises true,

THEN the argument form is VALID.

IF there is SOME truth-value assignment to the component sentential variables of the argument form that allows all premises to be true and the conclusion to be false,

THEN the argument form is INVALID.

Be sure you remember and understand this test of the validity of argument forms.

Note that the following deductive schema or argument forms all conform to the rule of DISJUNCTIVE SYLLOGISM, which says, simply:

GIVEN any disjunction of two sentences and the negation of one of the disjuncts,

We may deduce the other disjunct.

P V Q       ONE OR THE OTHER disjunct is true
-P          It's NOT the one

Q           So, it MUST be the other.
The disjuncts in question may themselves be complex or compound sentences, like:

\[
\begin{align*}
P \lor \neg Q & \quad (P \Rightarrow Q) \lor (R \lor S) & \quad -(P \land Q) \lor R \\
\neg Q & \quad -(R \lor S) & \quad \neg -(P \land Q)
\end{align*}
\]

\[
\begin{align*}
P & \\
(P \Rightarrow Q) & \\
R
\end{align*}
\]

I leave any proof of the validity of these instances of the DISJUNCTIVE SYLLOGISM form to you. Once you learn to correctly recognize the logical form of the rule and formula that correspond to it, no further proof should be necessary.

Rehearse both the SENSE and LOGICAL FORM of this rule so that you can recognize its occurrence and application in the context of ordinary language.

NOTE: This and other rules allow you to eliminate the step of DOUBLE NEGATION. Thus, for example, both of the following are allowed:

\[
\begin{align*}
P \lor \neg Q & \\
\neg Q & \\
\neg \neg Q & \\
P
\end{align*}
\]

The proof-checking programs will allow you to treat any formula, say, P, as if it were a double negation, \(\neg\neg P\).
3.8. THE MODUS PONENS AND MODUS TOLLENS RULES

Review of Conditionals: Necessary versus Sufficient Conditions

3.8.1. MODUS PONENS

The simple deductive schema (argument form) that represents the MODUS PONENS inference rule is:

\[ P \rightarrow Q \]

We can give a reading of the reasoning here as follows:

\[ P \rightarrow Q \quad \text{Assume that IF } P \text{ is true, } Q \text{ is true} \]

\[ P \quad \text{Now suppose } P \text{ IS true} \]

\[ Q \quad \text{Then, it follows that } Q \text{ MUST be true.} \]

Another way to read (1) \( P \rightarrow Q \)

(2) \( P \)

(3) \( Q \) is as follows:

The conditional (1) says that IF you have \( P \) that's SUFFICIENT to get \( Q \). (2) affirms that you have \( P \). So, given (1) and (2), you may conclude \( Q \). IF (1) and (2) are true, \( Q \) MUST be true.
A truth-functional CONDITIONAL of the form 'P \rightarrow Q' asserts two things in effect:

(1) P's being true is a SUFFICIENT CONDITION for Q's being true:

   IF P is true, that suffices to make Q true.

(2) Q's being true is a NECESSARY CONDITION for P's being true:

   P is true ONLY IF Q is true
   P is NOT true UNLESS Q is true.

Think about these two basic senses and variant translations of the truth-functional analysis of conditionals and REMEMBER them. If you understand what a CONDITIONAL asserts about the TRUTH-VALUES of its ANTECEDENT and CONSEQUENT, then it's easier to understand the reasoning behind both the MODUS PONENS and the MODUS TOLLENS rules.

The two truth-functional interpretations of what a conditional says are like two sides of the same coin. 'P \rightarrow Q' says, in effect:

(1) If the ANTECEDENT (P) is true, that is SUFFICIENT to make the CONSEQUENT (Q) true and

(2) For the ANTECEDENT (P) to be true it's NECESSARY that the CONSEQUENT (Q) be true:

   P is true ONLY IF Q is true
   P is NOT true UNLESS Q is true
   If Q is NOT true, then P is not true
3.8.2. **MODUS TOLLENS**

From this second reading (2) of a conditional 'P => Q' we see the simple sense of the MODUS TOLLENS rule. Consider:

\[ P \rightarrow Q \]

\[ P \text{ is true ONLY IF } Q \text{ is true} \]

(Or: \( P \text{ is NOT true UNLESS } Q \text{ is true} \))

\[ \neg Q \]

Well, \( Q \text{ is NOT true} \)

\[ \neg P \]

So, \( P \text{ is not true} \)

This rule simply says:

Given any CONDITIONAL sentence and the DENIAL of its CONSEQUENT

You may deny/negate its ANTECEDENT

In sum, reading 'P=>Q' in the first sense:

(1) IF P is true, THEN Q is true

we can make sense of the MODUS PONENS rule:

\[ P \rightarrow Q \]

\[ \text{IF you have the antecedent, you can have the consequent} \]

\[ P \]

We have the antecedent

\[ Q \]

So, we can have the consequent
Reading 'P⇒Q' in its other sense:

(2) P is true ONLY IF Q is true :: P is NOT true UNLESS Q is true

we can make sense of the MODUS TOLLENS rule:

P⇒Q  The antecedent is true ONLY IF the consequent is true
-Q    But we DENY the consequent
-P    So, we can DENY the antecedent
REMEMBER: The rules MP and MT are talking about logical connections that are posited between the ANTECEDENTS and CONSEQUENTS of conditionals.

**MODUS PONENS:** \( P \rightarrow Q \)

\[ \begin{align*}
&\quad P \\
\hline
&\quad Q
\end{align*} \]

If you can AFFIRM the ANTECEDENT of any conditional, then you can AFFIRM its CONSEQUENT.

This goes for conditionals with complex antecedents and consequents. Keep in mind not all conditionals are as simple as '\( P \rightarrow Q \)'. For example, the following argument forms correspond to the rule MODUS PONENS and, so, are valid deductions:

\[
\begin{align*}
(P \land Q) &\rightarrow \neg R \\
\neg (P \leftrightarrow Q) &\rightarrow (R \lor S) \\
\hline
\neg R &\quad R \lor S
\end{align*}
\]

**MODUS TOLLENS:** \( P \rightarrow Q \)

\[ \begin{align*}
&\quad \neg Q \\
\hline
&\quad \neg P \quad \text{(says)}
\end{align*} \]

If you DENY the CONSEQUENT of any conditional, then you can DENY the ANTECEDENT.

This rule goes for conditionals with complex antecedents and consequents. For example, the following all correspond to the MODUS TOLLENS rule:

\[
\begin{align*}
\neg P &\rightarrow \neg Q \\
\quad \neg Q &\quad (P \rightarrow Q) \rightarrow R \\
\hline
\quad \neg R &\quad ((P \land Q) \leftrightarrow R) \rightarrow (S \lor F) \\
&\quad (S \lor F) \\
\quad \neg P &\quad -(P \rightarrow Q) \\
\hline
&\quad -((P \land Q) \leftrightarrow R)
\end{align*}
\]
3.8.3. **EXAMPLE: NECESSARY VS. SUFFICIENT CONDITIONS**

To understand what sense the inference rules MODUS PONENS and MODUS TOLLENS make, it's helpful to understand the two senses in which 'P→Q' can be taken. Here's an example to help you remember the two senses of any conditional—the two ways in which ANTECEDENT and CONSEQUENT are logically related:

(1) **IF P is true, THEN Q is true:**  
   
   The antecedent is a **SUFFICIENT CONDITION** of the consequent

   E.g. IF it's raining, THEN it's wet out.

(2) **P is true ONLY IF Q is true:**  

   The consequent is a **NECESSARY condition** of the antecedent

   E.g. It's raining **ONLY IF** it's wet out.
   
   It's **NOT** raining **UNLESS** it's wet out.

**REMEMBER:** Any conditional can be taken in either sense (1) or (2). Both senses are depicted:  

P → Q
3.8.4. PROVING THE VALIDITY OF MP AND MT

While it's important to understand the SENSE of the rules, we can also PROVE that the argument schemas

\[
\begin{array}{c|c}
P \rightarrow Q & P \rightarrow Q \\
\hline
P & -Q \\
\hline
\end{array}
\]

are valid: in NO CASE whatsoever can an argument with either form have true premises and a false conclusion.

The strategy for testing the validity of an argument form is:

1. Construct a truth-table for the premises and conclusion of the argument form in question.

2. Start with the CONCLUSION: find those truth-value assignments (lines of the truth-table) which render the conclusion false.

3. Then proceed, premise by premise, to check whether any of the truth-value assignments that renders the conclusion false also allows ALL of the premises to be true.

IF there is NO truth-value assignment to the component sentential variables of the argument form that renders the conclusion false and all of the premises true, THEN the argument form is VALID.

IF there is SOME truth-value assignment to the component sentential variables of the argument form that allows all premises to be true and the conclusion to be false, THEN the argument form is INVALID.
We prove any argument form valid in sentential logic by constructing a truth-table to show that there is no case (no line in the truth-table) where an argument of the given form has all true premises when the conclusion is false.

The following tables prove the validity of argument forms corresponding to MODUS PONENS and MODUS TOLLENS.

<table>
<thead>
<tr>
<th>MODUS PONENS:</th>
<th>Conclusion</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td>P ( \rightarrow ) Q</td>
<td>Q</td>
<td>P ( \rightarrow ) Q</td>
</tr>
<tr>
<td>(1) T ( \rightarrow ) T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*(2) T ( \rightarrow ) F</td>
<td>F</td>
<td>F*</td>
</tr>
<tr>
<td>(3) F ( \rightarrow ) T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*(4) F ( \rightarrow ) T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MODUS TOLLENS:</th>
<th>Conclusion</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td>P ( \rightarrow ) Q</td>
<td>( \neg )P</td>
<td>P ( \rightarrow ) Q</td>
</tr>
<tr>
<td>*(1) T ( \rightarrow ) T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>*(2) T ( \rightarrow ) F</td>
<td>F</td>
<td>F*</td>
</tr>
<tr>
<td>(3) F ( \rightarrow ) T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4) F ( \rightarrow ) F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* NOTE: You need look at only those lines of the truth-table where the conclusion is false; you need proceed only until you find that at least one premise must be false when the conclusion is false.
3.9. THE CONSTRUCTIVE DILEMMA RULES

We are using two CONSTRUCTIVE DILEMMA rules, hence two valid argument forms representing these similar rules:

CDI: \[ P \lor Q \]
\[ P \Rightarrow R \]
\[ Q \Rightarrow R \]
\[ R \]

CDII: \[ P \lor Q \]
\[ P \Rightarrow R \]
\[ Q \Rightarrow S \]
\[ R \lor S \]

CDI applies to any argument form where BOTH disjuncts of the first premise are taken to imply ONE and the same statement.

CDII applies to any argument form where EACH disjunct of the first premise is taken to imply one of TWO different statements.

3.9.1. CONSTRUCTIVE DILEMMA I

\[ P \lor Q \]
\[ P \Rightarrow R \]
\[ Q \Rightarrow R \]
\[ R \]

IF at least one of two disjuncts is true \((P \lor Q)\) and it's true that both imply a certain statement \((P \Rightarrow R)\) and \((Q \Rightarrow R)\) THEN (we may conclude that) that statement \((R)\) is true.
The reasoning here goes roughly as follows:

Either one thing or another is true: \( P \lor Q \)
If the one is true, then \( R \) is true: \( P \Rightarrow R \)
If the other is true, then \( R \) is true: \( Q \Rightarrow R \)
So, in either case, \( R \) is true: \( R \)

Either you have \( P \) or you have \( Q \):
And with \( P \) you get \( R \):
And with \( Q \) you can get \( R \):
So, one way or the other, you can get \( R \):

Keep in mind: sentences of a complex form can stand in the place of the sentential variables without affecting the validity of an argument. Arguments with the following more complex components are valid according to the rule of CONSTRUCTIVE DILEMMA I:

\[
(P \land Q) \lor \neg P \\
(P \land Q) \Rightarrow (R \land F) \\
\neg P \Rightarrow (R \land F) \\
\hline
R \land F
\]

\[
R \lor \neg (F \lor L) \\
R \Rightarrow (G \lor S) \\
\neg (F \lor L) \Rightarrow (G \lor S) \\
\hline
G \lor S
\]

In each case above, each of the disjuncts of the first premise is taken to imply ONE AND THE SAME statement. Even though that statement is complex in form, CDI applies.

You can prove the argument forms above are valid according to the CONSTRUCTIVE DILEMMA I rule by deriving their conclusions from their given premises on BERTIE, using CDI.
3.9.2. **CONSTRUCTIVE DILEMMA II**

\[ P \lor Q \]
\[ P \implies R \]
\[ Q \implies S \]

\[ R \lor S \]

Either you have \( P \) or you have \( Q \):
And, given \( P \), you can get one thing, \( R \):
And, given \( Q \), you can get another thing, \( S \):

One way or the other, you can get either
one thing (\( R \)) or the other (\( S \)):

Again, as with any argument schema/inference rule, more complex/compound sentence forms may stand in for the sentential variables (\( P, Q, R, S \)) of the argument schema:

\[ P \lor Q \]
\[ P \implies R \]
\[ Q \implies S \]

\[ R \lor S \]

For example, the following argument forms correspond to the valid form of **CONSTRUCTIVE DILEMMA II**. You can prove this by BERTIE.

\[-J \lor -K \]
\[-J \implies (L \lor M) \]
\[-K \implies (S \& M) \]

\[ (R \implies Q) \lor F \]
\[ (R \implies Q) \implies (S \iff J) \]
\[ F \implies -P \]

\[ (L \lor M) \lor (S \& M) \]
\[ (S \iff J) \lor -P \]

In each case above, the argument form consists of:

(1) A DISJUNCTION
(2) A CONDITIONAL, whose antecedent is (1)'s left disjunct
(3) Another CONDITIONAL, whose antecedent is (1)'s right disjunct
(4) Another DISJUNCTION, whose left disjunct is (2)'s consequent and whose right disjunct is (3)'s consequent.
3.9.3. **TRUTH-TABULAR PROOF OF CDI AND CDII**

We can PROVE that the argument forms representing the two rules of CONSTRUCTIVE DILEMMA are each valid: that is, that NO argument of a corresponding form can have true premises and a false conclusion.

The following truth-table proves that an argument corresponding to the form of CONSTRUCTIVE DILEMMA I is valid: that it is IMPOSSIBLE for ANY argument with this form to have all true premises and a false conclusion.

**CDI:**

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>R</th>
<th>P V Q</th>
<th>P =&gt; R</th>
<th>Q =&gt; R</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F*</td>
</tr>
<tr>
<td>(3)</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F*</td>
</tr>
<tr>
<td>(4)</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F*</td>
</tr>
<tr>
<td>(5)</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>(7)</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8)</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td>F*</td>
</tr>
</tbody>
</table>

*NOTE:* We need examine only those cases/lines where the conclusion is false: in every case, the truth-value assignment that makes at least one premise false. Hence, no argument of this form with a false conclusion can have all true premises. Hence, EVERY argument with this form and true premises MUST have a true conclusion.
The case for CONJUNCTIVE DILEMMA II is the same: as shown by the following truth-table, NO argument with the form of CDII can possibly have all true premises and a false conclusion — the truth-table surveys EVERY case where the conclusion is false*:

CDII:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
<th>Conclusion</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>R V S</td>
<td>P V Q</td>
</tr>
<tr>
<td>(1)</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F*</td>
<td>T</td>
</tr>
<tr>
<td>(2)</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F*</td>
<td>T</td>
</tr>
<tr>
<td>(5)</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(7)</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F*</td>
<td>T</td>
</tr>
<tr>
<td>(8)</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F*</td>
<td>T</td>
</tr>
<tr>
<td>(9)</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(10)</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(11)</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(12)</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F*</td>
<td>T</td>
</tr>
<tr>
<td>(13)</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F*</td>
<td>T</td>
</tr>
<tr>
<td>(14)</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F*</td>
<td></td>
</tr>
<tr>
<td>(15)</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F*</td>
<td>T</td>
</tr>
<tr>
<td>(16)</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F*</td>
<td>F</td>
</tr>
</tbody>
</table>

*Wherever the conclusion is false, at least one premise is false!

Be sure you can explain why this shows the argument form to be VALID!
3.10. THE HYPOTHETICAL SYLLOGISM RULE

The rule schema is:

\[ P \rightarrow Q \]
\[ Q \rightarrow R \]

\[ P \rightarrow R \]

The rule says, in effect and roughly:

IF, when you have P you have Q: \[ P \rightarrow Q \]
and when you have Q you have R: \[ Q \rightarrow R \]

THEN when you have P you have R: \[ P \rightarrow R \]

Alternatively, since any conditional form

\[ P \rightarrow Q \]

may be read either as (1) IF P THEN Q

or as (2) P ONLY IF Q

the rule

\[ P \rightarrow Q \]
\[ Q \rightarrow R \]

\[ P \rightarrow R \]

may also be read, roughly:

IF you have P only if you've got Q
and you've got Q only when you've got R

THEN you have P only when you've got R.
The following are all valid by HYPOTHETICAL SYLLOGISM. You can prove this on BERTIE by deducing their conclusions from their given premises using the HYPOTHETICAL SYLLOGISM rule.

\[
\begin{align*}
(P \land Q) &\Rightarrow -R \\
-R &\Rightarrow (S \lor F) \\
(F \Rightarrow S) &\Rightarrow -R
\end{align*}
\]

\[
\begin{align*}
(P \land Q) &\Rightarrow (S \lor F) \\
-R &\Rightarrow -R
\end{align*}
\]

Each of the above argument forms consists of

1. A conditional
2. Another conditional, whose antecedent is the consequent of (1)
3. Yet another conditional, whose consequent is the consequent of (2) and whose antecedent is the antecedent of (1).

All arguments or deductions corresponding to HYPOTHETICAL SYLLOGISM have these three basic formal features in common.
We can easily PROVE that the argument form representing the HYPOTHETICAL SYLLOGISM rule is valid: that is, that EVERY argument with this form that has true premises is GUARANTEED to have a true conclusion.

The following truth-table analysis of the HYPOTHETICAL SYLLOGISM argument form proves that the form is VALID: in NO possible case (in no line of the truth-table) can an argument with this form have all true premises when its conclusion is false; hence, EVERY argument of this form with true premises MUST have a true conclusion. THIS, sweetheart, is the golden guarantee of VALIDITY!

<table>
<thead>
<tr>
<th>HYPOTHETICAL SYLLOGISM</th>
<th>Conclusion</th>
<th>Premises</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P =&gt; R</td>
<td>P =&gt; Q</td>
</tr>
<tr>
<td></td>
<td>Q =&gt; R</td>
<td></td>
</tr>
<tr>
<td>(1) T ! T ! T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*(2) T ! T ! F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>*(4) T ! F ! F</td>
<td>F</td>
<td>F*</td>
</tr>
<tr>
<td>(5) F ! T ! T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6) F ! T ! F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7) F ! F ! T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8) F ! F ! F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* NOTE: We need look at only those cases/lines where the conclusion is false: in each case, at least one premise must be false. So, NO argument of this form can have all true premises and a false conclusion.
3.11. PROVING VALIDITY BY VALID DERIVATION

Once we have proven to our satisfaction that certain ARGUMENT FORMS are valid, we can use those argument patterns as schematic models which, in effect, provide us with rules of valid inference. Inferences that formally conform to these valid argument schemas are themselves valid.

For example, inferences that formally conform to the following schemas are valid:

(MP) (1) $P \rightarrow Q$
(2) $P$
(3) $Q$

(DS) (1) $P \lor Q$
(2) $-P$
(3) $Q$

These schemas enunciate, in effect, rules of valid inference, as follows.

(MP) Modus Ponens says, in effect:

(1) Given a conditional of the form: $P \rightarrow Q$
(2) If you can affirm the antecedent: $P$
(3) You may then affirm/derive the consequent: $Q$

(DS) Disjunctive Syllogism says, in effect:

(1) Given a disjunction, which says that either one disjunct ($P$) or the other ($Q$) is true: $P \lor Q$
(2) If it's not the one that's true: $-P$
(3) You may then infer that the other is: $Q$
When we come upon more complex argument forms or longer sequences of inference, like:

(1) \( P \lor Q \)
(2) \( -P \)
(3) \( Q \Rightarrow R \)

Therefore: \( R \)

we can show that such argument forms are valid by deriving the conclusion from the given premises in a series of valid intermediate steps according to previously validated inference rules. The argument form above is proven to be valid by deriving its conclusion \((R')\) from its given premises \((1)-(3)\) according to the rules Modus Ponens and Disjunctive Syllogism, as follows.

(1) \( P \lor Q \) / PREMISE
(2) \( -P \) / PREMISE
(3) \( Q \Rightarrow R \) / PREMISE
(4) \( Q \) / INTERMEDIATE STEP: [Justified by the application of the DS rule to lines (1) and (2)]
(5) \( R \) / CONCLUSION: Justified by the MP rule applied to lines (3) and (4)

(Note: As in the computer-assist programs, a slash '/' indicates that what follows is the justification of the given line in the derivation.)

Once you become familiar with certain basic valid argument forms or valid inference rule schemas, you will be able to analyze the pertinent logical form of ordinary language arguments and prove arguments valid by deriving their conclusions from their given premises in a sequence of valid intermediate steps. Sometimes additional premises must be supplied in order to make an argument valid. You will get better at perceiving where an argument needs additional premises for validity as you become practiced in the application of valid rule schemas.)
4.1. LOGICAL EQUIVALENCE: TRUTH-TABULAR PROOF

Two sentences are LOGICALLY EQUIVALENT by virtue of having LOGICALLY EQUIVALENT FORMS. For example:

You are reading

It is NOT the case that you are NOT reading

have, respectively, the following logically equivalent forms,

symbolized:

\[ P \]

\[ \neg P \]

where we allow \( P \) to stand for 'You are reading.'

To say that two sentence forms are LOGICALLY EQUIVALENT means that any sentences with those forms always have THE SAME TRUTH-VALUE for any assignment of truth-values to their components.

