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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A CLASS OF TIME-DEPENDENT VOLterra EQUATIONS

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Abstract. The asymptotic properties of solutions to a time—dependent nonlinear Volterra integral equation are studied in a general Banach space. The concept of completely positive kernel plays a crucial role in the analysis.

1. Introduction. The purpose of this paper is to discuss the asymptotic behavior as $t \to \infty$ of solutions to the abstract Volterra equation

$$u(t) + \int_0^t b(t-s) (Au(s) + g(s) u(s)) \, ds \in f(t), \quad t \in \mathbb{R}^+ = [0, +\infty),$$

in a real Banach space $X$. Here $b: \mathbb{R}^+ \to \mathbb{R}^+$ is a completely positive kernel, $A$ is a nonlinear (possibly multivalued) $m$—accretive operator in $X$, $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a given function, $f$ maps $\mathbb{R}^+$ into $X$, and the integral is taken in the sense of Bochner.

General existence, uniqueness and continuous dependence results for $(V_{b,g,f})$ have been established by Crandall and Nohel [8] and Gripenberg [10]. The asymptotic properties of solutions of $(V_{b,g,f})$ have primarily been studied in the case when $g \equiv 0$. See e.g. [2,4,5,13,16]. Recently, Kato, Kobayasi and Miyadera [15] have discussed the asymptotic behavior of solutions to a class of functional—differential equations related to $(V_{b,g,f})$. When applied to $(V_{b,g,f})$, their theory requires that $0 \in \text{R}(A)$ and $g \in L^1(\mathbb{R}^+)$, being thereby restricted to bounded solutions.

The present work is mainly concerned with the "unbounded behavior", as $t \to \infty$, of solutions to $(V_{b,g,f})$, so that we generally assume that $\text{R}(A)$ is zero free and $g \notin L^1(\mathbb{R}^+)$. Our study can be viewed as an attempt to extend earlier results obtained by Israel and Reich [14], and Kobayasi [17] for $(V_{1,g,f})$ (that is, the case when $(V_{b,g,f})$ reduces to an evolution equation), as well as the asymptotic theory developed in [13,16] for $(V_{b,0,f})$. Although we consider $(V_{b,g,f})$ in a general Banach space, we emphasize that our results are new even in
Hilbert space. We also note that \((V_{b,g,f})\) is a special case of the more general equation

\[
u(t) + \int_0^t b(t-s) A(s)u(s)ds \geq f(t), \quad t \in \mathbb{R}^+,
\]

where \(\{A(t), t \in \mathbb{R}^+\}\) denotes a family of \(m\)-accretive operators in \(X\). An analysis of asymptotic properties of bounded solutions of \((V)\) has recently been carried out in [1], under the assumption that \(X\) is a Hilbert space, and \(A(t)\) is cyclically maximal monotone for each \(t \geq 0\).

The plan of the paper is as follows. In section 2 we recall for easy reference some basic facts about \(m\)-accretive operators and completely positive kernels, and we comment briefly on the existence and uniqueness of solutions to \((V_{b,g,f})\). The main asymptotic results are presented and proved in Sections 3 and 4, respectively. An application of physical interest is discussed in Section 5.

2. Preliminaries. Let \(X\) be a real Banach space of norm \(\|\cdot\|\), and dual \((X^*, \|\cdot\|_*)\). The duality pairing between \(X\) and \(X^*\) will be denoted by \(\langle \cdot, \cdot \rangle\). Let \(A\) be a set-valued operator in \(X\) with domain \(D(A)\) and range \(R(A)\). We say that \(A\) is accretive if

\[
\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda (y_1 - y_2)\|, \quad \forall \lambda > 0, \quad x_1, x_2, y_1, y_2 \in D(A),
\]

for all \(\lambda > 0\) and \(y_1, y_2 \in Ax\), \(i = 1, 2\). \(A\) is called \(m\)-accretive, if it is accretive and \(R(I + \lambda A) = X, \forall \lambda > 0\). (Here \(I\) stands for the identity on \(X\)). When \(A\) is \(m\)-accretive, one can define its Yosida approximation \(A_\lambda\) by \(A_\lambda = \lambda^{-1}(I - J_\lambda)\), with

\[
J_\lambda = (I + \lambda A)^{-1}, \quad \lambda > 0.
\]

It is easily seen that \(J_\lambda\) is nonexpansive on \(X\), \(A_\lambda\) is Lipschitz continuous on \(X\), and \(A_\lambda x \in A J_\lambda x, x \in H\).

