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# Behavior at a Corner for Solutions of the One Dimensional Heat Equation

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# Behavior at a Corner for Solutions of the One Dimensional Heat Equation

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Let us consider the behavior as  $(x, t) \rightarrow (0, 0)$  of solution of

$$u_t = u_{xx}, \quad 0 < x < a, \quad t > 0, \quad (1)$$

under the initial and boundary conditions

$$u(x, 0) = f(x), \quad u(0, t) = \varphi(t), \quad u(a, 0) = \Psi(t), \quad a > 0. \quad (2)$$

Our solution uses the error function in the form

$$v(x, t) = \frac{2}{\sqrt{\pi}} \int_0^{x/2t^{1/2}} e^{-\theta^2} d\theta. \quad (3)$$

We shall show that, if the data is continuous, but not necessarily matching at the corners, then as  $x \searrow 0$ ,  $t \searrow 0$ ,

$$u(x, t) = v(x, t) f(0) + (1 - v(x, t)) \varphi(0) + o(1). \quad (4)$$

In particular, if  $f(0) = \varphi(0)$ , then  $u$  is continuous at  $(x, t) = (0, 0)$ . Moreover, these results are true whether  $a$  is finite or infinite.

Let us prove this for the case  $a = +\infty$ . Then, as  $(x, t) \rightarrow (0, 0)$ , for any  $A > 0$ ,

$$\begin{aligned} u(x, t) &= \frac{f(0)}{\sqrt{4\pi t}} \int_0^A [e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}] dy \\ &+ \frac{\varphi(0)}{2\pi^{1/L}} \int_0^t \frac{x e^{-x^2/4(t-s)}}{(t-s)^{3/2}} ds + o(1). \end{aligned} \quad (5)$$

The trick now is to change variables so that each of the three integrands is the same, say  $e^{-\theta^2}$ . Then we have

$$\begin{aligned}
 u(x, t) = & \frac{f(0)}{\sqrt{\pi}} \int_{\frac{-x}{2t^{1/L}}}^{(A-x)/2t^{1/L}} e^{-\theta^2} d\theta - \int_{x/2t^{1/L}}^{(A+x)/2t^{1/L}} \\
 & + \frac{2\varphi(0)}{\sqrt{\pi}} \int_{x/t^{1/L}}^{\infty} e^{-\theta^2} d\theta + o(1).
 \end{aligned} \tag{6}$$

After noting that the contributions of the first two integrands from the interval  $(x/2t^{1/L}, (A-x)2^{1/2})$  cancel, we see that the result (4) follows.

In this case  $a < \infty$ , we use the representation in terms of the Jacobi's Theta Function,  $\theta_3$ , see Fulks [2], Hartman and Wintner [3] and Goursat [5], and Friedman [9].

Note that the contribution from the infinite series for  $k \geq 1$  and from the right boundary integral tend to zero uniformly. This leaves the expression (5).

Moreover, since  $\nu(x, t)$  satisfies the Heat Equations, we may solve the problem numerically for continuous data and then add on the easily compatible function  $\nu$ . Dr. Myron Sussman has informed me that he has found the solvability of continuous boundary value problems more stable.

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