For example: Whenever \( P \) is true, \( \neg P \) must be true; whenever \( P \) is false, \( \neg P \) must be false.
Logical equivalence in sentential logic can be shown by a truth-table test. For ALL POSSIBLE truth-value assignments to the sentential variable \( P \), \( P \) and \( \neg \neg P \) have the same truth-value*. We know this by the truth-functional rule for negation: when a sentence is true, its negation is false; and vice versa. So when a sentence is true, its double negation (the negation of its negation) is true. And when a sentence is false, its double negation is also false, as follows:

\[
\begin{array}{ccc}
P & \neg P & \neg \neg P \\
T & F & T \\
F & T & F \\
\end{array}
\]
4.2. RULES OF REPLACEMENT

If two sentences must ALWAYS have THE SAME truth-value, by virtue of having LOGICALLY EQUIVALENT forms, then two noteworthy points follow for constructing valid arguments:

1. Either sentence logically implies and can be validly DERIVED from the other.

2. Either sentence can be SUBSTITUTED for the other in any line of an argument or derivation without affecting VALIDITY.

Thus, LOGICALLY EQUIVALENT SENTENCE FORMS provide us with VALID RULES OF REPLACEMENT for constructing and evaluating arguments.

The foregoing points can be illustrated with the simple example of the DOUBLE NEGATION rule.

The DOUBLE NEGATION replacement rule, schematized as follows:

P :: --P

says, in short:

Wherever ANY sentence (P in this case) occurs in an argument, it may be REPLACED by its double negation; and vice versa.
Double Negation allows, specifically, two things:

1. From any sentence whatever (P for instance) you may derive its double negation; and vice versa:

   Thus: \[
   \begin{align*}
   \text{P} & \quad \rightarrow\neg \neg \text{P} \\
   \neg \neg \text{P} & \quad \rightarrow \text{P}
   \end{align*}
   \]

   are both VALID.

2. Wherever a sentence or sentence form (e.g., P) occurs within a more complex sentence or sentence form, it may be REPLACED by its double negation; and vice versa.

   Thus the following are all VALID derivations by Double Negation:

   \[
   \begin{align*}
   &P \rightarrow Q & &P \rightarrow Q & &\neg \neg (P \rightarrow Q) \& R \\
   \hline
   &\neg \neg P \rightarrow Q & &P \rightarrow \neg \neg Q & &\neg \neg (P \rightarrow Q) \& R \\
   &\neg \neg P \rightarrow Q & &P \rightarrow \neg \neg Q & &\neg \neg (P \rightarrow Q) \& R \\
   &P \rightarrow Q & &P \rightarrow Q & &\neg \neg (P \rightarrow Q) \& R
   \end{align*}
   \]

   NOTE: A REPLACEMENT RULE allows more than an INFERENCE RULE: when two sentences are logically equivalent:

   1. Either may validly be DERIVED FROM the other (the allowed inference goes both ways)

   2. Either may validly be SUBSTITUTED FOR the other in any line of an argument or derivation

   The double colon '::' in the Replacement Rule schemas stands for the relation of logical equivalence. Thus, the Double Negation Rule, schematized:
P :: --P

means that P and --P are logically equivalent and interchangeable.

But biconditionals like 'P if and only if --P,' (P <=> --P) can also be used to express logical equivalence. The biconditional (P <=> --P) is LOGICALLY TRUE: that is, there is no interpretation or truth-value assignment under which (P <=> --P) is false; it is true under all possible combinations or truth-values for its component atomic sentences. This is shown by the following truth-table: On every line of the truth-table, in every case, under every possible combination of truth-value assignments, (P <=> --P) is true. This is so because a biconditional is true if both sides of the biconditional have the same truth-value; and P and --P always have the same truth-value:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>-P</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

The biconditional (P <=> --P) is a TRUTH OF LOGIC, LOGICALLY TRUE, because under no possible truth-conditions can a sentence with this logical form be false. This biconditional expresses the law of double negation.

Given (P if and only if --P), we have, in effect, the conjunction of two conditionals, a two-way conditional:

(1) P IF --P [or: IF --P, then P]

AND

(2) P ONLY IF --P [or: IF P, then --P]

So, whenever we have P, we may infer or replace it with --P. AND whenever we have --P, we may infer or replace it with P. This law of logic, the rule of double negation, when expressed as a biconditional makes it clear why a RULE OF REPLACEMENT is more powerful than a RULE OF INSTRUCTION: we can do more with a rule of replacement than with a rule of inference because it allows inferences in both directions of the double arrow '=<>':
REMEMBER: A Rule of Replacement allows two things:

1. A rule of replacement (like P : : --P) allows us to infer each logically equivalent sentence from the other. Thus: P : : --P allows us two-way inferences (unlike Rules of Inference, which allow only one-way inferences):

   P and --P
   — — — —
   --P P

   (If P, then --P) and (If --P, then P)

2. And, again, rules of replacement, like P : : --P, allow us to replace either logical equivalent with the other inside complex molecular sentences without affecting the validity of an argument. Thus, the following replacement moves are allowed (by the double negation rule):

   R \rightarrow P     R \rightarrow --P
   — — — —
   R \rightarrow --P  R \rightarrow P
4.3. THE DE MORGAN RULE

EQUIVALENCE BETWEEN CONJUNCTIVE AND DISJUNCTIVE FORMS

Any conjunction has a logical equivalent in the form of a negated disjunction. And any disjunction has a logical equivalent in the form of a negated conjunction. The list of the possible permutations of such equivalents runs in the dozens. But the logic of the equivalence in question is quite commonsensical, so there is no need to memorize a long list of schematic equivalents: we can easily reason out the equivalence between any conjunction (or disjunction) and its negated disjunctive (or conjunctive) counterpart. The rule which allows us to make inferences or replacements of this sort is named after the logician De Morgan, who first proved the basic equivalence.

The following schema for the rule does not represent every possible permutation, but only two variations [(3) and (4)] on the two basic equivalences [(1) and (2)]:

(1) \(-P \& Q\) :: \(-P \lor -Q\)
(2) \(-P \lor Q\) :: \(-P \& -Q\)
(3) \(P \& Q\) :: \(-(-P \lor -Q)\)
(4) \(P \lor Q\) :: \(-(-P \& -Q)\)

Let's reason through the sense of each of these pairs of equivalents. If you can do this, you can easily apply the principle of De Morgan's rule without having to memorize the possible variations on it. Consider the following readings of the above pairs of equivalents. It should be intuitively clear that each sentence form in a pair has the same logical meaning as the other.
\[-(P \land Q) :: -P \lor -Q\]

\[-(P \land Q)\] says: It's not the case that both \(P\) and \(Q\) are true

(At least one is false)

\[-P \lor -Q\] says: Either \(P\) is false or \(Q\) is false

(At least one is false)

\[-(P \lor Q) :: -P \land -Q\]

\[-(P \lor Q)\] says: It's not the case either that \(P\) is true or that \(Q\) is true

(Neither \(P\) nor \(Q\) is true)

\[-P \land -Q\] says: \(P\) is false and \(Q\) is false

(Neither \(P\) nor \(Q\) is true)

\[P \land Q :: -(P \lor -Q)\]

\[P \land Q\] says: Both \(P\) and \(Q\) are true

(Both are true: Neither is false)

\[-(P \lor -Q)\] says: It's not the case either that \(P\) is false or that \(Q\) is false

(Neither is false: Both are true)
P v Q : : -(-P & -Q)

P v Q says: Either P is true or Q is true
(At least one is true)

-(P & -Q) says: It's not the case that both P is false and Q is false
(At least one is true)

The following sentence forms are also logically equivalent by De Morgan's rule. They are analogous to the equivalents above but simply have different distributions of negations. Notice that this rule obviates the need for double negation: any double negation of a sentence is equivalent to that sentence and is replaceable by the sentence itself. Thus, wherever an application of De Morgan's rule would result in a double negation of a sentence or variable (like --P, --Q), we may simply replace the double negation with the variable (P, Q) itself. In the examples below wherever there is a double negation (in the center column) we could as well cancel it out (as I have done in the right-hand column):

(a) P & -Q : : -(P v --Q) : : -(P v Q)
(b) P v -Q : : -(P & --Q) : : -(P & Q)
(c) -(P & -Q) : : -P v --Q : : -P v Q
(d) -(P v -Q) : : -P & --Q : : -P & Q

Now you try to match each of the sets of equivalent sentence forms (a) - (d) above with the intuitive readings (1) - (4) below which express their truth-functional meaning:
(1) It's not the case that P is true and that Q is false ::

   Either P is false or Q is true

(2) P is true and Q is false ::

   Neither is P false nor is Q true ::

   It's not the case either that P is false or that Q is true

(3) Either P is true or Q is false ::

   It's not the case that P is false and Q is true ::

(4) Neither is P true nor is Q false ::

   It's not the case either that P is true or that Q is false ::

   P is false and Q is true

[The answers are: (a)=(2), (b)=(3), (c)=(1), (d)=(4).]

You will find it useful to remember the truth-functional meaning, the logical force of 'neither-nor-' expressions. Consider the following series of equivalent expressions and sentence forms:

Neither P nor Q

-(P v Q)

It's not the case that either P or Q is true

Not P and not Q

-P & -Q

Both P and Q are false
Neither P nor Q: \neg(P \lor Q)

means something quite different from

Not both P and Q: \neg(P \land Q)

Neither P nor Q means that both P and Q are false: \neg(P \lor Q) is equivalent to \neg P \land \neg Q.

Not both P and Q means that not both are true, that at least one is false: \neg(P \land Q) is equivalent to \neg P \lor \neg Q.

That not both are true, that at least one is false is NOT equivalent to saying that both are false.

The truth-tables below demonstrate a typical logical equivalence within the scope of the De Morgan rule. Two sentence forms are logically equivalent by virtue of having the same truth-value under all possible assignments to their components: so, whenever one is true, its equivalent is true; and whenever one is false, its equivalent is false. Therefore, by allowing us to replace any part of an argument with its logical equivalent, rules of replacement (like the De Morgan rule) will never lead us from truth into falsehood: rules of replacement sanction only valid moves, valid inference forms. The following instance of the De Morgan rule:

\neg(P \lor Q) :\equiv -P \land \neg Q

sanctions inferences of the following forms as valid:

\begin{align*}
\neg(P \lor Q) & \equiv -P \land \neg Q \\
\hline
\neg P \land \neg Q & \equiv -(P \lor Q)
\end{align*}

and allows the following sorts of replacements within complex sentence forms:

\begin{align*}
R & \equiv -(P \lor Q) \\
\hline
R & \equiv (-P \land \neg Q)
\end{align*}

\begin{align*}
R & \equiv (-P \land \neg Q) \\
\hline
R & \equiv -(P \lor Q)
\end{align*}
The logical equivalence \(- (P \lor Q) :: -P \land -Q\) is proven by the following truth-table, which shows that a sentence of one form \(- (P \lor Q)\) is true when and only when a sentence of the other form \(-P \land -Q\) is true [line (4) of the table]; and whenever a sentence of the one form is false, a sentence of the other, equivalent form is false [lines (1)-(3) of the table].

\[- (P \lor Q) :: -P \land -Q\]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(- (P \lor Q))</th>
<th></th>
<th>(-P \land -Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>T</td>
<td>F</td>
<td>(F)</td>
<td>F</td>
</tr>
<tr>
<td>(2)</td>
<td>T</td>
<td>F</td>
<td>(F)</td>
<td>F</td>
</tr>
<tr>
<td>(3)</td>
<td>F</td>
<td>F</td>
<td>(T)</td>
<td>F</td>
</tr>
<tr>
<td>(4)</td>
<td>F</td>
<td>T</td>
<td>(T)</td>
<td>T</td>
</tr>
</tbody>
</table>

You can construct truth-tabular proofs of the logical equivalence of other instances of the De Morgan rule:

\[-(P \land Q) :: -P \lor -Q\]
\[(P \lor Q) :: -(\neg P \land \neg Q)\]
\[(P \land Q) :: -(\neg P \lor \neg Q)\]

\[-(P \lor \neg Q) :: -P \land Q\]
\[-(P \land \neg Q) :: -P \lor Q\]
\[P \lor \neg Q :: -(\neg P \land Q)\]
\[P \land \neg Q :: -(\neg P \lor Q)\]
4.4. THE TRANSPOSITION RULE

The schema for this rule of replacement is

\[ P \implies Q : : -Q \implies -P \]

This rule licenses inferences in the following two directions:

\[ P \implies Q \quad -Q \implies -P \]

\[
\begin{align*}
-\neg Q & \implies -P \\
\implies & \\
\neg Q & \implies -P \\
\implies & \\
P & \implies Q
\end{align*}
\]

The rule licenses replacements within more complex formulae like the following:

\[
\begin{align*}
(P \implies Q) & \& R \\
\implies & \\
(-\neg Q \implies -P) & \& R
\end{align*}
\]

\[
\begin{align*}
-(P \implies Q) & \\
\implies & \\
-(\neg Q \implies -P) & \\
\implies & \\
\neg P & \implies Q \\
\implies & \\
-P & \implies Q \\
\implies & \\
\neg Q & \implies P \\
\implies & \\
\neg Q & \implies \neg \neg P
\end{align*}
\]

The rule obviates the need to go through the step of double negation, if you wish to skip it, as follows—both the following are correct:

\[
\begin{align*}
-P & \implies Q \\
\implies & \\
-Q & \implies P \\
\implies & \\
\neg Q & \implies \neg \neg P
\end{align*}
\]

This rule of equivalence makes perfect sense in terms of our understanding of ordinary conditional statements. A conditional of the form \( P \implies Q \) may be read in either of the following two ways:
(1-a) If P, then Q: $P \Rightarrow Q$

(1-b) P only if Q: $P \Rightarrow Q$

Readings (a) and (b) are just two different ways of stating the same conditional relation between P and Q: they are two different sides of the same logical coin. 'If P, then Q' says, in effect, whenever you've got P, you've got Q. So, you've got P only if you have Q. Now, suppose you don't have Q. What can you expect about P? Clearly,

From:

(1) You have P only if you have Q

It follows that:

(2) If you don't have Q, you don't have P

A causal analogy, conditionally expressed, illustrates the equivalence:

From:

(1) It's raining out only if it's wet out

It follows that:

(2) If it's not wet out, then it's not raining out

And vice versa:

From:

(2) If it's not wet out, then it's not raining out

It follows that:

(1) It's raining out only if it's wet out
More abstractly, this rule of equivalence says:

Given any conditional: (1) $P \Rightarrow Q$

We may transpose* and negate its antecedent and consequent: (2) $\neg Q \Rightarrow \neg P$

Given any conditional: (2) $\neg Q \Rightarrow \neg P$

We may transpose and negate its antecedent and consequent: $\neg\neg P \Rightarrow \neg\neg Q$

which, by double negation, is equivalent to: (1) $P \Rightarrow Q$

* This is why this rule of equivalence is called the TRANSPOSITION rule: the antecedent and consequent are transposed (when the antecedent is negated because the consequent has been negated).

We can PROVE that a sentence of the form $P \Rightarrow Q$ is logically equivalent to (and validly replaceable by) a sentence of the form $\neg Q \Rightarrow \neg P$ by a truth-tabular analysis. Two sentence forms are logically equivalent if, for any and every assignment of truth-values to their component atomic sentences, they have the same truth-values: whenever one is false, the other is false; whenever one is true, the other is true.

So, two sentence forms are logically equivalent if it is a truth of logic that a sentence of one form is true if and only if a sentence of the other form is also true. The biconditional

$$(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)$$

is a truth of logic because under every possible truth-value assignment to its component atomic variables the biconditional is true: it is logically impossible for a biconditional of this form to be false.
The following truth-table shows that a sentence of the form $P \Rightarrow Q$ is true whenever a sentence of the form $\neg Q \Rightarrow \neg P$ is true; and false whenever $\neg Q \Rightarrow \neg P$ is false. If one is true, the other is true; and, the one is true only if the other is true. Hence, the corresponding biconditional stating this equivalence is a truth of logic, true under all possible truth-value assignments to its component variables:

<table>
<thead>
<tr>
<th></th>
<th>$P \Rightarrow Q$</th>
<th>$\neg Q \Rightarrow \neg P$</th>
<th>$(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$T \Rightarrow T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>(2)</td>
<td>$T \Rightarrow F$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>(3)</td>
<td>$F \Rightarrow T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>(4)</td>
<td>$F \Rightarrow F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
</tbody>
</table>
4.5. THE IMPLICATION RULE

The schema for this rule displays the logical equivalence between certain disjunctive and conditional sentence forms:

\[ P \lor Q : : -P \implies Q \]

This is to say that a disjunction may be equivalently expressed as a conditional whose antecedent is the negation of one of the disjuncts and whose consequent is the other disjunct.

We can reason this equivalence out in the following steps. The rule licenses inferences in two directions, from sentences of form (1) to sentences of form (2), and from (2) to (1) as follows:

(1) Either P is true or Q is true; one or the other is true: \[ P \lor Q \]

So, (2) If P is not true, Q is: \[ -P \implies Q \]

(2) If P is not true, then Q is: \[ -P \implies Q \]

Now, either P is true or it isn't

But (2) if P isn't true, then Q is

So, either P is true or P isn't true, (in which case, Q is)

So, (1) Either P is true or Q is: \[ P \lor Q \]
Here's a concrete example:

(1) Either I'll play or I'll quit; one or the other: \( P \lor Q \)

So, (2) If I don't play, then I'll quit: \(-P \Rightarrow Q\)

(2) If I don't play, then I'll quit: \(-P \Rightarrow Q\)

Now, either I'll play or I won't

But (2) If I don't play, I'll quit

So, Either I'll play, or else I won't (in which case, I'll quit)

So, (1) Either I'll play or else I'll quit: \( P \lor Q \)

We can start with (1) and reason our way intuitively to (2); and we can start with (2) and reason our way intuitively to (1): each implies the other; each is logically equivalent to (and validly replaceable by) the other. This we can prove by the following truth-table analysis:

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>( P \lor Q )</th>
<th>(-P \Rightarrow Q)</th>
<th>((P \lor Q) \leftrightarrow (-P \Rightarrow Q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
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<td>T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Each equivalent is true whenever the other is, and false whenever the other is false: It is always, necessarily and logically true that one is true if and only if the other is true. Hence, the biconditional expressing this equivalence is always, necessarily and logically true.
4.6. THE EQUIVALENCE RULE FOR BICONDIONALS

The schemas for this rule display the logical equivalence between a biconditional and (1), on the one hand, the conjunction of two conditionals and (2), on the other hand, the disjunction of two conjunctions.

(1) $P \iff Q : (P \implies Q) \land (Q \implies P)$

(2) $P \iff Q : (P \land Q) \lor (\neg P \land \neg Q)$

In other words, there are two logically equivalent ways of expressing a biconditional of the form $P$ if and only if $Q$:

$P$ if and only if $Q$

(1) $(P$ if $Q)$ and $(P$ only if $Q)$

$(If$ $Q,$ then $P)$ and $(If$ $P$ then $Q)$

$(If$ $P,$ then $Q)$ and $(If$ $Q$ then $P)$: $(P \implies Q) \land (Q \implies P)$

(2) Either both $P$ is true and $Q$ is true

Or both $P$ is false and $Q$ is false

$(P$ is not true and $Q$ is not true)

Either both are true or both are false: $(P \land Q) \lor (\neg P \land \neg Q)$

These two readings follow naturally from the syntax and truth-functionality of biconditionals, as follows:
4.6.1. BICONDITIONALS AS CONJUNCTIONS OF CONDITIONALS

On the one hand, P if and only if Q is, in effect, clearly the conjunction of two conditionals:

P if Q and P only if Q

which by a series of trivial commutative moves we can transform as follows:

P if Q, and P only if Q

If Q, then P; and P only if Q

P only if Q; and if Q, then P

Since P only if Q is equivalent to If P, then Q we have:

If P then Q, and If Q then P

\[ P \Rightarrow Q \quad \& \quad Q \Rightarrow P \]

as one analysis of the biconditional

P if and only if Q

\[ P \Leftrightarrow Q \]

in which the conditional arrow '⇒' points both ways:

P ⇒ Q and P ⇐ Q

We write these formulae for conditionals as:

\[ P \Rightarrow Q \quad \& \quad Q \Rightarrow P \]
Thus: \( P \iff Q \) is equivalent to \((P \Rightarrow Q) \land (Q \Rightarrow P)\)

That the biconditional \( P \iff Q \) is truth-functionally equivalent to the conjunction of two conditionals \((P \Rightarrow Q) \land (Q \Rightarrow P)\) is proven by the following truth-table:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \iff Q )</th>
<th>((P \Rightarrow Q) \land (Q \Rightarrow P))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>(T) \iff T \iff (T)</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>(F) \iff F \iff (T)</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>(T) \iff F \iff (F)</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>(T) \iff T \iff (T)</td>
</tr>
</tbody>
</table>

Notice that whenever the biconditional is true, so is the conjunction of \((P \Rightarrow Q)\) and \((Q \Rightarrow P)\); and whenever the biconditional is false, so is the conjunction of \((P \Rightarrow Q)\) and \((Q \Rightarrow P)\). Remember: a biconditional is true whenever both sides have the same truth-value (both are true, or both are false), as on lines (1) and (4) of the table above; otherwise it is false. A conjunction is true whenever both conjuncts are true; otherwise it is false.

