1995

There is an effective one-step cofinal reduction strategy for combinators

Richard Statman
Carnegie Mellon University

There is an effective one-step cofinal reduction strategy for combinators

by

Rick Statman

Carnegie Mellon University
Department of Mathematics
Pittsburgh, PA, USA

Research Report No. 95-177 2
June, 1995

There is an effective one-step cofinal reduction strategy for combinators

by
Rick Statman
Carnegie Mellon

Abstract:

In this note we shall construct an effective one-step reduction strategy for combinators. The construction clearly does not work for lambda calculus since it depends on the parallel residual property for combinators

Residuals of redexes are disjoint.

However, it holds for a variety of combinatory reduction systems.

Introduction:

In [1], page 350, exercise 13.6.6, Barendregt asks if there is an effective one-step cofinal reduction strategy. In this note we shall answer this question in the affirmative for combinators. We do not know the answer for lambda terms. The question arises form the work of Bergstra and Klop on effective Churh-Rosser strategies. The question of the existence of an effective one-step Church-Rosser strategy remains open, and is the most interesting question in the area.

The algorithm:

We adopt for the most part the notations of [1], especially those of chapter 12. If s is a reduction from M to N we write s : M ->> N. We also consider pairs (M,F) where F is a set of disjoint (non-overlapping) redexes of M. F/s is the set of residuals of members of F under the reduction s. It is also a set of non-overlapping redexes. We write s : (M,F) ->> (N,F/s) to show the action of s : M ->> N on F. cpl(M,F) is the complete reduction of M w.r.t. F.depth(M) is just the depth of M as a binary tree.

Let D be the digraph of the one-step reduction relation ->, and D(m) the subdigraph of D inducted by all combinators of depth $\leq$ m. Let D(M) be the weak component of D(depth(M)) containing M. We say (N,F) is in D(M) if N ε D(M). (N,F) in D(M) is said to be active for M if there is an s contained in D(M) such that s : N ->> M but there is no t contained in D(M) such that t : (N,F) ->> (M,ϕ). An s : (N,F) ->> (M,F/s) contained in D(M) with | F/s | as small as possible is said to be minimal.

Remark; if there is no active (P,F) in D(M) then M is recurrent ([3]). For if for each disjoint set of redexes F contained in M there is a reduction (M,F) ->> (M,ϕ) then by induction M ->> N => N->>M.

The pairs (N,F) can be ordered in type ω^*, the type of the non-positive integers. We refer to the non-positive integer corresponding to (N,F) as its priority.

The algorithm A:

Input; a combinator M.

Output; a combinator A(M) such that M -> A(M) unless M is normal in which case A(M) $\equiv$ M.

(1)Decide whether there is a pair (P,F) in D(M) which is active for M

(2)If the answer to (1) is yes then find such a (P,F) of highest priority and a minimal s : (P,F) -> (M,F/s) contained in D(M) else go to (5)

(3)If F/s = {R} then set A(M) := cpl(M,R) else select R ε F/s so that depth(M) $\leq$ depth (cpl(M,R)) and set A(M) := cpl(M,R).

(4)Exit

(5)If M is normal set A(M) := M else set A(M) := any N such that M -> N.

(6)Exit

Theorem: A is an effective one-step cofinal reduction strategy.

Proof; we must show cofinality.

Consider the iterations M -> A(M) -> A(A(M)) -> ... of the algorithm A on M. If this reduction sequence does not leave D(M) then it must cycle; in other words its contains a segment of the form N -> A(N) -> ... ->A^n(N) -> N. Let k be largest so that

depth (A^k(N)) = max { depth (A^i(N)) : i = 0,1,...,n }

and set N(i) := $A^{k+i \text{ (mod } n)}$(N). Thus the reduction sequence

N(0) -> N(1) -> ...-> N(n) -> N(0)

is contained in D(N(0)). By our previous remark, if instruction (5) in A is executed for any of the transitions N(i) -> N(i+1) then the corresponding N(i) is recurrent and the sequence of iterations of A on M is clearly cofinal. Otherwise for each i = 0,...,n we can find (P(i),F(i)) ε D(N(i)) and s(i) : (P(i),F(i)) ->> (N(i), F(i)/s(i)), contained in D(N(i)), obtained in the execution of instruction (2) in A on N(i).