We will frequently use the following characterisation of accretivity (cf.e.g. [7]). Let

\[\lambda : X \times X \to \mathbb{R}\] be defined for \(\lambda \neq 0\) by
\[ [y, x]_\lambda = (\| x + \lambda y \| - \| x \| ) / \lambda, \ \forall x, y \in X, \]

and set:

\[
[y, x]_+ = \lim_{\lambda \downarrow 0} [y, x]_\lambda = \inf_{\lambda > 0} [y, x]_\lambda,
\]

\[
[y, x]_- = \lim_{\lambda \uparrow 0} [y, x]_\lambda = \sup_{\lambda < 0} [y, x]_\lambda.
\]

(Note that \( \lambda \to \| x + \lambda y \| \) is convex, so that \([y, x]_\lambda \) is monotonically nondecreasing in \( \lambda \).)

Then \( A \) is accretive in \( X \) if and only if \([y_2 - y_1, x_2 - x_1]_+ \geq 0, \ \forall \ y_i \in A x_i, \ i = 1, 2. \) Also recall that the Yosida approximation \( A_\lambda (\lambda > 0) \) of an \( m \)-accretive operator \( A \) is strictly accretive, i.e. \([A_\lambda x - A_\lambda y, x - y]_- \geq 0, \forall x, y \in X \). Some of the basic properties of \([,]_\pm \) are summarized below.

Proposition 2.1. [7,9] Let \( x, y, z \in X \) and \( c \in \mathbb{R} \). Then

(i) \(|[y, x]_+| \leq \| y \| ,

(ii) \([cx, x]_+ = c \| x \| ,

(iii) \([-y, x]_- = -[y, x]_+ ,

(iv) \([y + z, x]_- \leq [y, x]_- + [z, x]_+ ,

(v) \([,]_+: XxX \to \mathbb{R} \) is upper semicontinuous.

If, in addition \( u: \mathbb{R}^+ \to X \) is such that \( u, \| u \| \) are differentiable at \( t > 0 \), then

(vi) \( \frac{d}{dt} \| u(t) \| = [u'(t), u(t)]_+ \) \((' = d/dt)\).

We assume throughout that \( A \) is an \( m \)-accretive operator on \( X \), and consider equation

(V\(_{b,g,f}\)) under the following minimal assumptions:

(H\(_b\)) \( b \in AC_{\text{loc}}(\mathbb{R}^+; \mathbb{R}), b(0) = 1, \ b' \in BV_{\text{loc}}(\mathbb{R}^+; \mathbb{R}) ,\)

(H\(_g\)) \( g \in C(\mathbb{R}^+; \mathbb{R}^+) ,\)

(H\(_f\)) \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}^+; X), f(0) \in D(A) .\)
Let $\lambda > 0$ and $A_{\lambda}$ be the Yosida approximation of $A$. Since $A_{\lambda} : X \rightarrow X$ is Lipschitzian, and $g$ is continuous, a simple contraction argument shows that the approximating equation

$$u_{\lambda}(t) + \int_0^t b(t-s)(A_{\lambda} u_{\lambda}(s) + g(s) u_{\lambda}(s)) \, ds = f(t), \quad 0 \leq t < \infty,$$  \hspace{1cm} (2.1)

has a unique solution $u_{\lambda} \in \mathcal{W}^{1,1}(\mathbb{R}_+; X)$. Moreover (cf. [8]), equation (2.1) is equivalent to

$$\frac{du_{\lambda}}{dt}(t) + b(t - u_{\lambda}(t) + A_{\lambda} u_{\lambda}(t) + g(t) u_{\lambda}(t) = k(t) f(0) + F(t),$$

a.e. on $\mathbb{R}_+$

$$u_{\lambda}(0) = f(0),$$

where $*$ denotes the convolution, $k$ satisfies

$$b(t) + k * b(t) = 1, \quad 0 \leq t < \infty,$$  \hspace{1cm} (2.3)

and $f$ is given by

$$F(t) = f'(t) + k * f'(t), \text{ a.e. } t \in \mathbb{R}_+.$$  \hspace{1cm} (2.4)

Note that (2.3) can be rewritten as $k + b' * k = -b'$, so that, by $(H_b)$, $k$ is uniquely determined in $\mathcal{BV}_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$. It also follows (see $(H_f)$) that $F \in L^1_{\text{loc}}(\mathbb{R}_+; X)$.