4.6.2. BICONDITIONALS AS EXPRESSING NECESSARY AND SUFFICIENT CONDITIONS

Conditionals can be read as expressing sufficient and necessary conditions as follows:

If \( P \), then \( Q \)  

\[ P \Rightarrow Q \] 

\( P \)'s being true is a sufficient condition for \( Q \)'s being true

\( P \) only if \( Q \)  

\[ P \Rightarrow Q \] 

\( Q \)'s being true is a necessary condition for \( P \)'s being true

All the above are equivalent expressions. Any conditional of the form
If P, then Q : \( P \Rightarrow Q \)

P only if Q : \( P \Rightarrow Q \)

may be taken to assert two things:

\( P \) is a **sufficient condition** for \( Q \): \( P \Rightarrow Q \)

and

\( Q \) is a **necessary condition** for \( P \): \( P \Rightarrow Q \)

Since a biconditional of the form \( P \) if AND only if \( Q \) is, in effect, the conjunction of two conditionals, namely:

If \( P \) then \( Q \): \( P \Rightarrow Q \)

and

If \( Q \) then \( P \): \( Q \Rightarrow P \)

where these conditionals can be rendered, respectively:

\( P \) is **sufficient** for \( Q \) and \( Q \) is **necessary** for \( P \): \( P \Rightarrow Q \)

and

\( Q \) is **sufficient** for \( P \) and \( P \) is **necessary** for \( Q \): \( Q \Rightarrow P \)

a biconditional, symbolized \( P \Leftrightarrow Q \), says, in effect:

\( P \) is **necessary and sufficient** for \( Q \)

or, equivalently:

\( Q \) is **necessary and sufficient** for \( P \)
4.6.3. BICONDITIONALS AS DISJUNCTIONS OF CONJUNCTIONS

A biconditional of the form \( P \text{ if and only if } Q \) is also equivalent to the disjunction of two conjunctions, schematically as follows:

\[
P \iff Q : \ (P \land Q) \lor (\neg P \land \neg Q)
\]

This reading of biconditionals follows naturally from what a biconditional asserts about the truth-values of its components, in effect:

- \( P \) if and only if \( Q \)
- \( P \) is true when and only when \( Q \) is true
- \( P \) and \( Q \) have the same truth value:
  - When \( P \) is true, so is \( Q \)
  - When \( P \) is false, so is \( Q \)
- Either both \( P \) and \( Q \) are true, or both are false
  \[
  (P \land Q) \lor (\neg P \land \neg Q)
  \]

The following truth-table shows the logical equivalence of biconditionals of the form \( P \iff Q \) and disjunctions of the form \( (P \land Q) \lor (\neg P \land \neg Q) \):

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \iff Q )</th>
<th>( (P \land Q) \lor (\neg P \land \neg Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>(T) \land T \lor (\neg T \land \neg T) \quad (T) \land T \lor (\neg T \land \neg T) \quad (T) \land T \lor (\neg T \land \neg T) \quad (T) \land T \lor (\neg T \land \neg T)</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>(T) \land T \lor (\neg T \land \neg T) \quad (T) \land T \lor (\neg T \land \neg T) \quad (T) \land T \lor (\neg T \land \neg T) \quad (T) \land T \lor (\neg T \land \neg T)</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>(F) \land F \lor (\neg F \land \neg F) \quad (F) \land F \lor (\neg F \land \neg F) \quad (F) \land F \lor (\neg F \land \neg F) \quad (F) \land F \lor (\neg F \land \neg F)</td>
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<td>F</td>
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<td>T</td>
<td>(F) \land T \lor (\neg F \land \neg F) \quad (F) \land T \lor (\neg F \land \neg F) \quad (F) \land T \lor (\neg F \land \neg F) \quad (F) \land T \lor (\neg F \land \neg F)</td>
</tr>
</tbody>
</table>

Where the biconditional is true, so also is the disjunction [lines (1) and (4) of the truth-table]. Where the biconditional is false, so also is the disjunction [lines (2)-(3)].
4.7. PROVING LOGICAL EQUIVALENCE BY MUTUAL DERIVATION

Just as we can prove that argument forms are VALID by DERIVATIONS (i.e., by deriving their conclusions from their premises by a series of valid INTERMEDIATE STEPS according to previously established rules of valid inference), we can also prove that two statements or sentence forms are LOGICALLY EQUIVALENT by DERIVATIONS (i.e., by deriving each statement from the other according to rules of valid inference, or by deriving a biconditional expression of their equivalence).

For this purpose, the techniques of CONDITIONAL or INDIRECT PROOF are often useful.

These special proof techniques are explained and illustrated in Chapter 7. The intuitive idea here is just this: If two sentence forms are logically equivalent, then we can mutually derive each from the other, or we can derive a biconditional expressing their equivalence, using only previously established rules of inference. Thus, without using anything but the previously established rules of inference, we can prove that P → Q and ¬Q → ¬P are logically equivalent by deriving ¬Q → ¬P from P → Q, and vice versa, or by deriving the biconditional expression of this equivalence, (P → Q) <=> (¬Q → ¬P).
5.1. QUANTIFICATIONAL LOGIC:

BEYOND SENTENTIAL LOGIC, SENTENCES, AND SENTENTIAL CONNECTIVES

A lot of what is accounted valid or invalid about arguments has to do, as you know, with the ways in which the compound sentences of the argument are negated and combined by means of sentential connectives like 'not,' 'and,' 'or,' 'if,' 'only if,' 'unless,' 'if and only if.' Truth-functional (sentential) logic shows us how the truth-value of a molecular sentence is a unique function of the truth-values of its component or atomic sentences. Given a truth-functional interpretation of the sentential connectives we can determine (by a truth-table analysis) whether it is possible for the premises of an argument to be true while its conclusion is false. Truth-functional logic thus provides a method for proving that certain argument forms are valid, and also that certain sentence forms are logically equivalent.

By providing us with provably valid argument forms and provable logical equivalents, sentential logic equips us with inference rules and replacement rules that we can use to construct valid arguments and to distinguish manifestly valid argument forms from apparently invalid argument forms.

But here a cautionary note is needed. While we can prove that an argument is valid if it conforms to some provably valid argument form, we cannot always prove that an argument is invalid just because it conforms to an invalid argument form.

This is so because more than one argument form may be attributable to a given argument. There can be different 'levels' of logical form in any sentence or argument. There may be more than one way to construe the logical form of a sentence or an argument.

Recall: an argument is valid if its logical form is valid. But, since an argument may have more than one identifiable argument form, one way of depicting its form may be invalid while another (perhaps not yet discovered) is valid.
Consider the following arguments, which you should know intuitively are valid (even though you may not yet know what about their logical form makes them valid):

(A) All Pittsburghers are Pennsylvanians

Preston is a Pittsburgher

Therefore, Preston is a Pennsylvanian

(B) All humans are mortal

President Carter is human

So, President Carter is mortal

(C) Only Pennsylvanians are Pittsburghers

President Carter is not a Pennsylvanian

So, President Carter is not a Pittsburgher

(D) Only mortals are human

Zeus is not mortal

So, Zeus is not human

In sentential logic, to symbolically depict the logical form of these—or any—arguments, we assign a sentential variable to each distinct atomic sentence and depict any negations and sentential connectives with the appropriate symbols.

In arguments (A) and (B) there are no sentential connectives. There are no logical operators that we can depict with the tools of sentential logic. From the point of view of sentential logic there are only unanalyzable atomic sentences in arguments (A) and (B). Arguments (C) and (D), of course, contain negations.
In sentential logic, the logical form of each of these arguments may be depicted as follows:

\[
\begin{array}{cccc}
(A') P & (B') P & (C') P & (D') P \\
Q & Q & \neg Q & \neg Q \\
\hline
R & R & \neg R & \neg R
\end{array}
\]

(Notice that the same sentential variables may be assigned to different sentences in the context of different arguments.)

The sentential logical forms of the above arguments are invalid: thus, argument forms (A'), (B'), (C'), (D') are invalid. Yet we know (and with simple diagrams could prove) that arguments (A), (B), (C), and (D) are in fact valid. This means that they each must have a valid logical form, but that sentential logic simply does not give us the tools to analyze and represent the relevant valid argument forms. This is because sentential logic only allows us to analyze and represent logical operations (negation, conjunction, etc.) that are performed on whole SENTENCES.

Sentential logic does not allow us to analyze the logical parts of atomic sentences. It does not allow us to go 'inside' atomic sentences in order to analyze their 'SUB-ATOMIC' logical structure. In particular, it does not allow us to analyze the logical connections between the SUBJECTS and PREDICATES of sentences or to represent the logical operations performed by such crucial terms as 'ALL,' 'ONLY,' 'SOME,' 'NONE.'

The validity of the four arguments above is a function of the logical relations between the SUBJECTS and PREDICATES of their premises and conclusions and the operations of the QUANTIFIERS 'all' and 'only.' To understand and represent these important 'sub-atomic' elements of logical form we need a system of logic that goes beyond the analysis of the connection among atomic sentences; we need an analysis of the logical structure of atomic sentences themselves. This system or domain of logic is sometimes called PREDICATE LOGIC because it studies the logical relations among subjects and predicates of sentences. And it is also called QUANTIFICATIONAL LOGIC because it also studies the use of terms like 'all,' 'only,' 'some,' 'none' that are used to state the quantity of things (e.g., All Pittsburghers, some Pittsburghers, . . .) to which certain properties (e.g., being a Pennsylvanian) are attributed (e.g., in sentences like, 'All Pittsburghers are Pennsylvanians').
5.2. INSIDE ATOMIC SENTENCES: SUBJECTS AND PREDICATES

It's useful to be able to analyze the internal logical structure of sentences, in particular, sentences that predicate properties of their subjects or that posit relations between their subjects. For example:

(a) Dracula is a fiend.
(b) The Christian God is more powerful than Dracula.
(c) Atlanta is the capital of Georgia.

We can analyze sentence (a) into two logical elements:

1. A noun phrase or SUBJECT ('Dracula') naming or describing something of which some property (being a fiend) is predicated.
2. A verb phrase or PREDICATE ('is a fiend') that describes the property attributed to the subject.

We can symbolize sentence (a) in a way that represents this internal structure by recourse to the following convention:

1. Let the lower case letters a through d serve as INDIVIDUAL CONSTANTS that stand for particular individual persons, places, or things.
2. Let the capital letters F through S serve as PREDICATE VARIABLES that may stand for predicates or verb phrases (just as in sentential logic we let these letters serve as sentential constants that may stand in for sentences).

Now, for purposes of representing the SUBJECT-PREDICATE STRUCTURE of sentence (a):

Let:  F stand for the predicate 'is a fiend'
      d stand for 'Dracula'

Then (a) may be symbolized as (a'):  Fd
We can analyze and represent the logical form of sentence (b) in a similar fashion; sentence (b) consists of:

1. Two subjects ('The Christian God' and 'Dracula') naming the things between which a relation (being more powerful than) is posited.

2. A predicate ('is more powerful than') by which this relation between the two subjects is expressed.

Let:  
- $P$ stand for 'is more powerful than'
- $c$ stand for 'the Christian God'
- $d$ stand for 'Dracula'

This sentence (b) may be symbolized as (b'-1):

$$Pcd$$

Sentence (b) could also be taken as attributing the property of being-more-powerful-than-Dracula to the Christian God, as follows:

Let:  
- $M$ stand for the predicate 'is more powerful than Dracula'
- $c$ stand for 'the Christian God'

such that we can symbolize sentence (b) as (b'-2):

$$Mc$$

We could also symbolize sentence (b) without representing any of its internal, subject-predicate structure, as (b'-3) below:

Let:  
- $G$ stand for the whole sentence 'The Christian God is more powerful than Dracula'

so that we have the rather unrevealing symbolization:

$$G$$

Note that (b'-1) represents more logical structure than does (b'-2), and both represent more logical structure than does (b'-3).
Likewise, sentence (c) could be symbolized in different ways:

\[(c'-1): \quad a = c\]

where \(a\) stands for 'Atlanta'

\(c\) stands for 'the capital of Georgia'

Here we take the 'equals' sign '=' to stand for the special relation of being-identical-with, and we take the verb 'is' in this case as meaning 'is identical with,' as asserting the special relation of identity between two things.

\[(c'-2): \quad Ga\]

where \(G\) stands for the predicate 'is the capital of Georgia'

\(a\) stands for 'Atlanta'

\[(c'-3): \quad G\]

where \(G\) stands for the whole sentence 'Atlanta is the capital of Georgia'

Again \((c'-1)\) displays more logical structure than \((c'-2)\), and both display more logical structure, more logical elements of sentence (c) than does \((c'-3)\).

As a general rule, we will want to depict all of the possibly relevant logical structure of the premises and conclusion of an argument before declaring the argument form invalid.

As you will see, if we represent only the sentential logical form of arguments \((A) - (D)\), we cannot show that these arguments are in fact valid; but if we display the quantificational logical form of the argument, representing the internal logical structure of their premises and conclusions, their validity either becomes evident or at least their validity can be proven by deriving their conclusions from their premises (by rules of quantificational logic together with rules of sentential logic). (You will learn simple quantificational inference rules by which this can be done.)
You've now seen how the internal logical form of statements about particular individual things can be analyzed into a subject-predicate structure. This allows us to depict more of the logical form of statements like the second premise and conclusion of the sample arguments above. For example:

(A) (1) All Pittsburghers are Pennsylvanians  
(2) Preston is a Pittsburgher
(3) Preston is a Pennsylvanian

(A') (1) — — —
(2') Pa
(3') Qa

where P stands for 'is a Pittsburgher'
Q stands for 'is a Pennsylvanian'
a stands for 'Preston'

But how do we analyze and represent the pertinent logical form of the first premise (A-1), wherefrom, given Pa, it logically follows that Qa?

(A-1) contains the crucial term 'ALL,' which is called a QUANTIFIER. How do we represent the logical force of such terms?
5.3. THE UNIVERSAL QUANTIFIER: 'ALL,' 'ONLY'

We often have recourse to universal generalizations or categorical statements that attribute properties (e.g., being a Pennsylvanian) to all or only those things that have certain other properties (e.g., being a Pittsburgher, living in the U.S.). Such statements are important in defining general concepts as well as in making factual generalizations. For example:

All sound arguments are valid

Only valid arguments are sound

And the general normative principles to which we appeal in our moral reasonings or assume as premises in our moral arguments often take these sorts of categorical logical form. For example:

All persons have rights

Only persons have rights

Words like 'ALL,' 'EVERY,' or 'ONLY' are called QUANTIFIERS because they are used to make statements about quantity, about how many things do or don't have certain properties.

In analyzing and symbolizing the logical force and form of sentences in which some quantifier is operating, we can draw and build upon our understanding of the logical force of the sentential connectives. In this way, quantified atomic sentences can often be analyzed and represented as quantified molecular sentences which we know how to handle by the rules of sentential logic. Examples follow.
5.3.1. TRANSLATING 'ALL,' 'EVERY,' 'ANY'

INTO UNIVERSALLY QUANTIFIED 'IF-THEN' EXPRESSIONS

Consider the universally quantified sentence (the universal generalization):

All Pittsburghers are Pennsylvanians

When we say that something (e.g., being a Pennsylvanian) is true of all things of a sort (e.g., Pittsburghers), we say, in effect, that the following goes for any thing:

IF it is a Pittsburgher, THEN it's a Pennsylvanian

Thus: For ANY arbitrary thing (let's call it x):

If x is a Pittsburgher, then x is a Pennsylvanian

Thus: where P stands for the predicate 'is a Pittsburgher'

Q stands for the predicate 'is a Pennsylvanian'

and where Px stands for 'x is a Pittsburgher' and

Qx stands for 'x is a Pennsylvanian'

we can translate the universally quantified sentence:

All Pittsburghers are Pennsylvanians

into the universally quantified conditional sentence form:

For all x: IF Px, then Qx

which we may begin to symbolize:

For all x: Px => Qx

We let the lower case x operate as an INDIVIDUAL VARIABLE which can stand in for any individual (person, place, thing . . .) whatever. We will employ the lower case letters x, y, z, w as individual variables to stand in for arbitrary individual entities in quantified sentences (just as we employ such letters in algebra to stand in for arbitrary numerical quantities in algebraic equations).
We let the individual variable \( x \) between parentheses \((x)\) stand for the universal quantifier expression 'for all \( x \)' and any of its ordinary English cognates ('all,' 'every,' 'any'). The parenthesized expression \((\ldots x \ldots)\) is the schema for the symbolization of whatever it is that is asserted to hold true for all \( x \). Algebraic expressions provide a good analogy; for example, in the numerical domain: \((x) \ (x.x = x)\).

Thus, we can symbolize the universally quantified sentence

All Pittsburghers are Pennsylvanians

as a universally quantified conditional sentence

\((x) \ (Px \Rightarrow Qx)\)

Consider now categorical sentences of the form:

All P's are Q's

Every P is a Q

Any P is a Q

These all say, in effect, that IF something belongs to the category of P's THEN it belongs to the category of Q's. The verb 'are' here has the sense of 'belong to the category of' or 'have the same properties as': the verb 'is' has the sense of 'belongs to the category of' or 'has the properties of.' There are different ways of formulating the logical force of such categorical expressions, but all could be symbolized:

\((x) \ (Px \Rightarrow Qx)\)

You will have to rely on your linguistic intuitions (and practice in translating and symbolizing such categorical statements) to understand when and why they can be analyzed and symbolized as universally quantified 'IF-THEN' CONDITIONALS. Sample symbolizations are provided in section 5.3.3.
5.3.2. TRANSLATING 'ONLY' INTO U-QUANTIFIED 'ONLY IF'

Consider the universally quantified sentence:

ONLY Pennsylvanians are Pittsbughers

This sentence says, in effect:

For any thing x:

x is a Pittsbugher ONLY IF x is a Pennsylvanian

And where Px stands for 'x is a Pittsbugher' and Qx stands for 'x is a Pennsylvanian' this sentence could be symbolized:

(x) (Px => Qx)

just as was the sentence

All Pittsbughers are Pennsylvanians

which says, in effect:

For all x:

IF x is a Pittsbugher THEN x is a Pennsylvanian

Compare now:

ONLY Pennsylvanians are Pittsbughers

ALL Pittsbughers are Pennsylvanians

When we analyze categorical statements of this sort, more schematically:

ONLY Q's are P's

ALL P's are Q's

as UNIVERSALLY QUANTIFIED CONDITIONALS, notice that they are symbolized in the same way:
ONLY Q's are P's

For all x: x is a P ONLY IF x is a Q

ALL P's are Q's

For all x: IF x is a P, THEN x is a Q

(x)(P(x) => Q(x))

This is to say that universally quantified conditionals have at least two equivalent formulations in English, just as do ordinary conditionals: Just as every 'IF-THEN' expression has an equivalent 'ONLY IF' expression, so also every 'ALL' expression has an equivalent 'ONLY' expression, as follows:

IF P THEN Q is equivalent to P ONLY IF Q or ONLY IF Q, P

ALL P's are Q'S is equivalent to ONLY Q's are P's

IF x is P, x is Q is equivalent to x is P ONLY IF x is Q
Notice that the **ANTECEDENT** of a universally quantified conditional is constituted out of the **ALL---'** phrase of the the categorical statement 'All P's are Q's'; the **ALL---'** phrase is converted into an equivalent 'IF---' clause of a universally quantified conditional:

**ALL** porpoises are mammals

For all \( x \): \( \text{IF } x \text{ is a porpoise, } x \text{ is a mammal} \)

\((x) (Px \Rightarrow Mx)\)

The 'ONLY---' phrase of the categorical statements translates into the 'ONLY IF' clause, the **CONSEQUENT** of a universally quantified conditional:

**ONLY** mammals are porpoises

For all \( x \): \( x \text{ is a porpoise ONLY IF } x \text{ is a mammal} \)

\((x) (Px \Rightarrow Mx)\)

Notice: these examples are not simply **CONDITIONALS** but rather **UNIVERSALLY QUANTIFIED** conditionals, just as \(-(P \Rightarrow Q)\) is not a conditional but rather a **NEGATION** of a conditional.

Not all universally quantified statements are appropriately analyzed and symbolized as universally quantified **CONDITIONALS**. Only those universally quantified statements like the examples above (that assert, in effect, that all [or only] those things of one category belong to another category), can be analyzed as universally quantified conditionals (conditionals that assert, in effect, that something belongs to one category **IF** [or **ONLY IF**] it belongs to some other category).
5.3.3. Quantifying Unconditionally

Statements Not Analyzable As Quantified Conditionals

Not every statement that asserts something about everything of a kind is a universally quantified conditional; it may be making an even broader assertion, attributing some property or set of properties to everything whatsoever. For example:

(1) Everything owes its existence to God.
(2) Everything is both good in some respects and bad in some respects.
(3) Everything is either dead or alive.
(4) Everyone should go to school.
(b) Everyone is afraid of Dracula.

We could symbolize these universally quantified statements as follows:

(1') (x)(Gx) where: Gx = 'x owes its existence to God'
(2') (x)(Gx & Qx) where: Gx = 'x is good in some respects'
Qx = 'x is bad in some respects'
(3') (x)(Px v Qx) where: Px = 'x is dead'
Qx = 'x is alive'
(4') (x)(Sx) where: Sx = 'x should go to school'
(5') (x)(Fx d) where: Fxy = 'x is afraid of y'
d = Dracula

At this point two crucial constraints on how we analyze and symbolize quantified statements need to be specified. You should, for example, be uncomfortable with the symbolization of sentences (4) and (5), even if not with any of the others. The next section explains why.
5.4. THE UNIVERSE OR DOMAIN OF DISCOURSE

How Much is Everything? What Domain Are We Talking About?

How Large a 'Universe' Are We Generalizing About?

When I say to my class, 'Everyone should have read Chapter Three by Monday,' I don't mean that literally everyone whosoever in the world should have read Chapter Three, nor even necessarily everyone in the classroom audience. I mean rather 'Everyone who is a student in my course should have read Chapter Three by Monday.' It is readily understood, implicitly, in such a situation that I am not referring to everyone at large in the whole universe but rather to a more limited 'universe' or 'domain,' namely the universe or domain constituted by persons who are students in my course.

