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Let us consider the behavior as $(x,t) \to (0,0)$ of solution of

$$u_+ = u_{xx}, \quad 0 < x < a, \quad t > 0, \tag{1}$$

under the initial and boundary conditions

$$u(x,0) = f(x), \quad u(0,t) = \varphi(t), u(a,0) = \Psi(t), \ a > 0. \tag{2}$$

Our solution uses the error function in the form

$$v(x,t) = \frac{2}{\sqrt{\pi}} \int_0^{x/2t^{1/2}} e^{-\partial^2} d\partial. \tag{3}$$

We shall show that, if the data is continuous, but not necessarily matching at the corners, then as $x \searrow 0$, $t \searrow 0$,

$$u(x,t) = v(x,t) f(0) + (1 - v(x,t)) \varphi(0) + o(1). \tag{4}$$

In particular, if $f(0) = \varphi(0)$, then u is continuous at $(x,t) = (0,0)$. Moreover, these results are true whether a is finite or infinite.

Let us prove this for the case $a = +\infty$. Then, as $(x,t) \to (0,0)$, for any $A > 0$,

$$\begin{aligned} u(x,t) &= \frac{f(0)}{\sqrt{4\pi t}} \int_0^A \left[e^{-(x-y)^L/4t} - e^{-(x+y)^L/4t} \right] dy \\ &\quad + \frac{\varphi(0)}{2\pi^{1/L}} \int_0^t \frac{x e^{-x^2/4(t-s)}}{(t-s)^{3/2}} ds + o(1). \end{aligned} \tag{5}$$

The trick now is to change variables so that each of the three integrands is the same, say $e^{-\partial^2}$. Then we have

$$
\begin{aligned}
u(x,t) &= \frac{f(0)}{\sqrt{\pi}} \int_{\frac{-x}{2t^{1/L}}}^{(A-x)/2t^{1/L}} e^{-\partial^2} d\partial - \int_{x/2t^{1/L}}^{(A+x)/2t^{1/L}} \\
&\quad + \frac{2\varphi(0)}{\sqrt{\pi}} \int_{x/t^{1/L}}^{\infty} e^{-\partial^2} d\partial + o(1).
\end{aligned} \tag{6}
$$

After noting that the contributions of the first two integrands from the interval $\left(x/2t^{1/L}, (A-x)\,2^{1/2}\right)$ cancel, we see that the result (4) follows.

In this case $a < \infty$, we use the representation in terms of the Jacobi's Theta Function, θ_3, see Fulks [2], Hartman and Wintner [3] and Goursat [5], and Friedman [9].

Note that the contribution from the infinite series for $k \geq 1$ and from the right boundary integral tend to zero uniformly. This leaves the expression (5).

Moreover, since $\nu(x,t)$ satisfies the Heat Equations, we may solve the problem numerically for continuous data and then add on the easily compatible function ν. Dr. Myron Sussman has informed me that he has found the solvability of continuous boundary value problems more stable.

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