The next result is a direct consequence of [8, Theorems 3 and 4] (cf. also [10, Theorem 5]).

**Proposition 2.2** Let $(H_b)$, $(H_g)$ and $(H_f)$ hold. Then there exists a (unique) function

$u \in \mathcal{C}(\mathbb{R}_+; X)$ such that $\lim_{\lambda \downarrow 0} u_{\lambda} = u$ in $\mathcal{C}([0,T]; X)$ for any $0 < T < \infty$, where $u_{\lambda}$ is the solution of (2.1) (equivalently, (2.2)).

**Definition 2.1** The limit function $u$, introduced in Proposition 2.2 is called the generalized
solution of \((V_{b,g,t})\).

To develop an asymptotic theory for generalized solutions of \((V_{b,g,t})\) we rely on the concept of completely positive kernel \([5,6]\). We confine ourselves to kernels satisfying \((H_b)\).

**Definition 2.2.** Let \((H_b)\) hold, and let \(k\) be defined by (2.3). Then \(b\) is said to be completely positive if \(k\) is nonnegative and nonincreasing on \(R^+\).

We next collect several important properties of completely positive kernels.

**Proposition 2.3.** \([5,19]\). Assume that \(b\) is completely positive. Then \(0 < b(t) < 1\) for all \(t \geq 0\), and \(\lim_{t \to \infty} b(t) = b(\infty)\) exists, with \(b(\infty) = (1 + \int_0^\infty k(s) \, ds)^{-1}\) if \(k \in L^1(R^+)\), and \(b(\infty) = 0\) if \(k \not\in L^1(R^+)\).

**Proposition 2.4.** (cf. e.g. \([13,15]\)). Let \(b\) be completely positive, and \(w \in W^{1,1}_{loc}(R^+; X)\). Then \(k \ast w\) and \(k \ast \|w\|\) are locally absolutely continuous and differentiable a.e. on \(R^+\). Moreover

\[
\frac{d}{dt} (k \ast w)(t), w(t) \geq \frac{d}{dt} (k \ast \|w\|)(t),
\]
for almost all \(t > 0\).

**Remark 2.1.** Let \(b\) be completely positive. Then, according to Proposition 2.3, \(b(\infty) > 0\) iff \(k \in L^1(0,\infty)\). Also, in this case, \(b \not\in L^1(R^+)\).

**3. Statement of Results.** Let \((H_b)\) and \((H_g)\) hold, and let \(k\) be given by (2.3).

For \(0 \leq s \leq t < \infty\), set

\[
a(t,s) = k(t-s) + g(s)
\]
and define the associated resolvent kernel \( r(t,s) \) by
\[
r(t,s) + \int_{s}^{t} a(t,\tau) r(\tau,s) d\tau = a(t,s). \tag{3.2}
\]

Since \( k \in L^p_{\text{loc}}(\mathbb{R}^+) \) and \( g \) is continuous, equation (3.2) has a unique solution \( r \), of class \( L^2(\mathbb{R} \times \mathbb{R}^+) \) (at least). See [11, chap. 9] or [20, chap. IV]. (We will often extend \( a \) and \( r \) loc by 0 for \( t < s \).) Define next
\[
R(t,s) = 1 - \int_{s}^{t} r(t,\tau) d\tau, \quad 0 < s < t < \infty. \tag{3.3}
\]

We need the following generalization of [18, Lemma 1.3].