We often employ implicitly understood limits to the range of things to which we mean to refer when using quantifiers like 'every' or 'all' or 'any.' If someone says, 'Everyone had better vote in this election,' he surely can't mean 'every person whosoever.' He may at maximum mean 'everyone franchised to vote in this election' (excluding from the domain to which he refers those who are underage, non-citizens, etc.). Or, within the particular context in which he is speaking, the domain of franchised voters to which he refers may be even more limited: in the context of his exhorting fellow party members, 'everyone' may mean 'everyone who is a member of this party,' in which case he may be perfectly pleased if everyone else (members of opposing parties) did not vote (so that his party could sweep the election).

We often mean to limit the domain to which we refer in using quantifiers like 'every' to some specifiable portion of everything in the whole universe. We will call the domain or range of things to which we wish to limit the reference of quantifiers 'The DOMAIN OF DISCOURSE or 'The UNIVERSE OF DISCOURSE.'

For purposes of clear and concise discourse we need to be able to refer to limited 'universes,' limited domains of things, to a limited range of all that there is.

Sometimes the context of discourse makes the domain of discourse perfectly clear: in the context of a given course and class meeting, it's clear that in 'Everyone should have read Chapter Three by Monday,' the quantifier 'everyone' refers to no one who is not a student in the course.
Sometimes pronouns themselves delimit the possible domain of discourse. 'Everyone' ordinarily means 'every person' and is understood to have reference only within the domain or limited 'universe' of all persons. Consider again sentences (4) and (5) at the end of the last section:

(4) Everyone should go to school
(5) Everyone is afraid of Dracula

Here the implicit DOMAIN OF DISCOURSE must be all persons, or else we could not symbolize (4) and (5) as we did:

(4') (x) (Sx) where: Sx = 'x should go to school'
(5') (x) (Fxd) where: Fxy = 'x is afraid of y'
\[ d = \text{Dracula} \]

Without supposing the domain of discourse to be all persons, (4') and (5') could be translated into statements much stronger than (4) or (5); respectively:

(4a) Everything should go to school
(5a) Everything is afraid of Dracula

Someone could still object that (4) and (5) are themselves too strong, overstated. It might be objected that what is meant is something more carefully qualified; like:

(4b) Everyone who is eligible should go to school
(5b) Everyone who is a warm-blooded mortal person is afraid of Dracula

(For surely, goes the objection to (5), Dracula is not afraid of himself, nor would be God, nor necessarily another vampire, nor a 'cold blooded' electronic cyborg, etc. etc.)
It all depends on what you want to say, of course, but a good rule of thumb, especially in logic and philosophy, is to be as explicit and precise as possible. If, in point of fact, (4b) and (5b) were what were meant by (4) and (5), logic gives us two options for making this clear and explicit.

OPTION 1: We could specify a distinct and limited DOMAIN OF DISCOURSE as an explicit part of the logical analysis and symbolization of (4) and (5);

(4) Everyone should go to school

Domain: All persons eligible for school
Let: Sx = 'x should go to school'

(4') (x)Sx

(5) Everyone is afraid of Dracula

Domain: All warm-blooded mortal persons
Let: Fxy = 'x is afraid of y'
d = Dracula

(5') (x)(Fxd)

The specification of the domain puts a limit to the range of things referred to by the quantifier ' (x). ' Since ' (x) ' itself simply means 'For any x' or 'For every x' or 'For all things x,' the universal quantifier ' (x) ' by itself will not distinguish 'everything (whatsoever)' from 'everyone' (every person). So,
Strictly speaking, an explicit specification of the domain of discourse should accompany the symbolization of 'everyone' as referring to the domain of persons rather than the domain of everything whatsoever. As a rule, when a more limited domain of discourse is not specified, the broadest possible range may be given the quantifier.

How we specify or understand a domain of discourse can make a difference as to whether we construe a quantified statement as true or as false. For example:

(5) Everyone is afraid of Dracula

is, supposedly, false; not everyone is afraid of Dracula: Dracula supposedly is not afraid of Dracula. Specifying the domain of discourse is one way of being clear and explicit and, so, avoiding misunderstanding or objections like the above. There is another option.

OPTION 2: We can allow the domain of discourse to range as widely as the universe and all its imaginable alternatives, but classify what is meant precisely by what we say or by how we analyze what is said. When faced by a debatable case like

(5) Everyone is afraid of Dracula

where we hardly suppose that literally everyone could be afraid of Dracula (for Dracula supposedly is not) -- in such cases we might explicate and reformulate (5) as follows:

(5b) Everyone who is a warm-blooded mortal person is afraid of Dracula

Now we've effectively qualified 'everyone' as limited to those-who-are-warm-blooded-and-mortal.

Structurally, (5b) is no longer a statement about simply everyone in the universe. It is now a statement about some category of things (things that are warm-blooded mortal persons) to which a property is attributed (being afraid of Dracula). (5b), in effect, lays down conditions for being afraid of Dracula, whereas (5) simply states categorically that everyone is afraid of him. (5b) is a universally quantified conditional that says, in effect:
The following goes for everything:

**IF** it's a warm blooded mortal person, **it's afraid of Dracula**

and would be symbolized something like:

\[(5b') \ (x)((Rx \& Mx) \Rightarrow Fxd)\]

Where:  
\( Rx = 'x \text{ is red/warm-blooded}' \)  
\( Mx = 'x \text{ is a mortal person}' \)  
\( Fxy = 'x \text{ is afraid of } y' \)  
\( d = \text{Dracula} \)

Notice how we can simplify the explication and symbolization of the original statement

\[(5) \text{ Everyone is afraid of Dracula}\]

by just specifying an appropriate domain of discourse:

**Domain:** All warm-blooded mortal persons

**Let:**  
\( Fxy = 'x \text{ is afraid of } y' \)  
\( d = \text{Dracula} \)

Then: \((5') \ (x)Fxd\)

Given the above interpretation, \((5')\) says the same thing as \((5b')\). Because \((5')\) is making an assertion about literally everything within its specified domain of discourse, it does not need to specify the conditions or properties by virtue of which something fears Dracula [as does the universally quantified conditional \((5b')\)]. Just as, when I know that the class knows the domain of discourse, I can say to the class, 'Everyone should have read Chapter III by Monday' rather than 'Everyone-who-is-a-student-in-this-course . . .'. 
5.5. The SCOPE of a Quantifier: BOUND Variables

You are familiar with the importance of parentheses in correctly parsing logical form when symbolizing sentences in sentential logic. For example, you know that the sentence

(1) Sentence a is not both true and false

would be symbolized

(1'a) \(-(Sa \& Fa)\)

where: \( Sa = 'sentence a is true' \)
\( Fa = 'sentence a is false' \)

It would be incorrect to symbolize sentence (1) as follows:

(1'b) \(-Sa \& Fa\)

Sentence (1) is a NEGATION denying that a certain sentence a is both true and false, thus negating the CONJUNCTION 'Sentence a is true and sentence a is false.' We might say that the SCOPE of the negation sign is the whole conjunction, as depicted by (1'a), rather than just the left conjunct, as depicted by (1'b). The use of parentheses around the conjunction 'Sa & Fa' allows us to show that everything within the PARENTHESES is within the SCOPE of the negation sign.

Similarly, we must use parentheses to make the SCOPE of a quantifier explicit. This is best explained by example.
Consider the following case:

Let:  The domain of discourse - All sentences
      \( S_x = 'x \text{ is a true}' \)
      \( F_x = 'x \text{ is a false}' \)

Then we have:

\( (3') \ (x)(S_x \lor F_x) \)

which says:

(3) Any sentence is either true or false.

[Notice that if we had not specified the domain of discourse as above, (3') would be a false statement, to the effect: Any thing whatsoever is either true or it is false. This is not true of physical things, or questions, etc.]

Now consider carefully the function of the parentheses in formula (3'), given the above interpretation:

\( (3') \ (x)(S_x \lor F_x) \)

Contrast the meaning of (3') with the following:

\( (3'a) \ (x)S_x \lor (x)F_x \)
\( (3'b) \ S_x \lor F_x \)

OK. Now... What's wrong with (3'a) and (3'b) as symbolizations of the universally quantified disjunction (3)? Let's take each case in turn.
The parentheses in (3') indicate that the scope of the universal quantifier \( (x) \) is the whole disjunction: for every (sentence) \( x \), \( x \) is true or \( x \) is false: the quantifier BINDS the individual variable \( x \) in both disjuncts within its SCOPE [just as the negation sign applies to both disjuncts in \(-(P \lor Q)\)]. Formula (3') represents the true statement: For any sentence \( x \), either that sentence is true or that sentence is false.

Formula (3'a) represents a very different statement, and one that is clearly false;

\[
\begin{align*}
(3a) & \text{(i) Either every sentence is true or} \\
& \text{every sentence is false.} \\
& \text{(ii) For every (sentence) \( x \), \( x \) is true} \\
& \lor \text{or} \\
& \text{(ii) For every (sentence) \( x \), \( x \) is false}
\end{align*}
\]

Formula (3') represents a universally quantified disjunction, whereas (3'a) represents the disjunction of two universally quantified statements (i) and (ii). (3') and (3'a) are different in LOGICAL FORM, hence different in MEANING, and, as it happens, different in truth-value.

The difference in logical form and meaning between (3') and (3'a) lies in the differences in the SCOPE of the universal quantifier \( (x) \) as it is employed in each formula.

Formula (3'b): \( Sx \lor Fx \)

is NOT WELL-FORMED: Any occurrence of the variable \( x \) by itself is meaningless unless and until it is BOUND within the scope of some quantifier. Consider the following case:

\( Sx \)

This formula says only that 'unknown quantity' \( x \) is true; but by itself the variable \( x \) refers to nothing, so we can't make any sense out of it, let alone determine whether \( Sx \) represents a true sentence or a false one. When an individual variable occurs outside the scope of a quantifier, we say that it is a FREE variable, that it is UNBOUND. A formula with a free or unbound variable is meaningless because we have no idea of what portion of the universe of discourse or how many individuals it might refer to.
On the other hand, when an individual variable $x$ is **BOUND** within the scope of a quantifier, say, the universal quantifier $(\forall x)$, then we can assign a clear reference to the variable. For example:

$$(\forall x) \ Sx$$

This formula refers to *every* and *any* thing $x$; it tells us that the variable $x$ may refer to *any* or *all* things within the domain of discourse. Knowing what portion of the universe of discourse the variable $x$ may refer to allows the formula in which it occurs to have both meaning and truth-value. 'Unknown quantity $x$ is true' means nothing, whereas 'Every or any sentence $x$ is true' both means something and has a determinate truth-value (namely, it is false).

Every quantifier must have a determinate **SCOPE** in any quantified formula (its scope being defined by the placement of right and left parentheses -- just as the scope of a negation sign is determined).

For example, the formula

$$(\forall x) (Sx \lor Fx)$$

is not well-formed because the parentheses are not closed: the scope of the quantifier $(\forall x)$ is not determinate; it is ambiguous whether the quantifier takes in and binds only one or both of the variables that occur in the formula.
When the scope of a quantifier is unambiguous because it ranges only over a variable in an atomic formula that immediately follows it, then parentheses are unnecessary. For example:

\[(x) \ Sx\]
\[(x) \ (Sx)\]

are both well-formed formulae. Whereas the following:

\[(x)(Sx) \ V \ Fx\]

is not well-formed because the second occurrence of the variable x is not bound within the scope of any quantifier. Likewise, none of the following is well-formed:

\[(x)(y)Fxyz\]
\[(x)Fy\]
\[(x)Fxy\]

because each contains an unbound variable (z, y, and y, respectively.)
Remember: not only must all occurrences of individual variables \((x, y, z, w)\) in any formula be bound within the scope of a corresponding quantifier \([x], (y), (z), (w),\) respectively, but the way in which the scope of a quantifier is defined by parentheses can make a big difference to the logical form and meaning of a formula (and, so, to the correctness of a symbolization). Here are some more examples to help you remember this vital point; note the differences in meaning between the various pairs of sentences and their symbolizations.

\[(4a)\] Everything is either human or non-human.

\[(4a')\] \((x) (Hx \lor Nx)\)

\[(4b)\] Either everything is human or everything is non-human

\[(4b')\] \((x) Hx \lor (x) Nx\)

\[(5a)\] Not everything is human

\[(5a')\] \(-(x) Hx\)

\[(5b)\] Everything is not-human. [Nothing is human.]

\[(5b')\] \((x) \neg Hx\)

The following section contains many examples with which to exercise your sense of how to translate and symbolize universally quantified statements and their negations.
5.6. SAMPLE SYMBOLIZATIONS: THE UNIVERSAL QUANTIFIER

Predicate variables are indicated in brackets within each sentence after the respective predicate. Sometimes alternative symbolizations are provided: one with, one without the domain of discourse specified.

(1) All persons [P] have rights [R]

(1'a) (x)(Px => Rx) :: (x)(If x is a person, x has rights)

(1'b) (x)Rx Where the Domain = Persons

(2) Only persons have rights

(2'a) (x)(Rx => Px) :: (x)(x has rights only if x is a person)

(2'b) (x)Px Where the Domain = Things with rights

(3) Every living object [O] has the right to life [L]

(3'a) (x)(Ox => Lx)

(3'b) (x)Lx Where the Domain = Living things

(4) A thing is not precious [P] unless it's living [L]

(4'a) (x)(Px => Lx)

(4'b) (x)Px Where the Domain = Living things

(5) A person has rights [R] only if she's male [M] [Px='x is a person']

(5'a) (x)(Px => (Rx => Mx))

(5'b) (x)(Rx => Mx) Where the Domain = Persons

(6) All and only persons [P] have rights [R]

(6') (x)(Rx <=> Px)
(7) A thing has rights \([R]\) if and only if it's a person \([P]\)

(7') \((x)(Rx \iff Px)\)

(8) Something has rights \([R]\) only if it's living \([L]\)

(8') \((x)(Rx \implies Lx)\)

(9) Barb shaves all and only those who do not shave themselves

\([Sxy='x\ shaves\ y']\) \(b = Barb\)

(9'a) \((x)((\neg Sxx \implies Sbx) \& (Sbx \implies \neg Sxx))\)

(9'b) \((x)(Sbx \iff \neg Sxx)\)

(10) If a person is a member of a group that has been unjustly discriminated against \([M]\), justice demands that that person be compensated \([J]\)

(10') \((x)(Mx \implies Jx)\)

\[\text{Domain= Persons}\]

(11) If one is female \([F]\) then one is a member of a group that has been unjustly discriminated against \([M]\)

(11') \((x)(Fx \implies Mx)\) Where the Domain = Persons

(12) No one has been unjustly discriminated against \([G]\)

(12') \((x)-Gx\) Where the Domain = Persons

(13) Not everyone has been unjustly discriminated against \([\text{It's not the case that everyone has been . . .}]\)

(13') \((-x)Gx\) Where the Domain = Persons
Consider the following argument [by Lewis Carroll]. Is it valid?

The only animals in this house are cats. Every animal is suitable for a pet, that loves to gaze at the moon. When I detest an animal I avoid it. No animals are carnivorous, unless they prowl at night. No cats fail to kill mice. No animals ever take to me, except what are in this house. Kangaroos are not suitable for pets. None but carnivora kill mice. I detest animals that do not take to me. Animals that prowl at night always love to gaze at the moon. Therefore, I always avoid a kangaroo.

In fact, the argument is valid. How could this be proven? By deriving the conclusion from the given premises, of course. For this, it would help to depict (symbolize) the logical form of the premises and conclusion. I have done this in items (14) through (24) below, after altering the sequence of the premises so that they fall into a more apparently logical order. Carroll's argument contains some interesting exercises in symbolization. I have rendered all of the premises in the form of universally quantified conditionals.

See if, given the following symbolizations, you can figure out the intermediate steps by which the conclusion can be derived.
(14) The only animals in this house are cats.
Let: $Ix = \text{'x is an animal in this house'}$
$Px = \text{'x is a pussycat/cat'}$
(14') $(x)(Ix \Rightarrow Px) : : (x)\ (Ix\ only\ if\ Px)$

(15) No cats fail to kill mice. Let: $Mx = \text{'x kills mice'}$
(15'a) $(x)(Px \Rightarrow Mx)$
(15'b) $(x)-(Px & \neg Mx)$
(15'c) $(x)(\neg Px V Mx)$

(16) None but carnivora kill mice. Let $Ox = \text{'x is an carnivore'}$
(19'a) $(x)(Mx \Rightarrow Ox)$
(19'b) $(x)(\neg Mx V Ox)$
(19'c) $(x)-(Mx & \neg Ox)$

(17) No animals ever take to me, except what are in this house.
[No animal likes me unless it is an animal in this house]
[An animal likes me only if it is an animal in this house]
[It is not the case both that an animal likes me and that it is not an animal in this house]
Let: $Lx = \text{'x likes/takes to me'/'x is an animal that takes to me'}$
(17'a) $(x)(Lx \Rightarrow Ix)$
(17'b) $(x)(\neg Lx V Ix)$
(17'c) $(x)-(Lx & \neg Ix)$
(18) I detest animals that do not take to me.

Let: \( Hx = \text{\textquoteleft}x \text{ is an animal I hate/detest}\) 

(18'a) \((x)(-Lx \Rightarrow Hx)\)

(18'b) \((x)(Lx \lor Hx)\)

(19) When I detest an animal, I avoid it.

Let: \( Rx = \text{\textquoteleft}x \text{ is an animal I reject/avoid}\) 

(19') \((x)(Hx \Rightarrow Rx)\)

(20) No animals are omnivorous, unless they prowl at night.

Let: \( Nx = \text{\textquoteleft}x \text{ is an animal that prowls at night}\) 

(20'a) \((x)(Ox \Rightarrow Nx)\)

(20'b) \((x)(Nx \lor -Ox)\)

(21) Animals that prowl at night love to gaze at the moon.

Let: \( Gx = \text{\textquoteleft}x \text{ is an animal that loves to gaze at the moon}\) 

(21') \((x)(Nx \Rightarrow Gx)\)

(22) Every animal is suitable for a pet, that loves to gaze at the moon.

Let: \( Sx = \text{\textquoteleft}x \text{ is an animal suitable for a pet}\) 

(22') \((x)(Gx \Rightarrow Sx)\)

(23) Kangaroos are not suitable for pets. \( Kx = \text{\textquoteleft}x \text{ is a kangaroo}\) 

(23') \((x)(Kx \Rightarrow -Sx)\)

(24) I always avoid a Kangaroo.

(24') \((x)(Kx \Rightarrow Rx)\)

**QUESTION:** Do you see how sentence (24) follows logically from (14) through (23)? The next chapter (6) will explain.
5.7. **THE EXISTENTIAL QUANTIFIER: 'SOME'**

As you know, in addition to sentential connectives there are other logical operators, logical terms, that we need to understand in order to decide the validity of arguments or to determine the logical connections between statements. Among these are terms called QUANTIFIERS, terms used to refer to certain quantities of things to which we wish to attribute properties or amongst which we wish to posit relations. You are familiar with terms like ALL, EVERY, ANY, ONLY that are used to refer to everything whatsoever in the universe of discourse and that are symbolized by the UNIVERSAL QUANTIFIER, \( (x) \). We also use terms like SOME to refer to one or more, but fewer than all things in the universe of discourse, as in statements like the following:

(a) **Some humans have rights**

Sentence (a) can be analyzed into two components as follows:

(a') (i) **There is some (at least one) thing x such that**

(ii) **x is a human AND x has rights**

The first component of (a'), (i), states what seems to be implied when we refer to SOME members of a class (in this case, the class of humans): To talk about SOME things of a kind seems, in effect, to posit that some such things exist.

Now, universally quantified statements may seem equally to carry existential import: to talk about ALL things of a kind may seem to posit the existence of such things. But the existential implications that universally quantified statements seem to have is at least muted when they are analyzed as universally quantified CONDITIONALS: 'All P's are Q's' may seem to imply that there 'exist' P's and Q's; but we lessen this impression when we analyze the statement as 'For all x, IF x is a P, then x is a Q' which does not commit us to the existence of either P's or Q's; the latter statement does not assert that THERE IS any x that is either a P or a Q. Does "Some vampires hate the night" imply the existence of vampires any more or less than "All vampires hate the day"?
Perhaps because of the existential implications that the use of terms like SOME seem to carry, the symbol used to represent the logical force of SOME is called the EXISTENTIAL QUANTIFIER. However, the function of the existential quantifier is not to beg questions about the 'existence' of anything, but rather to allow us to refer to a certain QUANTITY of things within our universe of discourse, within the domain of what we want to talk about (irrespective of their existential status in the actual universe).

The symbol we will use to represent the logical force of SOME consists of a capital 'E' along with an individual variable contained within parentheses, as follows:

(Ex) (Ey) (Ew) (Ez)

any variation of which, say, '(Ex),' may be read as follows:

There exists an x such that . . .
There is some x such that . . .
Some x is such that . . .
For some x it is the case that . . .
For some x, . . .

[Note: Logic texts often employ a backwards 'E' in the symbol for the existential quantifier. We will resort to the normal capital E simply because there is no backwards E on the keyboard of most computer terminals on which the logic programs which accompany this text run. Hence, the programs accept and use only the regular E in the existential quantifier.]
Clearly, the existential quantifier need not be taken literally or exclusively to posit actual or physical existence in the so-called 'real' world. It can as well be used to refer to abstract or imaginary entities that we conceive and whose properties and relationships we wish to talk about (say, entities that 'exist' or are defined only within some conceptual system, imaginary 'world' or system of human conventions -- things like numbers, rules, words, hobbits, golden mountains . . . ). So, we can make sense out of existentially quantified statements like the following, without worrying much about their 'existential' implications:

Some numbers are even
There exists an x such that x is a number and x is even
Some Hobbits are too hairy
There is an x such that x is a Hobbit and x is too hairy
Some rules are made to be broken
For some x, x is a rule and x is made to be broken

What concerns us about SOME at present is not its existential implications but its quantificational force.

We interpret SOME as AT LEAST ONE, allowing but not requiring that it refer to more than one thing. This is the minimal quantity that 'some' can refer to and, hence, is consistent with reference to classes which contain only one member as well as with stronger interpretations, where 'some' could refer to any quantity greater than one but fewer than all.