By induction on i we see that depth(N(0)) = depth (N(i)) and F(i)/s(i) has some residuals in N(0) after the reduction

N(i) -> N(i+1) ->...-> N(0).

For i = 0, F(0)/s(0) has residuals in N(0) since (P(0),F(0)) was active. For i > 0, since F(i-1)/s(i-1) has residuals in N(0) we have depth(N(i)) $\geq$ depth(N(i-1)). In particular, D(N(i)) = D(N(i-1)) =...= D(N(0)). If F(i)/s(i) has no residuals in N(0) then the reduction s -> N(i+1) -> ... -> N(0) -> ... -> N(i+1) is contained in D(N(i)) and this contradicts the activity of (P(i),F(i)).

Next by induction on i we see that each (P(i),F(i)) = (P(0), F(0)). For let k be smallest so that (P(k),F(k)) =/= (P(0),F(0)). Then (P(k),F(k)) has higher priority than (P(0),F(0)) since (P(0),F(0)) is active for N(k). But then (P(k),F(k)) cannot be active for N(0), so there is a reduction t: (P(k),F(k)) ->> (N(0), ϕ) contained in D(N(0)). But then the reduction t -> N(1) -> N(2) -> ... -> N(k) is contained in D(N(k)) and this contradicts the activity of (P(k), F(k)) for N(k).

Now it is easily seen that | F(i+1)/s(i+1) | < | F(i)/s(i) | for i = 0,...,n-1.Thus the reduction s(n) -> N(0) leaves fewer residuals of F(0) in N(0) than s(0) does. This contradicts the minimality of s(0). Thus we conclude that the sequence of iterations of A on M is cofinal by cycling in D(M) or it exits from D(M) at some A^m(M) with depth (A^m(M)) > depth (M).

Let M(i) := $A^i(M)$ and suppose that the sequence M(0) -> M(1) -> .... -> M(m) -> ... never cycles. Then there is an infinite increasing sequence f(0), f(1),... such that M(0) -> ... -> M(f(i)) is contained in D(M(f(i))). We prove by induction on |priority| that each (P,F) is active for at most finitely many of the M(f(i)). Suppose that this is true for all pairs of priority higher than (P,F). Let k be so large that all pairs of priority higher than (P,F) are inactive past M(f(k)) in the subsequence of M(f(i))'s. If (P,F) is active for M(f(k+1)) it is of highest priority and the number of residuals of F is reduced by the transition M(f(k+1)) - > M(f(k+1)+1). Moreover depth (M(f(k+1))) $\leq$ depth (M(f(k+1)+1) ,if any residuals remain after the transition . This remark can be repeated for M(f(k+1)+1), M(f(k+1)+2),..., etc. Thus (P,F) will be inactive for any M(f(j)) with j > k+ the number of residuals of F in M(f(k+1)). We can now show that the sequence

$$M \to A(M) \to \dots \to A^m(M) \to \dots$$

is cofinal.

Suppose that s : M ->> N. We shall prove by induction on s that for some m , N ->> $A^m(M)$. Again we let M(m) := $A^m(M)$. Let M ->> N -> P. By induction hypothesis there is an m such that N ->> M(m). Suppose that

$$N \overset{R}{\to} P,$$

and let F be the set of residuals of R in M(m). By the previous paragraph we can find n > m so that the pair (M(m),F) belongs to D(M(n)) and (M(m),F) is not active for M(n). Then ,by the strip lemma, P ->> M(n), and this completes the proof.

References:

[1] Barendregt, The Lambda Calculus
North Holland, 1984

[2] Bergstra&Klop, Church-Rosser strategies in the lambda calculus
Preprint 62
University of Utrecht, Department of Math.

July 1977
[3] Jacopini&Venturini-Zilli, Equating for recurrent terms of lambda calculus and combinatory logic
Pubblicazioni dell' Istituto per le Applicazioni del Calcolo
Ser.III 85
CNR, Rome, 1978