Thus: 'SOME humans have rights' can be true whether it is ONE human, SEVERAL humans, MOST humans, or however many that have rights. Of course, 'SOME humans have rights' is consistent with, indeed, logically implied by, ALL humans have rights,' even though the converse is not the case.
Let's now return to (a'), our analysis of the logical force of (a):

(a) Some humans have rights

(a') (i) There is some (at least one) x such that

(ii) x is a human and x has rights

You now know how to interpret and symbolize the first component (i) using the existential quantifier. You should also be able to symbolize the second component (ii), given, say: Hx = 'x is a human'; Rx = 'x has rights'.

(i) (Ex)
(ii) Hx & Rx

Putting these two components of the logical analysis of (a) together, we must enclose (ii) in parentheses to indicate the scope of the quantifier.

(a'') (Ex)(Hx & Rx)

Notice the difference in how we analyze and symbolize the existentially quantified

(a) Some humans have rights

versus the universally quantified

(b) All humans have rights

Respectively: (a) is an existentially quantified CONJUNCTION:

(a') For SOME x, x is a human AND has rights

(a'') (Ex) (Hx & Rx)

as versus (b), a universally quantified CONDITIONAL:
(b') For ALL x, IF x is a human, x has rights

(b'') (x) (Hx => Rx)

Consider the consequences of analyzing (a) or (b) otherwise. Suppose, for example, we were to try to treat (a) as an existentially quantified conditional

(a*') There is some x such that, IF x is a human, then x has rights

Now, what sense does it make to say THERE IS something so 'IFFY'???

Suppose, on the other hand, we treated (b) as if it were a universally quantified conjunction:

(b*'') For all x: x is a human AND has rights

EVERYTHING both is human and has rights? Clearly this is not what (b) means!

Clearly, (a*'') and (b*'') are incorrect analyses of the logical force or meaning of (a) and (b), respectively. (a*'') hardly makes sense; indeed, we hardly ever have occasion to assert existentially quantified statements with this logical form:

(a*'') (Ex) (Hx => Rx)

Can you think of any statement with this logical form that makes sense?

While (b*'') makes perfect sense, on the other hand, it means something quite different from (b), and accordingly, it would be symbolized quite differently

(b*'') (x)(Hx & Rx) Compare with: (Ex)(Hx & Rx) !!

[Review section 5.3.3 for a discussion of when universally quantified statements are not to be analyzed as universally quantified conditionals.]
For the present, you will have to rely on practice in the computer exercises with your linguistic and logical intuitions for deciding how to analyze and symbolize existentially quantified statements. For starters, the next section provides some sample exercises with symbolizations for you to think about and practice on.

The remarks about specifying the UNIVERSE or DOMAIN OF DISCOURSE (UD) in section 5.4 apply equally to existential quantification. Clearly, 'Some things are mammals' would be TRUE where the UD was as wide as possible, but FALSE where the UD was, say, the domain of all insects. The remarks about quantifier SCOPE in section 5.5 also obviously apply to existentially quantified formulas.
5.8. SAMPLE SYMBOLIZATIONS:

UNIVERSAL AND EXISTENTIAL QUANTIFICATION

(1) Some humans have rights.  (1') (Ex)(Hx & Rx)

(2) Some things have rights.  (2') (Ex)Rx

(3) Something has rights.  (3') (Ex)Rx

(4) Some humans have rights. Where UD = All humans

                (4') (Ex)Rx

(5) Something that is human has rights.  (5'a) (Ex)(Hx & Rx)

                (5'b) (x)(Hx => Rx)

[Note: (5'a) and (5'b) mean very different things; but both are possible readings of (5). This is to say that 'something' can sometimes have the logical force of 'anything'/'everything', just as 'someone,' like 'one,' can be used to refer to ANY arbitrary person whatsoever. Which reading of (5) seems more likely to you? Why? How could (5) be rephrased so as to more clearly express the meaning of (5'a) and (5'b), respectively? Other examples of the kind follow: Ask yourself these same questions in each case.]

(6) Something is either human or it doesn't have rights.

                (6'a) (x)(Hx V -Rx) :: (x)(-Hx => -Rx)

                (6'b) (Ex)(Hx V -Rx)

(7) Someone has rights only if he is a person

                (7') (x)(Rx => Px)

(8) One has rights only if he/she is a person.

                (8') (x)(Rx => Px)
(9) No one who is not a person has rights.

(9'a) \((x)(\neg Px \rightarrow \neg Rx)\) ::

(9'b) \(- (Ex)(\neg Px & Rx)\)

[Note: In this case (9) the two symbolizations are logically equivalent! Can you see why? See examples 17-21 below for more of the same. Logical equivalence between or among universally and existentially quantified formulae is discussed in the next chapter.]

(10) Some humans have no rights.

(10') \((Ex)(Hx & \neg Rx)\)

[Can you come up with an equivalent formulation using the universal quantifier?]

(11) Some humans who have no rights are fetuses.

(11') \((Ex)((Hx & \neg Rx) & Fx))\)

(12) Some who have no rights are fetuses.

(12') \((Ex)(\neg Rx & Fx)\) Where the Domain = Humans

(13) It's false that some politicians are honest.

(13') \(- (Ex)(Px & Hx)\)

(14) Some things are neither moral nor immoral.

(14') \((Ex)-(Mx V Ix) :: (Ex)(\neg Mx & \neg Ix)\)

(15) Some human actions are neither intelligent nor stupid.

(15') \((Ex)(Hx & -(Ix V Sx)) :: (Ex)(Hx & -(Ix & -Sx))\)

Where the Domain = Human actions:

(15'') \((Ex)-(Ix V Sx) :: (Ex)(-Ix & -Sx)\)

(16) Some non-humans have rights.

Where \(Hx = 'x \text{ is human}'\): \((16')(Ex)(\neg Hx & Rx)\)

Where \(Nx = 'x \text{ is non-human}'\): \((16')(Ex)(Nx & Rx)\)
(17) No non-human is inhumane.
   (17'a) -(Ex)(Nx & Ix)

(17'b) (x)-(Nx & Ix) ::
       (x)(-Nx V -Ix) ::
       (x)(Nx => -Ix)

[Read the alternative phrasings of (17) out loud to yourself in ordinary English. Can you reason out the logical equivalence of these various formulations? You may be beginning to suspect that any initial symbolization has any number of logical equivalents. If so, you are right -- though some are better, more natural translations of the original sentence than others. This situation is similar to that in sentential logic wherein one can begin with one sort of formula, say, a conjunction, and transform it into any number of logical equivalent disjunctions, conditionals, etc. You will learn the rules for making these equivalent transformations in quantificational logic in the next chapter.]

(18) No politician is either honest or even intelligent

(18'a) -(Ex)(Px & (Hx V Ix))

(18'b) (x)-(Px & (Hx V Ix)) :: (x)(-Px V -(Hx V Ix)) ::
       (x)(Px => -(Hx V Ix))

(19) No one's a politician unless he's neither honest nor intelligent

[Politicians are not either honest or intelligent.]

(19'a) (x)(Px => -(Hx V Ix)) :: (x)(-Px V -(Hx V Ix)) ::
       (x)-(Px & (Hx V Ix))

(19'b) -(Ex)(Px & (Hx V Ix))

[There is no politician who is either honest or intelligent]
(20) None but the rich are truly happy

[One is happy only if one is rich.]

(20'a) \( (x)(Hx \Rightarrow Rx) :: (x)(\neg Rx \Rightarrow \neg Hx) :: (x)(Rx \lor \neg Hx) :: (x)(\neg (Rx \land Hx)) :: \)

(20'b) \( \neg (\exists x) (\neg Rx \land Hx) \)

(21) None except the happy are truly 'rich.'

[It's false that there is someone who is not happy and yet is 'rich.']

(21'a) \( \neg (\exists x) (\neg Hx \land Rx) :: \)

(21'b) \( (x)(\neg Hx \land Rx) :: (x)(Hx \lor \neg Rx) :: (x)(\neg Hx \Rightarrow \neg Rx) \)

Look again now at example (9). Is it any easier to see or reason out in some clearly valid step-wise fashion the logical equivalence of (9'a) and (9'b)? Compare these with (21'a) and (21'b).

If you can explain or reason out why it is the case that in examples 9 and 17-21 symbolization (a) is logically equivalent to symbolization (b) you are well on your way to understanding the RULE OF REPLACEMENT called QUANTIFIER NEGATION. This replacement rule for quantificational logic is illustrated and discussed in the next chapter.
QUANTIFICATIONAL LOGIC is so named because it is concerned with quantifiers, expressions that we use to talk about quantities of things. Quantificational logic studies two sorts of quantifier: (1) the universal quantifier ' (x),' which represents the logical force of quantifier terms like 'all,' 'every,' 'any,' 'only' by which we refer to everything in our universe of discourse; (2) the existential quantifier ' (Ex),' which represents the logical force of quantifier terms like 'some' by which we refer only to some thing or things within our universe of discourse and by which we mean, at minimum, 'at least one.'

In quantificational logic we analyze the logical form of sentences into subjects, predicates, and quantifiers, which we represent using the following symbolic conventions.

PREDICATE VARIABLES: The upper case letters F through S will stand for predicates. A predicate is a verb phrase used to attribute a property to a thing or things, or a relation between things.

For example, '____ is far away,' '____ is farther than ____' and '____ is midway between ____ and ____' are predicate expressions containing the predicates 'is far away,' 'is farther than,' 'is midway between.' These predicates are 'one-place,' 'two-place' and 'three-place' predicates, respectively, so-called because they have places for one, two and three subjects, respectively. (The subjects of each predicate would occur in place of the blanks in the English expressions above.

We add lower case letters following a predicate letter in order to represent the subjects of the predicate. The following are examples of sentences containing the above one-, two- and three-place predicates, symbolized with the lower case letters a, b, c, d standing in for their respective subjects:

Fa : 'a is far away'
*Fab : 'a is farther than b'
*Fca : 'c is farther than a'
*Mbcd : 'b is midway between c and d'
*Mdbc : 'd is midway between b and c'

* Notice: the lower case letters follow the predicate letters in the order in which their corresponding subjects occur in the English expression.
THE IDENTITY RELATION: We will use the equals sign '=' to stand for the special relation of identity between two things. 'a = c' means 'a is c' in the sense of 'a is identical with c.'

INDIVIDUAL CONSTANTS: The lower case letters a, b, c, d will stand for the names or descriptions of particular individual things, persons, places, states of affairs, events. For example, we could let

a stand for 'Pittsburgh, Pa.'
b stand for 'The decision to go to the ball game last night'
c stand for 'The Christian God'
d stand for 'The worst decision I ever made'

And 'b = d' would read: 'The decision to go to the ball game last night is the worst decision I ever made.' Where L stands for the predicate 'lives in,' '-Lea' would read 'God doesn't live in Pittsburgh, Pa.'

INDIVIDUAL VARIABLES: The lower case letters w, x, y, z will serve as variables which can stand for the name or description of any thing whatever. Variables serve as place-holders for individual constants (just as in algebra variables serve as place-holders for numerical values).

For example, the formula 'Hx' could stand for the predicate expression 'x has parents' where x stands for some unspecified subject. (The expression 'x has a parent' by itself is not a meaningful sentence, because as it stands the expression refers to no one in particular. Variables like x have to occur within the scope of a corresponding quantifier in order to refer to some meaningful portion of the universe of discourse.)
THE UNIVERSAL QUANTIFIER: The lower case individual variables, (letters w, x, y, z) when placed by themselves between parentheses [as follows: (w), (x), (y), (z)] will stand for universal quantifiers by which we generalize over the corresponding variables that occur within their scope in a formula. For example, where 'Hx' stands for the expression 'x has parents' and 'Px' for the predicate 'x is a parent,' then:

\[(x) (Px \Rightarrow Hx)\]

may be read:

For any x if x is a parent, then x has parents
For every thing x, if x is a parent, x has parents
Everything that is a parent has parents

The SCOPE of a quantifier, say \((x)\), includes any corresponding variable, x, that occurs either in a formula outside parentheses immediately to the right of the quantifier; for example:

\[(x) Px\]

or in a formula closed within parentheses immediately to the right of the quantifier; for example:

\[(x) (Px \Rightarrow Hx)\]

The DOMAIN or UNIVERSE OF DISCOURSE is the limit we place on the range of all things extant or conceivable in order to specify or limit the range of things we wish to talk about. Sometimes the universe of discourse (that limited range within the actual or conceivable universe we wish to focus on) is clearly implicit within the context of discussion. Thus, for example, when I say 'Everyone should study hard for the final exam,' I mean 'everyone who is taking this course,' not 'everyone whosoever in the world': the implicit domain of discourse limits the range of things to which 'everyone' refers to people taking this course. Often we need to explicitly specify the domain of discourse to avoid confusion about the overall quantity or range of things to which a quantifier is meant to refer.
The EXISTENTIAL QUANTIFIER: The lower case individual variables (letters \( w, x, y, z \)) when placed within parentheses preceded by the upper case \( E \) [as follows: \( (Ew), (Ex), (Ey), (Ez) \)] will stand for the existential quantifier by which we existentially generalize over the corresponding variables that occur within the scope of the quantifier in order to attribute properties to some (rather than all) things in a given universe of discourse. For example, where \( 'Lx' \) stands for the predicate 'x is a living thing' and \( 'Rx' \) stands for the predicate 'x has rights,' then:

\[ (Ex) (Lx \& Rx) \]

may be read:

Some living things have rights.

For some \( x \), \( x \) is a living thing and \( x \) has rights.

There is (at least one) \( x \) such that

\( x \) is a living thing and \( x \) has rights

PEUDO-NAMES: The lower case letters \( t \) and \( u \) are used as pseudo-names for referring to arbitrary, unspecified individual things, persons, places or events. Their function is best explained in the context of the quantificational rules of inference in the next chapter.

Notice: Different ranges of the letters of the alphabet have different symbolic functions. We make these restrictions so that the computer programs know that a given letter has a specific function. Because the proof-checking programs know the symbolic function of each part of the alphabet, they are tolerant in allowing you to enter any letter in either upper or lower case. In setting-up or listing an argument, the programs will print out letters \( F \) through \( S \) in upper case, and letters \( a \) through \( d \), \( t \), \( u \), and \( w \) through \( z \) in lower case, in order to keep predicates and their subjects clearly distinguished. So, if you type \( '(X) PX,' \) in listing your proof the program will print this out as \( '(x) Px.' \)
CHAPTER 6
QUANTIFICATIONAL RULES

RULES OF INference AND REPLACEMENT

6.1. RULES FOR THE UNIVERSAL QUANTIFIER: UI AND UG

6.1.1. THE RULE OF UNIVERSAL INSTANTIATION: UI

The rule, UI, tells us on what conditions we are allowed to make inferences from a universally quantified statement (say, a universal generalization or principle) to an instance of the universally quantified statement (say, a particular case covered by the generalization or principle).

Roughly speaking, the rule allows that if some proposition is true for every thing, then we may infer that it is true for any particular individual thing that we might name.

In particular, from a universally quantified statement, the rule, UI, allows us to derive a statement about a particular individual or thing. For example:

Let:  a = 'Alexander the Great'

Gx = 'x is a Greek'

Mx = 'x is mortal'

Given:  (x)(Gx => Mx)

Then we may infer:  Ga => Ma, by the rule UI

[GIVEN that it's true for every one x that

IF x is a Greek, then x is mortal

THEN the same goes for ANY one we might name.  So:

IF Alexander is a Greek, Alexander is mortal.]

The following derivations are valid by the rule of universal instantiation, UI, where each and every occurrence of a variable (e.g. x) is replaced by the same individual constant (e.g. a):
(1) (x) Px / Premise (or previously derived line)

(2) Pa / I, UI

(1) (x)(Gx => Mx) / Premise (or previously derived line)

(2) Ga => Ma / I, UI

In each case the inference by UI to (2) results from (1) by replacing every occurrence of a variable x (or y, etc.) with an individual constant (a or b, etc.).

The letters t and u serve as pseudo-names that we can use to refer to any arbitrary individual thing or unit in the universe of discourse. We might read the logical meaning of the pseudo-names t or u in terms of the following sorts of description: Let t (or u) stand for 'this, that or whatever arbitrarily chosen thing t'; in other words, t stands for some unit or individual chosen at random from the universe of discourse and thus a thing that is representative of all other individuals in the universe of discourse.

Given: (x)(Gx => Mx)

Then we may infer: GT => Mt

or: Gu => Mu by UI

[Given that it's true for every person that IF x is a Greek, x is mortal, then the same goes for ANY arbitrarily chosen person u: IF some arbitrarily chosen person u is a Greek, then that person u is also mortal.]

Thus, we can make inferences by the rule of universal instantiation either to statements that contain individual constants (referring to particular individual things) or to statements that contain pseudo-names (referring to any arbitrarily chosen individual whatsoever). Thus, the following are valid inferences by UI, where each occurrence of the variable x is replaced by a pseudo-name (t or u, referring to any arbitrary individual we might name):
From universal generalizations or principles the rule of universal instantiation (UI) allows us to infer statements about particular instances or individual cases that are covered by the universal generalization.

But what use is there to being able to infer from a universal generalization (e.g., 'All persons have rights') that any arbitrarily chosen person (to which we assign a pseudo-name like t) has rights? The utility of being able to make inferences to arbitrary things to which we give pseudo-names derives from the fact that this practice allows us to ensure the validity of certain inferences: inferences from universal generalizations to other universal generalizations, as follows.
6.1.2. THE RULE OF UNIVERSAL GENERALIZATION: UG

The rule of universal generalization (UG) allows us explicitly to derive universal generalizations, like

Human fetuses have a right to life

from other universal generalizations, like

All persons have a right to life

Human fetuses are persons

In deriving universal generalizations from other generalizations in a valid, step-by-step deduction the use of UNIVERSAL INSTANTIATION to PSEUDO-NAMES is crucial. Notice first of all how universal instantiation uncovers the sentential logical form within universally quantified statements. Consider the following deduction by UI:

Let: \( t \) = 'ANY arbitrarily chosen THING in the universe'

\( Fx \) - 'x is a human fetus'

\( Px \) = 'x is a person'

\( Rx \) = 'x has a right to life'

(1) \( (x)(Px=>Rx) \) / Premise
    [All persons have a right to life]

(2) \( (x)((Fx=>Px) \) / Premise
    [All human fetuses are persons]

(3) \( Pt \Rightarrow Rt \) / 1, UI
    [If any arbitrary thing \( t \) is a person, then \( t \) has a right to life]

(4) \( Ft \Rightarrow Pt \) / 2, UI
    [If any thing \( t \) is a human fetus, then \( t \) is a person]
Once we have (4) Pt => Pt
and (3) Pt => Rt

it's easy to see that we can get the following by applying
a familiar rule of sentential logic:

(5) Pt => Rt / 3,4 HS

The crucial role played by inferring statements containing pseudo-names
from universal generalizations is that this application of universal
instantiation allowed us to reveal the underlying sentential logical
structure from which we could derive (5) by Hypothetical Syllogism. Now,
the logical function of t is to refer to 'any arbitrary thing in the
universe whatsoever.' So:

(5) Ft => Rt

says, in effect:

If any arbitrary thing whatsoever is a human fetus,
then that thing has a right to life.

Because the pseudo-name t refers to ANY THING whatsoever, statement (5)
can be universally generalized to hold for every thing whatsoever.

This is what the rule of UNIVERSAL GENERALIZATION (UG) allows: When
you have a statement containing a pseudo-name that was obtained by UI from
previous universal generalizations, you may universally quantify that
statement, as follows:

(5) Ft => Rt / Derived (by HS) from lines obtained
by previous applications of UI

(6) (x) (Fx => Rx) / 5, UG
[All human fetuses have a right to life]

Notice: each occurrence of the pseudo-name t in (5)
must be replaced by a variable x bound within the scope
of a corresponding universal quantifier (x).

UNIVERSAL GENERALIZATION (UG) allows us to infer universally quantified
Statements from statements containing pseudo-names that were previously obtained by universal instantiation.

The use of universal instantiation to obtain statements with pseudo-names allows us (1) to reveal the underlying sentential structure of universally quantified statements in order (2) to apply the rules of sentential logic in the derivation of the universally quantified statements. The following is a simple model of the use of universal generalization:

1. Given a universal generalization: *(x) Px*

2. Derive a statement containing pseudo-names by universal instantiation, *(Pt / 1, UI)*

3. Which may then be manipulated using any rules of sentential logic, for example: *(Pt v - Pt / 2, ADD)*

4. Then you may universally quantify the resulting pseudo-name statement by the rule UG: *(x)(Px v - Px) / 3, UG*

The intuitive justification of the inference from (3) to (4) by UG is simply:

Given that something is true for any particular thing t you might name

Then it is true for every thing you might name
6.1.3. **RESTRICTIONS ON THE USE OF UNIVERSAL GENERALIZATION (UG)**

1. You may NOT universally generalize from constants

You may universally generalize only from pseudo-names, never from constants. Generalizing from an individual case is obviously INVALID by commonsense, and proven to be so by the following argument schema and interpretation, whereby the premise (1) is obviously true but the conclusion (2) is obviously false:

Let: $P_x = x$ was president of the U.S.; $a = Abe Lincoln$

1 $P_a$ Abe Lincoln was president of the U.S.

INVALID: 2 $(x)P_x$ Everything was president of the U.S.

2. You may NOT UG from pseudo-names that occur in a line obtained by EI

When a given pseudo-name (say, $t$) ever occurs in a line that is obtained by EI (as $t$ does in line 3 below), you may not universally generalize from it -- even if the pseudo-name was itself originally obtained by UI (as $t$ was at line 2 below). The following derivation and interpretation shows that this move can lead from truth (Everyone has a parent) to falsity (Someone is everyone's parent). Therefore, it is INVALID:

Let: The Domain = people; $P_{xy} = x$ is a parent of $y$
1 $(x)(Ey)P_{yx}$ Everyone has a parent
2 $(Ey)Pyt$ / UI, 1
3 Put / EI, 2 [Cannot UG from t]

INVALID >>>
4 $(x)P_{ux}$ Some person $u$ is everyone's parent
5 $(Ey)(x)Pyx$ / EG, 4 Someone's everyone's pare
6.2. THE RULES OF IDENTITY:

At any point in a deduction one may introduce a sentence of the form: \((x) (x = x)\)

Identity Elimination (IE) for individual constants:

Where \(a\) is any constant and \(b\) is any constant

Given a sentence in which \(a\) occurs: \((1) Pa\)

And given an identity statement: \((2) a = b\)

Then you may derive a corresponding sentence: \((3) Pb / 1, IE\)
6.3. RULES FOR THE EXISTENTIAL QUANTIFIER: EG AND EI

6.3.1. The Rule of Existential Generalization: EG

1. Schematic Statement of the Rule

Arguments of the following forms are valid, where the schemas '(...a...)', '(...t...)', '(...x...)' stand, respectively, for formulae in which some constant [a], or pseudo-name [t], or variable [x] occurs:

\[\begin{array}{c}
(...a...) \\
(Ex)(...x...)
\end{array} \quad \begin{array}{c}
(...t...) \\
(Ex)(...x...)
\end{array}\]

For example, you can see that the following are clearly valid:

\[\begin{array}{c}
Fa \land Pa \\
(Ex)(Fx \land Px)
\end{array} \quad \begin{array}{c}
Ft \land Pt \\
(Ex)(Fx \land Px)
\end{array}\]

Let:
- \(a\) = The 'A' student in the class
- \(t\) = Some arbitrary person that failed the final but passed the course
- \(Fx\) = x failed the final
- \(Px\) = x passed the course

GIVEN that the 'A' student [or some person] who failed the final passed the course

THEN THERE IS SOME x such that x failed the final and x passed the course

SOMEONE failed the final but passed the course
3. Explicit Statement of the Rule EG

GIVEN any sentence $P$ that contains a constant $[a, b, c, d]$ or a pseudo-name $[t, u]$

you may derive an EXISTENTIALLY GENERALIZED sentence $Q$ from $P$ by the following two-step procedure:

1. Introduce an existential quantifier [e.g., (Ex)] that has scope over, and only over, $P$

2. Replace each occurrence of the constant or pseudo-name in $P$ with the individual variable [e.g., $x$] represented in the quantifier introduced at (i)

4. Sample Derivation With EG

1. $Fa & Pa$
2. $(Ex) (Fx & Px) / EG, 1^*$
3. $(Ex)-(-Fx V -Px) / DEM, 2$
4. $(Ex)-(Fx => -Px) / IMPL, 3$
5. $-(x) (Fx => -Px) / QN, 4$

This sample derivation shows that interesting inferences can be made by the use of EG. Given the same interpretations of $Fx$ and $Px$ as on the previous page, the example above shows formally that

GIVEN a case where a person $[a]$ failed the final but passed the course

IT FOLLOWS THAT not everyone who failed the final failed to pass the course.

This shows formally how a given case refutes a generalization like $(x)(Fx => Px)$. By use of EG in conjunction with the quantifier negation rule QN (and other equivalence rules) we can formally show that a given case constitutes a counter-example to, and thereby refutes, a given generalization. We will be using this formal strategy to derive the negation of general principles from putative counter-examples.
6.3.2. The Rule of Existential Instantiation (EI)

It is convenient within certain derivations to be able to instantiate from an existentially quantified expression. The rule which allows such a move is called Existential Instantiation (EI).

The rule of Existential Instantiation allows that from any existentially quantified formula, say

\[(\exists x)Px\]

you may instantiate to a pseudo-name but ONLY to a pseudo-name that has NOT been used in a previous line of your derivation:

\[(\exists x)Px \rightarrow \]
\[Pt\]

(where the pseudo-name \( t \) has NOT been used in a previous line). Note that you may NOT existentially instantiate to constants. Can you see why this would be invalid? Consider the examples on the next page, and remember these restrictions on the use of the Existential Instantiation rule!
6.3.3. Restrictions on Existential Instantiation (EI)

1. You may NOT existentially instantiate to constants

You may existentially instantiate only to pseudo-names, never to
constants: the following argument schema, existentially instantiating to a
constant, is shown to be INVALID by the following interpretation, whereby
the premise (1) is obviously true but the conclusion (2) is obviously
false.

Let: \( P_x = x \) is president of the U.S.; \( d = \) Princess Diana

\[
\begin{align*}
1 & \quad (\exists x)P_x \quad \text{Someone is president of the U.S.} \\
\text{INVALID:} & \quad 2 \quad P_d \quad \text{Princess Diana is president of the U.S.}
\end{align*}
\]

2. You may NOT EI to a pseudo-name already introduced in the derivation

When a given pseudo-name (say, \( t \)) has been previously introduced in a
derivation (say, by UI at line 2 in the example below), you must
existentially instantiate to a different pseudo-name (say, \( u \)). The
following interpretation shows that existentially instantiating to a
pseudo-name already introduced can lead from truth (Everyone has a parent)
to falsity (Someone is his own parent). The INVALID move is at line 3;
line 4 is legal by EG.

Let: The Domain = people; \( P_{xy} = x \) is a parent of \( y \)

\[
\begin{align*}
1 & \quad (x)(\exists y)P_{xy} \quad \text{Everyone has a parent} \\
2 & \quad (\exists y)P_{yt} \quad / \ UI, 1
\end{align*}
\]
3 Ptt Someone is his own parent
4 (Ex)Pxx There is someone who's his own parent
6.4. THE QUANTIFIER NEGATION RULE: QN

LOGICAL EQUIVALENCE: UNIVERSAL AND EXISTENTIAL QUANTIFICATION

You saw from examples in the last section of Chapter 5 that certain universally quantified formulae are logically equivalent to certain existentially quantified formulae — and vice versa, of course. Because of the variety of formulaions that such equivalences can take, it will do little good for you to try simply to memorize them. As with the transformations that are possible with DeMorgan's replacement rule, it is best to be able to reason out the transformations that are possible by the QUANTIFIER NEGATION replacement rule (QN).

The rule schema for QUANTIFIER NEGATION (QN), where 'Px' stands for any formula, is as follows:

(x) -Px :: -(Ex) Px  (x) Px :: -(Ex) -Px
-(x) Px :: (Ex) -Px  -(x) -Px :: (Ex) Px

The proof-checking programs (BERTIE, ARGUE -- RECON contains no arguments requiring quantification) will allow you to make inferences or replacements that are of the same basic form as the above. The rationale for these moves should be intuitively apparent:

(1) (x) -Px  It's true for everything x that x is not P
(2) -(Ex) Px  Therefore, there is no thing x such that x is P
(3) -(Ex) Px  There is not something x that is P
(4) (Ex) -Px  Therefore, for all x, it is not the case that x is P
(5) Therefore, for all x, it is not the case that x is P
It should be intuitively clear that (1) and (2) mean the same thing.

(3) -(x) Px  It's not the case for all x that x is P
(4) (Ex) -Px  Therefore, there is some thing x that is not P
(4) (Ex) -Px There exists some x such that x is not P

(3) -(x) Px Therefore, not everything is P

It should be intuitively clear that (3) and (4) mean the same thing.

In order to make the following sorts of deductive moves, you may but need not employ the DOUBLE NEGATION rule (DN): you can apply QN directly in the proof-checking programs:

(A) (x) Px Everything is P

-----------

-(Ex) -Px It is not the case that there is some thing x that is not P

(B) (Ex) Px There exists some thing x that is P

-----------

-(x) -Px It is not the case that nothing is P

These inferences are valid. We can show this by the following derivations in which steps (A-3) and ((B-3) conform formally to what is allowed by the QUANTIFIER NEGATION rule. Here the step of Double Negation is included:

(A) (1) (x) Px / Premise
(2) --(x) --Px / 1, DN

-----------

(3) -(Ex) -Px / 2, QN

(B) (1) (Ex) Px / Premise
(2) --(Ex) --(Px) / 1, DN

-----------

(3) -(x) -Px / 2, QN
What follows is a series of logically equivalent sentences and quantified symbolizations: you can try, at first, to reason through these equivalences; then try to derive any from any other using the QUANTIFIER NEGATION rule plus the other requisite rules. You can enter these exercises into the BERTIE or ARGUE program if you wish monitored practice in the use of QN. (Enter a symbolized formula and apply QN.)

All of the following sentences are logically equivalent — and provably so by mutual derivation. They are followed by logically equivalent symbolizations on the following page.

Consider: Which of the formulae (1) - (8) (next page) is the best symbolic translation of which of the sentences (a) - (l)?

(a) Only mortals are human.
(b) All humans are mortal.
(c) If something's human, then it's mortal.
(d) None but mortals are human.
(e) Nothing's human that isn't mortal.
(f) A thing is human only if it's mortal.
(g) A thing's not human unless it's mortal.
(h) Either a thing is mortal or it's not human.
(i) If a thing is not mortal, it's not human.
(j) It's not the case that there is something that is both human and not also mortal.
(k) There is nothing human that isn't mortal.
(l) No human isn't mortal.
Here are symbolizations of the above sentences (a) - (1).

Notice that in this sequence we arrive eventually (10) back at the formula with which we began (1). Enter (1) in BERTIE or ARGUE in 'BEGIN' mode and derive the following transformations by applying the respective rules.

1. (x) (Hx => Mx)
2. (x) (-Mx => -Hx)  Derive from (1) by TRANS
3. (x) (-Hx V Mx)  Derive from (2) by IMPL and COM
4. (x) -(Hx & -Mx)  Derive from (3) by DEM
5. -(Ex) (Hx & -Mx)  Derive from (4) by QUANTIFIER NEGATION (QN)
6. -(Ex) -(Hx V Mx)  Derive from (5) by DEM
7. -(Ex) -(Hx => Mx)  Derive from (6) by IMPL
8. -(Ex) -(Mx => -Hx)  Derive from (7) by TRANS
9. (x) (-Mx => -Hx)  Derive from (8) by QUANTIFIER NEGATION
10. (x) (Hx => Mx)  Derive from (9) by TRANS

This sequence of logical equivalents shows that any of the sentences (a) - (1) above COULD logically be translated simply as

(x) (Hx => Mx)

or as

-(Ex) (Hx & -Mx)

Why translate any of the sentences one way rather than another?
CHAPTER 7

SPECIAL PROOF STRATEGIES

CONDITIONAL PROOF and INDIRECT PROOF

7.1. Conditional Proof: The Hypothesis and CP Rules

When we take some proposition as a PREMISE in an argument, we are in effect treating it as if it were true for the course of the argument, even though we may not actually take it to be true ourselves.

We often take premises as givens, to see whether by assuming them certain conclusions can be validly derived. We can do this as a logical or analytical exercise, without necessarily assenting to the premises ourselves.

All valid arguments have a conditional 'if-fy' import: IF the premises are true, then the conclusion is also true. But this is not to claim that the premises are in fact true -- even though we grant them the special status of givens for the sake of the argument.

Sometimes it's useful to make assumptions without granting them the special albeit 'if-fy' status of premises. Sometimes it's useful to make a supposition for expressly hypothetical purposes -- just to see what follows logically from the supposition. We'll call this sort of merely hypothetical supposition a HYPOTHESIS in order to clearly distinguish its function from that of premises.

For our purposes of argument analysis, a HYPOTHESIS is even more 'if-fy' than a premise, because what we derive by use of a hypothesis will always be conditional in form.

For example, if by assuming some hypothesis P we derive some consequence Q, we will allow ourselves only the conditional conclusion:
P => Q

If we suppose P, then Q follows

We thus make the 'if-fy' status of hypotheses explicit by deriving from them only conditionals in which the hypothesis is the antecedent.

Because we employ hypotheses only to derive conditional conclusions, this argumentative strategy is called conditional proof.

The strategy of conditional proof has a multitude of uses. Its use and the constraints we must observe in using it are perhaps best explained in the context of some concrete and schematic examples.

Consider: Is God really omnipotent? By the customary definition of 'God' omnipotence, the power to do anything whatsoever, is thought to be one of God's essential properties. But suppose we don't wish to commit ourselves to this proposition; suppose we don't want simply to assume it. Suppose we're not at all sure that it makes sense to hold that God is omnipotent, but we wish to investigate the logical consequences of this proposition and our commonsense understanding of what it means. Now, commonsense would seem obviously to hold the following: (1) IF [O] God is omnipotent, then it's not the case that [Q] there is something that God cannot do. But (2) IF [S] God can create a stone that He cannot lift, then [Q] there is something that God cannot do (namely, lift the stone). And (3) IF it's not the case that [S] God can create a stone that He cannot lift, then likewise [Q] there is something that God cannot do (namely, create the stone.)

Now, we have not stipulated that God is omnipotent. We've only posited three commonsense premises, to wit:

1. O => -Q / Premise
2. S => Q / Premise
3. -S => Q / Premise

We grant these propositions the status of premises because they seem so obvious to commonsense. They do not commit us to accepting God's omnipotence. Granted these premises, we can investigate what follows logically from the further assumption of God's omnipotence: What now would follow IF, just for hypothetical purposes, we were to suppose that God is omnipotent? Here we can employ the hypothesis and conditional proof rules.
to see what follows IF God is omnipotent: Conditional Proof will always result in a conditional conclusion, in this case of the form:

IF God is omnipotent, then ________________
Given our original premises:

1  O => -Q / Premise
2  S => Q / Premise
3  -S => Q / Premise

We may introduce the supposition that God is omnipotent using the
HYPOTHESIS rule:

4  O / HYPOTHESIS

From the original premises and this hypothesis there follows a disturbing
contradiction—namely that there is something God cannot do and that it is
not the case that there is something God cannot do:

5  -Q / MP, 1, 4
6  -S / MT, 2, 5
7  S / MT, 3, 5
8  S & -S / CONJ, 6, 7

The CONDITIONAL PROOF rule now allows us to make the 'IF-fy' results of
supposing, purely hypothetically, that God is omnipotent explicit: We have
shown that IF we assume God is omnipotent, THEN a contradiction follows; so
we may conclude this conditional:

9  O => (S & -S) / CP 4-8

IF we suppose God to be omnipotent, given our original seemingly
commonsensical premises, we land ourselves in contradiction. This may give
us pause about the proposition that God is omnipotent. We will discuss the
strategic uses of Conditional Proof more later. For now, attend carefully
to the procedure and form of Conditional Proof:
Notice that the **ANTECEDENT** of the resultant conditional is our hypothesis and that the **CONSEQUENT** of the conditional is what we derived on the basis of our hypothesis.

Notice that in citing the Conditional Proof rule (CP) we also cite lines 4 through 8 (inclusive) and use the dash '-' to indicate that we have used all the lines between 4 and 8 (inclusive) to show that the given consequence follows from our hypothesis.

In our logical system we employ **hypotheses** only within the context of a conditional proof, in which our aim is to derive a **conditional** whose antecedent is the hypothesis in question.

Remember: A conditional proof must result in a **conditional**; hence its name!

A conditional proof may consist of several steps (as in the example above), but it is basically a three-phase procedure:

1. Assume as your **HYPOTHESIS** the **ANTECEDENT** of the target conditional.

2. Derive the **CONSEQUENT** of the conditional you wish to derive.

3. **DISCHARGE** your hypothesis by conditionalization:

   Form the desired conditional, citing the CP rule and the line number of its antecedent (your hypothesis) through the line number of its derived consequent.

   **THUS**, where you wish to derive, say, \( P \rightarrow Q \)

   1 \( P \) / **HYPOTHESIS**
   
   :  
   
   :  

   2 \( Q \) / [However derived]

   3 \( P \rightarrow Q \) / **CP** 1-2
Phase 3, what's called DISCHARGING the hypothesis, is absolutely crucial to the use of hypotheses in conditional proofs. By analogy, you might think of a hypothesis as a kind of loan, something you may use for purposes of deriving whatever you wish with it — but which must be paid back or discharged eventually (by incorporation into a conditional). Closing a conditional proof by conditionalization is analogous to closing a loan account by paying the loan back: you can make no use of the hypothesis itself or anything derived from it by itself, except the resultant conditional which you obtain by discharging your hypothesis. The resultant conditional — and the conditional alone — is yours to keep and use however you wish after you have discharged your hypothesis and closed your conditional proof. The conditional results of conditional proofs are often useful in the larger contexts of arguments or derivations in which premises are employed. For example, we can use conditional proof in showing that the following argument form is indeed valid. (This would be much more tedious using only our previous rules — you may wish to try deriving the following conclusion without using CP.)

\[
P \implies (Q \land R)
\]

\[
R \implies S
\]

\[
P \implies S
\]

Here are two derivations using conditional proof:

1. \(P \implies (Q \land R)\) / Premise
2. \(R \implies S\) / Premise
3. *** P / HYPOTHESIS
4. * Q & R / SIMPR 1, 3
5. * R / SIMPR 4
6. * S / SIMPR 2, 5

************

7. \(P \implies S\) / CP 3 - 6

The BERTIE and ARGUE programs will list conditional proofs (in response to the LIST command) with asterixes (*) setting the proof off from the body of the rest of the derivation as displayed above. This is to help you to remember when you are operating in the context of an 'open' conditional
proof whose hypothesis has not yet been discharged, and to show clearly when the proof has been 'closed.' You may then use the results of your conditional proof in further derivation, as in the alternative derivation below:

1  P => (Q & R) / Premise
2  R => S / Premise
3  ***> P / HYPOTHESIS
4  *  Q & R / MP 1, 3
5  *  R / SIMPR 4

P => R / CP 3 - 5
P => S / HS 3, 6

Conditional proof may also be employed within the context of another conditional proof.

Here it is imperative to remember to discharge hypotheses: where more than one hypothesis has been introduced, you must discharge the most recently introduced hypothesis before you can discharge any previous hypothesis!

For example, suppose we wanted to derive

(P v Q) => (-P => Q)

We can do so -- without using any premises -- by two applications of the conditional proof strategy, as follows:
DERIVE: \((P \lor Q) \Rightarrow (-P \Rightarrow Q)\)

Strategic note: Since our target conclusion is a conditional, we try assuming its antecedent as a hypothesis.

1 \(P \lor Q\) / HYPOTHESIS

But we can't get anything from this disjunction alone. Yet, if we had \(-P\) we could get \(Q\) by Disjunctive Syllogism. This is what the consequent of the target conditional says, in effect: \(-P \Rightarrow Q\)

Since the desired consequent is also a conditional, why not try CP:

2 \(-P\) / HYPOTHESIS

3 \(Q\) / DSR 1, 2

*************** Now we can discharge the most recent hypothesis:

4 \(-P \Rightarrow Q\) / CP 2 - 3

*************** Now we can discharge the first hypothesis:

5 \((P \lor Q) \Rightarrow (-P \Rightarrow Q)\) / CP 1 - 4

The hypothetical reasoning here simply states a logical truth:

IF we suppose \(P \lor Q\) :: Either P is true or else Q is

THEN: IF \(-P\) :: If it's not the case that P is true

THEN \(Q\) :: Then Q must be true

IF \((P \lor Q)\) THEN \((\text{IF } -P \text{ THEN } Q)\)

The 'IF' clauses in the schematic statement above are represented by the two hypotheses in the derivation: the embedding of one conditional proof within another is indicated by the embedding of one 'IF' clause within the above conditional. The structure of the formula to be derived in this case gives the clues for the structure of the conditional proof strategy.
7.2 Indirect Proof: The Reductio Rule

You know that if a sentence implies a contradiction, then the sentence is false. In fact, if it's true that P logically implies Q, and it's also true that Q is false, then P must be false. Or, schematically, the following is a valid argument form:

\[ P \implies (Q \land \neg Q) \]

\[-P\]

You can prove that the above argument form is valid -- that is, that its premise cannot be true while its conclusion is false -- by the standard truth-table method: For the conclusion \(-P\) to be false P must be true; if P is true, then the antecedent of the premise is true; since the consequent of the premise is a contradiction and, so, must be false, the premise itself is then false when the conclusion \(-P\) is false.

7.2. Indirect Proof: CP + The Reductio Rule

You know that if a sentence implies a contradiction, then the sentence is false. In fact, if it's true that P logically implies Q, and it's also true that Q is false, then P must be false. Or, schematically, the following is a valid argument form:

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truth-table method: For the conclusion \(-P\) to be false, \(P\) must be true; if \(P\) is true, then the antecedent of the premise is true; since the consequent of the premise is a contradiction and, so, must be false, the premise itself is then false when the conclusion \(-P\) is false.

Recall the example of Conditional Proof about God's supposed omnipotence in the last section:

\[
\begin{align*}
1. & \quad O \Rightarrow -Q & / \text{Premise} \\
2. & \quad S \Rightarrow Q & / \text{Premise} \\
3. & \quad -S \Rightarrow Q & / \text{Premise} \\
4. & \quad O & / \text{HYPOTHESIS} \\
5. & \quad -Q & / \text{MP, 1, 4} \\
6. & \quad -S & / \text{MT, 2, 5} \\
7. & \quad S & / \text{MT, 3, 5} \\
8. & \quad S \& -S & / \text{CONJ, 6, 7} \\
9. & \quad O \Rightarrow (S \& -S) & / \text{CP 4 - 8}
\end{align*}
\]

We could at this point employ the REDUCTIO rule to derive the further conclusion that God is not omnipotent:

\[
\begin{align*}
10. & \quad -O & / \text{RED, 9}
\end{align*}
\]

In fact, we could have set out with the strategy to derive that God is not omnipotent by use of Conditional Proof coupled with Reductio, whereby we show that, given certain accepted premises (1-3), IF we suppose, for purely hypothetical purposes, that God is omnipotent, THEN we land in contradiction. So, we conclude, God is not omnipotent.

This strategy is called INDIRECT PROOF because we derive the negation of a proposition (say, \(-O\)) by assuming the proposition (\(O\)) and showing that (in this case, on the basis of other acceptable premises) we are led into contradiction. Put more positively, we indirectly 'prove' a proposition (say, \(-O\)) by assuming its negation (\(-\neg O\)) as a hypothesis and showing that
this hypothesis leads us into contradiction: we 'prove' some proposition indirectly by assuming its contradictory.

The INDIRECT PROOF strategy employs CONDITIONAL PROOF (and HYPOTHESIS) coupled with REDUCTIO Ad Absurdum (reduction to contradiction), in outline, as follows:

Where you wish to derive some sentence, say: P

1. Assume as your HYPOTHESIS the negation of the sentence to be derived:
   
   1 -P / HYPOTHESIS

2. Derive a contradiction (say, Q & -Q):
   
   2 Q & -Q / [From whatever previous lines]

3. Apply the CONDITIONAL PROOF rule to obtain the resultant conditional:
   
   3 -P => (Q & -Q) / CP 1 - 2

4. Apply the REDUCTIO rule to obtain the negation of the Hypothesis:
   
   4 P / RED, 3

For example, by INDIRECT PROOF, you can prove: P V -P:

1 -(P V -P) / HYPOTHESIS

2 -P & P / DEM, 1

3 P & -P / COM, 2

4 -(P V -P) => (P & -P) / CP 1 - 3

5 P V -P / RED, 4
1. APPENDIX: ARGUMENT ANALYSIS

APPLICATIONS OF FORMAL LOGIC TO PHILOSOPHIC ANALYSIS

1.1. FOUR TASKS OF ARGUMENT ANALYSIS

This section contains brief illustrations of four basic tasks in the analysis of arguments, particularly philosophic arguments about normative issues. Normative issues are issues about the norms, rules or principles which are the logical bases for our arguments and judgments about what is right and what is wrong.

The following sections offer an extended illustration of these four basic tasks of argument analysis applied to a sample argument about social policy, the policy of preferential hiring. These illustrations demonstrate how the formal logic you are learning can be applied in the systematic reconstruction of arguments. Careful reconstruction of arguments is a prerequisite for their analysis; and it provides a guiding framework for philosophic analysis, the analysis of the normative principles that are crucial premises of our arguments about what is right and what is wrong. These illustrations will demonstrate how the formal reconstruction of arguments is relevant to philosophic analysis and will introduce you to basic tools and techniques of philosophic analysis, in particular, the concept of plausibility and the use of counter-examples in testing the plausibility of normative principles. In the sections that follow you will begin to see how formal validity can serve as a guide in the pursuit of truth and justified belief in philosophy.

1. ANALYZING LOGICAL FORM: CHECKING FOR VALIDITY

One way to get clear about a given line of reasoning, about the assumptions that must be made for a conclusion to follow logically, is to reconstruct the line of reasoning in the precise form of a deductively valid argument. This enables one to identify key assumptions, to see the precise logical form these must take in order to support a given conclusion, to uncover tacit assumptions (unstated premises needed to make the argument valid), and to single out inessential assumptions (logically superfluous premises, premises not needed for the argument to be valid).

In many of the arguments we encounter, especially in philosophic disputes, there will be tacit premises (assumptions that are not stated but that must be made explicit for the argument to be valid). Before we can adequately evaluate any argument, we must make any tacit premises (unstated but necessary assumptions) explicit. The first step in our reconstruction of any argument will be to analyze and represent the logical form of the
argument's stated premises and conclusion: this provides an important clue to the form and content that any tacit premises must have in order to make the argument valid.
Consider the following argument.

As the first step in reconstruction, I've represented the logical form of the argument symbolically to its right:

If you have ambition, you're in for a lot of frustration. And, you're apt to be miserable if you're often frustrated.

So, your life will take on purpose only if you're apt to be miserable.

Once the logical form of the argument is represented as above we can tell by inspection that the argument is invalid: There's no way to derive the conclusion from the given premises. But, in this case, it should also be clear from inspection of the argument's logical form that a premise of the following logical form would be sufficient to make the argument valid:

\[ P => H \]

\[ H => F \]

\[ F => M \]

\[ P => M \]

The derivation above shows that a premise of the form

\[ P => H \]

Your life will take on purpose only if you have ambition

is sufficient to make the argument valid.

Once you've depicted the logical form of an argument in some clear and explicit (symbolic) way, it's easier to inspect the argument for validity and to see what forms of additional premise(s) will make it valid. This is why our first task in reconstructing and analyzing an argument will be to analyze and represent (symbolize) its logical form; to check the argument for validity and unstated premises needed for validity.
This first step in the analysis of arguments (ANALYZING LOGICAL FORM and CHECKING FOR VALIDITY) will help us with one of the most crucial tasks of PHILOSOPHIC ANALYSIS, which is the second basic task of argument analysis: identifying underlying principles and 'hidden' or tacit (unstated but necessary) premises, illustrated below.

2. IDENTIFYING UNDERLYING PRINCIPLES AND TACIT PREMISES

To identify the GENERAL PRINCIPLE behind a position on some issue, it helps to try various formulations of the likely candidate principles as explicit premises in a valid deductive argument whose conclusion is the position in question. Consider, for illustration, the following widely held position:

[P] We owe it to future generations to control population size.

A likely principle behind this position [P] would be the following:

[0] We have an obligation to future generations to bequeath to them the means for the best possible life.

An additional tacit premise in support of [P] might be:

[0] => [P] IF we have an obligation to future generations to bequeath them the means for the best possible life, THEN we owe it to future generations to control population size.

We can see by inspection, or prove by the derivation on the right below, that the resulting argument form is valid:

<table>
<thead>
<tr>
<th></th>
<th>1 O</th>
<th>/ Premise [Unstated Principle]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>O  =&gt; P</td>
<td>2 O  =&gt; P / Premise [Unstated]</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>3 P  / 1, 2 MP</td>
</tr>
</tbody>
</table>
3. SORTING OUT AMBIGUITIES IN PRINCIPLES AND PREMISES

There is usually more than one way of construing the general principle behind a given position. We want to beware, in particular, of AMBIGUOUS principles, principles that can be taken to mean either or both of two different things. Once we've identified a likely candidate principle (like [O]) behind a position (like [P]) and reconstructed an argument for that position with the likely principle and other likely assumptions needed for validity stated as explicit premises (as above), we need to be aware of other possible formulations of the principle in question and, in particular, to beware of any AMBIGUITY that may be lurking in the principle we've chosen in reconstructing the argument. This is the next task in argument analysis: SORTING OUT AMBIGUITIES IN UNDERLYING PRINCIPLES AND PREMISES.

Can you think of at least two different things that the principle [O] (Premise 1 below) might be taken to mean?

1. 0
   We have an obligation to future generations to bequeath them the means for the best possible life

2. 0 => P
   IF so, THEN we owe it to future generations to control population size

3. P
   So, we owe it to future generations to control population size

Here are three things Premise 1 might be taken to mean:

[O1] We have an obligation to future generations to ensure that the future population, however it be constituted, has the means for the best possible life of which it is capable

[O2] We have an obligation to future generations to ensure that the future population is so constituted as to be capable of enjoying the best possible life

[O1 & O2] We have BOTH these obligations to future generations
WHICH of the above versions of the original principle \( O \) we incorporate into the argument for the call for population control \( P \) will make a decided difference to the SOUNDNESS or PLAUSIBILITY of the argument, as follows.

Using \( O_2 \) as a premise (or the conjunction of \( O_2 \) and \( O_1 \)), it's pretty easy to construct an argument with \( P \) as a conclusion that is VALID. But it is not so easy—it may well be impossible—to construct an argument using \( O_2 \) that BOTH (1) is VALID and (2) has all true or PLAUSIBLE premises, for this reason: \( O_2 \) (or any premise containing \( O_2 \)) is objectionable/IMPLAUSIBLE because it has objectionable logical consequences: Not only can \( O_2 \) be used to support a policy controlling the SIZE of any future population, but it can also be used to support a number of other quite objectionable policies, such as policies of genetic manipulation, selective breeding and sterilization of humans, consumer preference conditioning, and the like.

On the other hand, while \( O_1 \) by itself may be perfectly plausible (and have no untoward consequences if adopted), \( O_1 \) by itself (without additional premises) will not constitute a VALID argument in support of the control of population SIZE \( P \). It will be difficult—if not impossible—to use \( O_1 \) along with additional PLAUSIBLE premises to construct a VALID argument in support of the control of population SIZE \( P \).

TRY IT! Try to construct an argument for \( P \) using \( O_1 \) (but not \( O_2 \)) that BOTH (1) is VALID and (2) all of whose premises are true or PLAUSIBLE.

How, actually can we assess the PLAUSIBILITY or [TRUTH] of the premises (normative or factual) of philosophic arguments. This is the next crucial task in argument analysis, and a basic task in all philosophic analysis: testing the truth or plausibility of general principles and premises. An important tool for this task is the COUNTER-EXAMPLE, illustrated in the next sections.
Scientists refute empirical hypotheses by citing counter-evidence—instances in which the hypotheses are false. Somewhat similarly, philosophers standardly refute putative definitions of important concepts (justice, knowledge, etc.), as well as general normative principles, by citing counter-examples. Arriving at a satisfactory definition by a series of successive formulations and counter-examples is the most characteristically philosophical of reasoning techniques. Attributed to Socrates, it is an important feature of philosophic analysis.

To illustrate the philosophic use of counter-examples, consider the distinction commonly cited by medical personnel between killing a patient and letting him die. Some try to explain it thus:

To let someone die, as opposed to killing him, is to be in a position to save his life but deliberately to refrain from doing so.

This formulation may be shown to be incorrect by the following counter-example. Suppose someone smothers you by pressing a pillow to your face for a period of several minutes. Once the pillow is in place, he is in a position to do something that would save your life—viz., lift the pillow—but he deliberately refrains from doing so. Yet, contrary to the proposed explanation, it would be natural to say that such a person killed you, not that he merely let you die.

Sometimes it is said that the distinction between killing and letting die is an instance of the 'active/passive' distinction. Putative counter-examples to this proposal are ready to hand. Removing a respirator from a critically ill patient is surely 'active' rather than 'passive.' Yet such an action could well be described as 'letting die' rather than 'killing.' Or suppose an anesthesiologist deliberately fails to make the necessary adjustments in certain life-support and monitoring systems attached to a patient undergoing surgery, thereby deliberately ensuring that the patient dies. Although such failures to act are 'passive,' it would be natural to accuse the anesthesiologist not merely of letting the patient die but of killing him.
The four basic tasks illustrated in the analysis of the sample argument below are:

1. The Reconstruction of an Argument in VALID Deductive Form.

2. The Explication or Revision of the UNDERLYING NORMATIVE PRINCIPLES that are Crucial Premises of the Argument.

3. The Analysis of any AMBIGUITY in the Crucial Premises of the Argument.

4. Testing the PLAUIBILITY or Truth of the Crucial Premises of the Argument Against Putative COUNTER-EXAMPLE

These tasks are not exhaustive of what-all is involved in the analysis of arguments, but they are basic and important. Tasks (3)-(4) often require a reformulation of an argument, which in turn requires the repetition of tasks (1) and (2) in order to preserve the validity of the argument and the plausibility of its crucial premises.

The analysis of arguments requires the coordination of logical analysis and philosophic analysis.

Logical analysis primarily concerns the logical form the premises must take in order to support conclusion and maintain validity.

One function of philosophic analysis is to test the plausibility of the normative principles underlying our judgments and arguments; thereby, to articulate and develop the various issues underlying our arguments by adducing pertinent objections to the crucial premises of an argument (counter-examples or problem-cases), possible replies to those objections, possible rejoinders to the replies. The procedure, in rough outline, is: (1) to try to capture in explicit premises the 'intuitions' (and tacit principles) to which we appeal in our particular judgments by (2) abstracting those principles from clear 'paradigm' cases (cases, precedents or common practice where we are especially confident in our judgments); then (3) to adduce the logical consequences of our principles and test our premises against conflicting intuitions about putative counter-examples and (4) to reformulate these premises in order to overcome refutation by counter-example, to adjudicate or explain away problem-cases; and, so, (5) to repeat this process of careful formulation, testing and reformulation . . . until, ideally, we have made our arguments clearly valid and rendered our principles (a) perfectly explicit, (b) logically consistent among themselves, (c) evidently immune to further refutation by counter-example and (d) sufficient for adjudicating or explaining away problem-cases.
The analysis of the sample argument that follows is an illustration of logical and philosophical analysis coordinated in the service of the four basic tasks outlined above.
I.2. A SAMPLE ARGUMENT: PREFERENTIAL HIRING

Justice surely demands that someone unjustly deprived of something to which he had a right be compensated. Of course, normally, it's wrong to discriminate among job applicants on the basis of racial or sexual characteristics. But there are exceptions (as to any general rule). Blacks and women, for example, have a right to equal opportunity for advancement in education and employment. Yet both have been unjustly discriminated against in these areas. Not only does justice require that victims of such discrimination and right-violation be compensated, but by hiring blacks and women in preference to white males we do not thereby discriminate in a morally objectionable way. We rather compensate the victims of job discrimination as justice demands.

The foregoing is an argument from alleged requirements of justice. People often appeal to considerations of compensatory justice in defense of preferential hiring. One point we want to make as explicit and precise as possible by constraining the argument in valid form is exactly what justice is supposed to require.

Reconstructing this line of argument in deductively valid form will not produce a single argument or principle of justice; many valid reconstructions are possible employing any one of several possible formulations of the alleged requirements of justice.

But by constraining the argument in deductively valid form we force ourselves to specify some principle explicitly connected to the policy in question. This begins the dialectical program of successive reconstructions of the principle to take account of objections to it, and successive reconstructions of the argument providing the logical connection between the principle and the policy of preferential hiring.
I.3. AN INITIAL RECONSTRUCTION

Any simple, plausible formulation of the argument will do for starters. Whatever formulation we begin with, it can be made progressively more precise under the fire of counter-examples and within the constraint of deductively valid form.

Consider the following generalized reconstruction of the argument. Replacing 'Xs' by 'blacks' or 'women' and 'Ys' by 'white males' will render the intended conclusion of the original argument. In brackets I will assign a variable letter to each statement in the argument so that its sentential-logical form can be readily depicted.

(A) 1. If [0] Xs have been unjustly deprived of something (e.g., equal employment opportunity) to which they had a right, then [J] justice demands that Xs be compensated

(2) [0] Xs have been unjustly deprived of something (i.e., equal employment opportunity) to which they had a right

(3) [M] Preferential hiring of Xs over Ys is on balance morally permissible if [S] preferential hiring of Xs serves to compensate them as victims of job discrimination

Therefore: [P] Justice demands preferential hiring of Xs over Ys

There are at least three problems with the argument as stated that come out in the course of reconstruction. First, whatever quarrel one might have with the accuracy of the initial reconstruction (A) (anyone may try his own), one can see that the logical form of the argument is, in any case, not manifestly valid: the conclusion does not follow. There are important unstated assumptions. Sentential logic will suffice, for starters, to find and fill the gross logical gaps in the argument. Second, the argument is rife with ambiguity. Third, once one has sorted out some of the ambiguity, it turns out to be remarkably difficult to reconstruct the argument so that it both is valid and has all true or plausible premises, premises at least immune to obvious counter-example. From these lessons of reconstruction philosophic lessons are also to be learned. I will deal with them in turn.
I.4. **THE FIRST PROBLEM: INVALIDITY AND UNSTATED ASSUMPTIONS.**

The argument as it stands is invalid. This is easily seen by inspection of its logical form, abstracted symbolically:

\[
\begin{align*}
(A') & \quad (1') \text{ If } O \text{ then } J \\
(2') & \quad O \\
(3') & \quad M \text{ if } S \\

(A'') & \quad (1'') \text{ If } O \text{ then } J \\
(2'') & \quad O \\
(3'') & \quad S \Rightarrow M
\end{align*}
\]

Therefore: \( P \)

One advantage of being able to depict the logical form of an argument in abbreviated notation is analogous to the advantage of having an x-ray device: it allows us to look at the bare skeletal structure apparently supporting the conclusion, to detect distinctly structural flaws underneath the enveloping verbal flesh and musculature. From our x-ray of argument (A) it's clear that the conclusion \( P \) is in no way explicitly connected to any of the stated premises. Nor is any explicit connection between premises (1) and (2) and premise (3) yet apparent: from (1) and (2) we can conclude \( J \); but what connections are presumed to exist among \( J \), premise (3) and the conclusion \( P \)? These connections, in some form, must be made explicit, so as to make explicit use of the stated premises of the argument and also render it valid. Here we need to consider the content as well as the form of the argument.

Symbolic logical form and validity serve, respectively, as clues and guiding constraints in the search for tacit premises; but they are not sufficient grounds for generating sensible additional premises, or for deciding among competing premises where any number might make an argument valid. It is necessary to introduce other guiding constraints in the reconstruction of an argument. Validity remains a powerful minimal condition of the enterprise nonetheless: insisting on manifest validity keeps us honest about what exactly is or must be assumed and exactly what follows from what. In this case it requires us to produce some further assumptions on which the conclusion tacitly rests. Once laid out explicitly, these assumptions are open to question; and they may force us to change the shape of the argument or even abandon it. One obvious tacit assumption is:

\[ [R] \text{ Ys (white males) have received undue preferential treatment over Xs (blacks or women) in hiring practice.} \]

Without assuming at least some such condition it would make no sense to assert that it is morally permissible to compensate Xs at the expense of Ys. Moreover, without the addition of some such condition as R to premise (3), this premise is open to obvious counter-example and, so, is false.
That is, the truth of S is not always a sufficient condition for the truth of M, for surely the following interpretation of premise (3) is false:

If preferential hiring of Mexican-Americans (X's) compensates them . . . then preferential hiring of Mexican-Americans (X's) over blacks (Y's) is morally permissable.
The logical form of our further reconstruction now looks like this:

(A')

(1') If O then J
(2') O

Therefore: J

(3')* If S and R, then M
(4') R

Therefore: P

Explicitly assuming the condition S as an additional premise

(5') S

we may draw the further intermediary conclusion

Therefore: M

from (3'), (4') and (5'). We have now gotten so far as to conclude that [J] justice demands compensation and that [M] preferential hiring is on balance a morally permissible way to compensate. We have yet explicitly to complete the connection to the ultimate conclusion [P] to the effect that justice in turn demands preferential hiring as the mode of compensation. Any of the following additional premises connecting the conclusion to the foregoing results would render the argument valid:

(6') (a) If J and M, then P
(b) If J then P
(c) If M then P

Both (b) and (c) are objectionable, on similar grounds. That [J] justice demands compensation is not sufficient grounds for asserting that [P] justice demands that compensation take a particular form, namely, preferential hiring. That [M] preferential hiring (or anything else, say, singing in the shower) is on balance morally permissible is not sufficient
grounds for holding that [P] justice requires it. So, (b) and (c) are implausible or false. Moreover, their addition to the argument, while making it valid, would be to cast adrift other presumably relevant premises as logically superfluous.

(6') (a) seems the best of the three alternatives. Choosing it has been an exercise in the reconstruction of a normative principle, an attempt to specify sufficient grounds on which justice would require and, so, justify a particular policy. The reconstruction of principles and the reconstruction of arguments go hand-in-hand in the moral-philosophic forum, because general normative principles are always among the (stated or tacit) assumptions of a moral-philosophic argument. Hence, the reconstruction of arguments can play a heuristic role in the explication and analysis of the normative principles underlying our reasonings.

Once the argument, with its tacit underlying principles, has been reconstructed in valid form, we are at least assured that if the premises are acceptable, so must be the conclusion. But are they?
I.5. THE SECOND AND THIRD PROBLEMS: AMBIGUITY AND VULNERABILITY TO COUNTER-EXAMPLE

For purposes of illustration, I will focus on the first premise of the argument only. Where ambiguities are discerned or counter-examples found premises must be reformulated, jettisoned, or added. Premise revision involves further, alternative reconstructions of the argument to preserve its validity. This may be difficult, but to that extent instructive.

Consider: The principle of compensatory justice to which argument (A) appeals, premise (A-1), can of course be applied quite generally. So, the original line of reasoning and policy based on this principle can be applied quite generally. How generally? To whoever can be counted among the X's. Who might be counted among the X's? On grounds of premise (A-1), anyone who has ever been deprived of something to which she had a right, say, 'equal' employment opportunity. (He could well be a highly competent white bank executive from a wealthy family who has been denied 'equal' consideration for jobs many times because of his religious or political views.)

There is a serious ambiguity in premise (A-1). How are we to interpret the demand for compensation of Xs? There are at least two possibilities where Xs are members of some identifiable group:

A distributive interpretation: a person is to be compensated if he is an X (black, woman, atheist . . .) and he has himself been unjustly discriminated against . . .

A collective interpretation: a person is to be compensated if he is an X and Xs (blacks, women, Irish-Catholics, communists . . .) have in general been unjustly discriminated against . . .

The collective interpretation of the demand for compensation for Xs does not require that any given X have been unjustly discriminated against, but rather that other Xs as a group have been unjustly deprived: under this interpretation Xs as such are to be compensated.

A wealthy Jewish or Irish Catholic businessman who had never himself been deprived of anything could qualify under the collective interpretation for compensation where X's were Irish Catholics or Jews. A wealthy white male who had himself been unjustly discriminated against because of his atheism could qualify for compensation under a distributive interpretation. Presumably the purpose of the preferential hiring policy in question is neither to compensate wealthy people nor to compensate just anybody for any unjust deprivation she may have suffered. Under either interpretation the
general demand for compensation could be applied to practically anybody; whereas the specific demand for compensatory preferential hiring is on behalf of certain presently and unfairly disadvantaged groups, namely, certain racial minorities and women.

We need to specify the conditions of premise (A-1) so as to justify compensatory treatment in the form of preferential hiring for all and only those whom the policy is meant to compensate. For a sense of these two possibly conflicting constraints—the justice and purpose of the policy in question—we need appeal to our intuitions, our tacit conceptions of both, as yet imperfectly captured in premise (A-1) (and yet to be tested against limiting counter-examples).

We need first to specify more precisely the grounds on which justice demands compensation. We will consider five candidate criteria. These will be sufficient to delineate some of the major ambiguities of our original principle of compensatory justice.

(a) Membership in a group whose members have been widely and unjustly discriminated against and thereby deprived of something to which they had a right.

This criterion would qualify blacks and women, but would it qualify all and only those actually deserving compensatory treatment? As already suggested, it would not. The criterion is too inclusive. Who would not qualify for compensation? Consider the cases of well-to-do Catholics, Protestants or Jews who have never been unjustly deprived of anything but who are members of groups which have (somewhere) suffered great injustice. Does justice require that they be compensated?

It is evidently not mere membership in some identifiable class of persons many of whose members have suffered injustice at some time in the past that recommends a given member for compensation. Yet the policy we are seeking to justify on the basis of the requirements of justice designates its beneficiaries according to racial or sexual characteristics.

Perhaps it is rather the likelihood of having herself suffered injustice, given effective, recent and widespread prejudice and discrimination against Xs as such, that recommends any given X for compensatory treatment.

(b) Likelihood of having suffered unjust deprivation oneself because of membership in a group whose members have been recently and widely discriminated against.
This criterion would include blacks and women and exclude consideration of white Catholics or Jews for compensation. But against this suggestion stands the case of any well-to-do black woman who has never been deprived of anything. Does justice demand that a person who has never suffered any injustice be compensated? What is it for which she would be compensated? Analogously, should courts award compensatory damages to a person on the likelihood that he suffered defamation of character when in fact he hasn't—or because it is established that he was actually wronged and harmed in a way penalizable by law? The latter case is more problem-case than counter-example. Consider then:

(c) Having in fact been unjustly discriminated against and thereby deprived of something to which one had a right.

Whatever justice demands in the way of compensation, it would seem that justice demands it only for persons who have themselves been unjustly harmed or deprived, not for persons who haven't in fact been wronged but who happen to have certain characteristics (e.g., race) in common with others who have.

But while actually having been wronged oneself may be a necessary condition for claiming compensation on grounds of justice, this requirement of justice would not justify preferential hiring of blacks or women as such. On the other hand, a principle stipulating criterion (c) as a sufficient condition for compensatory treatment would not justify preferential hiring of all and only those (certain minority groups and women) whom the policy seems intended to benefit. Such a principle would justify compensatory treatment of a person irrespective of her race, sex or socio-economic status. A policy of preferential hiring based on such a criterion, designating beneficiaries according to their personal histories rather than race or sex, would seem impracticable.

There are further ambiguities in the position regarding the qualifications for compensatory treatment. Some of these are made explicit in the multiple-choice reconstruction, argument (E), below.

One instructive difficulty with the line of argument under consideration has clearly emerged in our reconstructive effort: the problem of fitting (logically connecting) the desired policy (compensatory preferential hiring of certain racial minorities and women) to the requirements of justice that seemed initially to demand such a policy.

The grounds on which justice might demand and distribute compensation are not obviously the grounds on which the policy in question would distribute preferential treatment.
A fairly superficial examination of the ambiguities of our initial formulation of what justice requires has produced a fair array of questions. Never mind objections to other premises for now. We find that the alleged requirements of justice are themselves clearly questionable. We can map out the issues and strategic options confronting us by reformulating the original deductive argument to take account of the ambiguities and objections raised. We might consider this endeavor a kind of game, a game of argumentative reconstruction and counter-example.

Attempting to reconstruct the argument in expressly valid form, while taking account of ambiguities and counter-examples, makes the philosophic problem of fitting the desired policy to the demands of justice more acute. While this reconstructive exercise may well make the argument less persuasive, the exercise is nonetheless instructive regarding some of the philosophic issues underlying the policy in question. This is one educational objective of the task of reconstructing arguments in deductively valid form, and one rationale for the CAI programs in argument construction and reconstruction, which enforce the constraint of validity while allowing the student to view and manipulate both the logical form and the content of an argument side-by-side.
I.6. THE GAME OF ARGUMENT RECONSTRUCTION AND COUNTER-EXAMPLE

The object of this game is to construct an evidently valid and plausible argument supporting the policy of preferential hiring in question from the requirements of justice.

The first phase of the game is to construct a deductive argument whose conclusion is the position on the policy in question by selecting those premises required to make manifest the validity of the argument.

The second phase of the game is to test the plausibility of the selected premises, by explicating ambiguities and adducing putative counter-examples or problem-cases. A given premise remains plausible only so far as it is at least immune to obvious counter-examples.

Our two concerns are the logical connection between premises and conclusion, a matter of logical form, and the plausibility of the premises, a function of their actual content and the argumentative context. As we shall see, these concerns are not unrelated in the game of argumentative strategy that follows. It is often very difficult to satisfy both constraints at once. In this respect deductive validity, a matter of logical form, is indeed related to the pursuit of truth. Where the task is apparently impossible, we have good reason to abandon an argument and seek alternative strategies.

Consider now a multiple-choice reformulation of the original argument from justice, argument (B), below. Premises may be constructed by selecting one (or more) of the lettered options (and providing suitable logical connectives). The options (a)-(e) given under each premise are intended to take account of the ambiguities already detected in the principle of justice employed in the original argument (A). At this stage of reconstruction it is useful to have recourse to quantificational logic. The logical form of each premise is symbolized to its right so that validity can be readily assessed by inspection or derivation.
(1) IF a person

(a) is a member of a group whose members have been widely and unjustly discriminated against and thereby deprived of something to which they had a right

(b) is likely himself to have been unjustly discriminated against and thereby deprived of something to which he had a right

(c) has in fact himself been unjustly deprived of something to which he had a right

(d) has himself suffered harm or serious disadvantage as a result of having been unjustly discriminated against

(e) presently is suffering harm or serious disadvantage as a result of having been unjustly discriminated against

THEN justice demands that person be compensated.

(2) ALL women

(a) are members of some group whose members have been widely and unjustly discriminated against and thereby deprived . . .

(b) are likely to have been unjustly deprived

(c) have in fact been unjustly discriminated against and thereby deprived . . .

(d) have suffered harm or serious disadvantage as a result of having been unjustly discriminated against

(e) are presently suffering harm or serious disadvantage as a result of having been unjustly discriminated against

THEREFORE: Justice demands that women be compensated
We are considering now just the first stage of the original argument, to the first intermediate conclusion that justice demands compensation for, say, women. If we can't construct a valid and plausible argument to this intermediate conclusion, we can hardly justify the policy in question on the basis of premise (A-1).

There are at least five different sets of premises, consisting of alternative versions of premises (A-1) and (A-2), that will each provide a valid argument to the first conclusion. They are: (A-1a), (A-2a); (A-1b), (A-2b); (A-1c), (A-2c); (A-1d), (A-2d); (A-1e), (A-2e). The list could be lengthened by including sets of compound premises, such as: (A-1e or d), (A-2e). However, nothing would be gained thereby. Keeping in mind the questions raised previously regarding criteria (a) through (e), consider the arguments resulting from these sets of premises. Which of the optional arguments is most evidently sound? As it happens, none is sound (i.e., both is valid and has all its premises immune to obvious counter-example).

Conditions (1c), (1d) and (1e) seem to provide the most plausible grounds for justice to demand compensation, namely: that a person herself has suffered an injustice, or harm as a result of injustice, in order that she actually have something to be compensated for. The difference between (1c) and (1d) or (1e) concerns whether we wish to compensate persons who were in fact treated unjustly but who suffered no harm or disadvantage on that account. Is it the mere fact of injustice or rather the resultant harm that demands compensation? If the latter, is it present or past harm?

By contrast, condition (1a) seems an implausible basis for justice to demand anything, let alone compensation. Mere membership in a group does not suffice to establish that a person suffered any injustice. If a person suffered no injustice, what is she to be compensated for? Some further condition seems necessary to establish an evidentiary connection between group membership and injustice. (A-1b) makes this condition explicit, asserting a probable connection. The implausibility of (A-1a) can be shown by an appeal to the untoward consequence that would result from its general application: cases where justice demanded compensation but where there was no victim of injustice to be compensated. The same objection could be lodged against (A-1b), with counter-examples.

Whereas premises (A-1c), (A-1d) and (A-1e) provide plausible grounds for justice to demand compensation, the factual assumptions respectively required to entail the desired conclusion are very likely false. On the other hand the factual assumptions (A-2a) and (A-2b) while true, seem to provide insufficient grounds for justice to demand compensation, as shown by counter-examples to (A-1a) and (A-1b) above.

If we revise this stage of the argument in order to demand compensation, discriminately, only for those (blacks, women or Xs) who
qualify on conditions (lc), (ld) or (le), we would vitiate the validity of the argument to the final conclusion (which calls for compensation for ALL blacks or women as such). If we revise the final conclusion, to preserve validity, and thereby discriminately demand preferential hiring for only those who qualify on conditions (lc), (ld), or (le), we cannot justify compensatory hiring of blacks and women as such. We would then be arguing for a very different, and probably impracticable, policy. The difficulty is to provide a manifestly valid argument with plausible premises to the specified conclusion. This difficulty would be compounded if we were to examine other premises in the argument. More subtle refinement or reformulation of the premises will not eliminate the basic difficulty.

We have reached an apparent impasse in our game of argument reconstruction. There are some lessons of argumentative strategy to be gained at this impasse.

It is not obvious that an appeal to the requirements of compensatory justice is, after all, viable in behalf of preferential hiring of blacks, women or other minority members as such. This has been shown by making certain alternative connections between the policy and the presumed requirements of justice explicit in valid deductive form. Making the supposed logical connections between policies and principles explicit in this way forces us to clarify our normative assumptions and provides us with clear departure points for further dialectical analysis of policy issues and matters of principle.

We also get clearer on exactly what position we want or need to hold on the policy in question. When we reach an impasse such as we have in our appeal to compensatory justice, we would be well-advised at least to consider other lines of argument. Perhaps mere compatibility with the requirements of justice and an appeal to 'social utility' would suffice to support the policy in question. Perhaps the correction of certain social ills (disproportionate poverty, unfair competitive disadvantage or unemployment among certain minorities) or the provision of certain social benefits (positive role models and career incentives) is the proper aim of the policy in question. Perhaps these ends, if achievable with negligible infractions of justice, would justify preferential hiring of blacks or women as such. Perhaps. This, in any case, is a strategy different from the one with which we began. To get clear on exactly what would have to be true in the way of both normative and factual assumptions in order to support preferential hiring along these lines we would do well to make those assumptions explicit within the frame of a valid deductive argument.

And the game of argument reconstruction and counter-example would resume. A game that is, after all, a serious form of philosophic inquiry within strict logical constraints.
II. SUMMARY: RULES OF INference AND REPLACEMENT

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SENTENTIAL RULES OF INFERENCE

SENTENTIAL RULES OF REPLACEMENT

CONDITIONAL PROOF RULE & INDIRECT PROOF STRATEGY

QUANTIFICATIONAL RULE SCHEMAS

RESTRICTIONS ON THE USE OF UNIVERSAL GENERALIZATION

RESTRICTIONS ON THE USE OF EXISTENTIAL INSTANTIATION
II.1. SENTENTIAL LOGIC: RULES OF INFERENCE

Premise (PREM or P): A premise may be introduced on any line of a derivation -- except within a Conditional Proof.

Excluded-Middle Introduction (E-MI or EMI): On any line of a derivation one may introduce a sentence of the form: P V ~P.

<table>
<thead>
<tr>
<th>Conjunction (CONJ)</th>
<th>Addition (ADD)</th>
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<tbody>
<tr>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Q</td>
<td>P V Q</td>
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<tr>
<td>P &amp; Q</td>
<td>P V Q</td>
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<td>Simplification (SIMPL)</td>
<td>Simplification (SIMPR)</td>
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<td>P &amp; Q</td>
<td>P &amp; Q</td>
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<td></td>
<td>P</td>
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<td>Disjunctive Syllogism (DSL)</td>
<td>Disjunctive Syllogism (DSR)</td>
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<tr>
<td>P V Q</td>
<td>P V Q</td>
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<td>~Q</td>
<td>~P</td>
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<tr>
<td>P</td>
<td>Q</td>
</tr>
<tr>
<td>Modus Ponens (MP)</td>
<td>Modus Tollens (MT)</td>
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<tr>
<td>P =&gt; Q</td>
<td>P =&gt; Q</td>
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<td>P</td>
<td>~Q</td>
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<td></td>
<td>~P</td>
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<td>Q</td>
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<td>Constructive Dilemma (CDI)</td>
<td>Constructive Dilemma (CDII)</td>
</tr>
<tr>
<td>P V Q</td>
<td>P V Q</td>
</tr>
<tr>
<td>P =&gt; R</td>
<td>P =&gt; R</td>
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<td>Q =&gt; R</td>
<td>Q =&gt; S</td>
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<td></td>
<td>R V S</td>
</tr>
<tr>
<td>Hypothetical Syllogism (HS)</td>
<td>Reductio Ad Absurdum (RED)</td>
</tr>
<tr>
<td>P =&gt; Q</td>
<td>P =&gt; (Q &amp; ~Q)</td>
</tr>
<tr>
<td>Q =&gt; R</td>
<td></td>
</tr>
<tr>
<td>P =&gt; R</td>
<td>~P</td>
</tr>
</tbody>
</table>
II.2. SENTENTIAL LOGIC: RULES OF REPLACEMENT

Note: The dual colon ' :: ' indicates that the respective formulae are logically equivalent -- i.e., that either formula may be replaced by the other in any line of a derivation, OR that either may be derived from the other in a derivation.

Double Negation (DN):
\[ P :: - - P \]

Commutation (COM):
\[ P & Q :: Q & P \]
\[ P V Q :: Q V P \]
\[ P \leftrightarrow Q :: Q \leftrightarrow P \]

Transposition (TRANS):
\[ P \rightarrow Q :: - Q \rightarrow - P \]

Equivalence (EQUIV):
\[ P \leftrightarrow Q :: (P \rightarrow Q) & (Q \rightarrow P) \]
\[ P \leftrightarrow Q :: (P & Q) \lor (-P & -Q) \]

Implication (IMPL):
\[ P \rightarrow Q :: - P V Q \]

De Morgan (DEM):
\[ - P & - Q :: - (P V Q) \]
\[ - P V - Q :: - (P & Q) \]
\[ P & Q :: -( - P V - Q) \]
\[ P V Q :: - -(P & -Q) \]

Exportation (EXP):
\[ (P & Q) \rightarrow R :: P \rightarrow (Q \rightarrow R) \]

Tautology (TAUT):
\[ P :: P V P \]
\[ P :: P & P \]
II.3. **CONDITIONAL PROOF RULE & INDIRECT PROOF STRATEGY**

**CONDITIONAL PROOF STRATEGY: HYPOTHESIS + CP**

With conditional proof your aim is to derive a conditional, say: \( P \rightarrow Q \)

1. **Assume the ANTECEDENT** of the conditional as a HYPOTHESIS.

   [Be sure to enter any premises BEFORE you enter your hypothesis!]

2. Derive the CONSEQUENT of the conditional you wish to derive.

3. **Discharge** your hypothesis by conditionalization:

   Form the desired conditional (\( P \rightarrow Q \)), citing the CP rule and the line of your hypothesis (\( P \)) through the line of the consequent (\( Q \)):

   \[
   \text{THUS: } \begin{align*}
   1 & P & / \ HYPOTHESIS \\
   & & \\
   & 2 & Q & / \ [\text{Cite rules/lines by which derived}] \\
   & & \\
   & 3 & P \rightarrow Q & / CP \ 1-2 \ [\text{Note use of dash '->'}]
   \end{align*}
   \]

**INDIRECT PROOF STRATEGY: CP + REDUCTIO**

Where you wish to derive some sentence, say: \( P \)

1. Assume as a HYPOTHESIS the NEGATION of the sentence to be derived:

   \[
   \text{1 } -P & / \ HYPOTHESIS \\
   \text{2 } Q & -Q & / \ [\text{Rule/line cit.}] \\
   \text{3 } -P \rightarrow (Q & -Q) & / CP \ 1-2 \\
   \text{4 } P & / \ REDUCTIO, \ 3
   \]

   \[
   \text{THUS: } \begin{align*}
   1 & -P & / \ HYPOTHESIS \\
   & & \\
   & 2 & Q & -Q & / \ [\text{Cite rules/lines by which derived}] \\
   & & \\
   & 3 & -P \rightarrow (Q & -Q) & / CP \ 1-2 \\
   & & \\
   & 4 & P & / \ REDUCTIO, \ 3
   \end{align*}
   \]
II.4. QUANTIFICATIONAL RULE SCHEMA SUMMARY

\[ (x)Px \rightarrow (x)(Px \Rightarrow Qx) \]
\[ Pa \rightarrow Pa \Rightarrow Qa \]
\[ (x)Px \rightarrow (x)(Px \Rightarrow Qx) \]
\[ Pt \rightarrow Pt \Rightarrow Qt \]

UG
--

ONLY from pseudo-names NOT occurring in lines obtained by EI:

\[ Pt \rightarrow Pt \Rightarrow Qt \]
\[ (x)Px \rightarrow (x)(Px \Rightarrow Qx) \]

QN
--

\[ (x)Px :: - (Ex)-Px \]
\[ (x)-Px :: - (Ex)Px \]
\[ -(x)Px :: (Ex)-Px \]
\[ -(x)-Px :: (Ex)Px \]

\[ (x)(Px \& Qx) :: -(Ex)-(Px \& Qx) \]
\[ (x)-(Px \& Qx) :: -(Ex)(Px \& Qx) \]
\[ -(x)(Px \& Qx) :: -(Ex)-(Px \& Qx) \]
\[ -(x)-(Px \& Qx) :: -(Ex)(Px \& Qx) \]

Steps:
1. Change quantifier: \( (x)-Px \rightarrow (Ex)-Px \)
2. Negate quantifier: \( -(Ex)Px \)
3. Negate after quantifier: \( -(Ex)--Px \)
4. Drop any double negation: \( -(Ex)Px \)

EG
--

From constants:

\[ Pa \rightarrow Pa \& Qa \]
\[ (Ex)Px \rightarrow (Ex)(Px \& Qx) \]

or pseudo-names:

\[ Pt \rightarrow Pt \& Qt \]
\[ (Ex)Px \rightarrow (Ex)(Px \& Qx) \]

EI
--

ONLY to pseudo-names not used in a previous line:

\[ (Ex)Px \rightarrow -(Pu \& Qu) \]
II.5. RESTRICTIONS ON UNIVERSAL GENERALIZATION (UG)

1. You may NOT universally generalize from constants

You may universally generalize only from pseudo-names, never from constants. Generalizing from an individual case is obviously INVALID by commonsense, and proven to be so by the following argument schema and interpretation, whereby the premise (1) is obviously true but the conclusion (2) is obviously false:

Let: \( P_x = x \) was president of the U.S.; \( a = \) Abe Lincoln

1. \( P_a \) Abe Lincoln was president of the U.S.

INVALID:

2. \( (x)P_x \) Everything was president of the U.S.

2. You may NOT UG from pseudo-names that occur in a line obtained by EI

When a given pseudo-name (say, \( t \)) ever occurs in a line that is obtained by EI (as \( t \) does in line 3 below), you may not universally generalize from it -- even if the pseudo-name was itself originally obtained by UI (as \( t \) was at line 2 below). The following derivation and interpretation shows that this move can lead from truth (Everyone has a parent) to falsity (Someone is everyone's parent). Therefore, it is INVALID:

Let: The Domain = people; \( P_{xy} = x \) is a parent of \( y \)

1. \( (x)(Ey)P_{yx} \) Everyone has a parent

2. \( (Ey)P_{yt} \) / UI, 1

3. \( P_u \) / EI, 2 [Cannot UG from \( t \)]

INVALID >>>

4. \( (x)P_{ux} \) Some person \( u \) is everyone's parent

5. \( (Ey)(x)P_{yx} \) / EG, 4 Someone's everyone's p
II.6. RESTRICTIONS ON EXISTENTIAL INSTANTIATION (EI)

1. You may NOT existentially instantiate to constants

You may existentially instantiate only to pseudo-names, never to constants: the following argument schema, existentially instantiating to a constant, is shown to be INVALID by the following interpretation, whereby the premise (1) is obviously true but the conclusion (2) is obviously false.

Let: \( P_x = x \) is president of the U.S.; \( d = \) Princess Diana

\[
\begin{align*}
1 & \quad (\exists x)P_x \quad \text{Someone is president of the U.S.} \\
\text{INVALID:} & \quad 2 \quad P_d \quad \text{Princess Diana is president of the U.S.}
\end{align*}
\]

2. You may NOT EI to a pseudo-name already introduced in the derivation

When a given pseudo-name (say, \( t \)) has been previously introduced in a derivation (say, by UI at line 2 in the example below), you must existentially instantiate to a different pseudo-name (say, \( u \)). The following interpretation shows that existentially instantiating to a pseudo-name already introduced can lead from truth (Everyone has a parent) to falsity (Someone is his own parent). The INVALID move is at line 3; line 4 is legal by EG.

Let: The Domain = people; \( P_{xy} = x \) is a parent of \( y \)

\[
\begin{align*}
1 & \quad (x)(\exists y)P_{yx} \quad \text{Everyone has a parent} \\
2 & \quad (\exists y)P_{yt} \quad / \ UI, 1 \\
\text{INVALID:} & \quad 3 \quad P_{tt} \quad \text{Someone t is his own parent} \\
4 & \quad (\exists x)P_{xx} \quad \text{There is someone who's his own parent}
\end{align*}
\]
Preston

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