Lemma 3.1. Let \((H_b)\) and \((H_g)\) be satisfied and \( k,R \) be given by (2.3) and (3.1) – (3.3), respectively. If also \( b \) is completely positive, then
\[
0 < R(t,s) < 1, \forall 0 < s \leq t < \infty. \tag{3.4}
\]

Our first important result for solutions of \((V_b,g,t)\) is:

Theorem 3.1. Let \((H_b)\) and \((H_g)\) hold. Suppose that \( f(\hat{f}) \) satisfy \((H_\hat{f})((H_\hat{f}^t))\), and that \( F \) \((F)\) are associated to \( f(\hat{f}) \) by (2.4). Let \( u \) and \( \hat{u} \) be the generalized solutions of \((V_{b,g,t})\) and \((V_{b,g,\hat{t}})\), respectively. If, in addition, \( b \) is completely positive, then
\[
\| u(t) - \hat{u}(t) \| \leq \| u(0) - \hat{u}(0) \| \left( 1 - \int_{0}^{t} R(t,s) g(s) ds \right) + \int_{0}^{t} R(t,s) [F(s) - \hat{F}(s), u(s) - \hat{u}(s)] d\tau ds, \tag{3.5}
\]
for all \( t \geq 0 \).

As an immediate consequence, we obtain

Corollary 3.1. Suppose that \((H_b), (H_g)\) and \((H_\hat{f})\) hold, and \( b \) is completely positive. Let \( u \) be the generalized solution of \((V_{b,g,t})\), and \( F \) be given by (2.4). Then
for all $z \in A y$, and all $s \geq 0$. In addition, for any $s$, $t > 0$,

$$
\| u(s) - y \| \leq \| u(0) - y \| (1 - \int_0^s R(s, \tau) g(\tau) d\tau) \\
+ \int_0^s R(s, \tau) [F(\tau) - g(\tau)y - z, u(\tau) - y]_+ d\tau,
$$

(3.6)

where $u = u(0)$.

We are now in a position to state our main asymptotic result. Here and in the sequel, $u$
denotes the generalized solution of $(V, b, g, F, t)$.

**Theorem 3.2.** Let $(H_b), (H_g)$ and $(H_f)$ be satisfied. Also assume that $b$
is completely positive with $b(\infty) > 0$, and $F$ verifies

$$
\int_0^t F(s) ds = 0, \\
H(t) = \int_0^t h(s) ds.
$$

(3.8)

where

$$
h(t) = \exp \left( \int_0^t g(s) ds \right), \quad H(t) = \int_0^t h(s) ds.
$$

(3.9)

If either

$$
g \in L^1(\mathbb{R}^+),
$$

(3.10)

or
$g \in L^1(R^+)$, $g$ positive, $\lim_{t \to \infty} g(t) = 0$, 
then there exists an element $\Theta \in S(X^*) = \{z \in X^* : \|z\| = 1\}$,
such that

$$\lim_{t \to \infty} <u(t), \Theta > = \lim_{t \to \infty} \|u(t)\| = d(O, R(A)), \quad (3.12)$$

where $d(O, R(A))$ denotes the distance from $0$ to $R(A)$.

A key tool in the proof of theorem 3.2 is

**Lemma 3.2.** Let the assumptions of Theorem 3.2 be satisfied. Then

$$\lim_{t \to \infty} \int_0^t R(t, \tau) d\tau = +\infty, \quad (3.13)$$

and

$$\lim_{t \to \infty} \int_0^t R(t, \tau) g(\tau) d\tau = 0. \quad (3.14)$$

The following consequence of Theorem 3.2 can easily be deduced (see[17]).

**Corollary 3.2.** Let the assumptions of Theorem 3.2 hold.

(i) If $X$ is reflexive and strictly convex, then

$$w - \lim_{t \to \infty} u(t)/\int_0^t R(t, s) ds = -v,$$

where $\|v\| = d(O, R(A))$ and $w$-$\lim$ stands for weak convergence.

(ii) If $X^*$ has a Fréchet differentiable norm, then
lim_{t \to \infty} \frac{u(t)}{\int_0^t R(t,s)ds} = -v,

where $v$ is the unique point of least norm in $\mathbb{R}(A)$.

Remark 3.1. It is easily verified (see (3.1) - (3.3)) that $R(t,s) = b(t,s)$ if $g \equiv 0$, and $R(t,s) = \frac{h(s)}{h(t)}$ if $b \equiv 1$ (with $h$ defined by (3.9)). Consequently, our Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 are natural generalizations of [13, Theorem 5.3], [16, Theorems 2.1, 2.4 and Corollaries 2.2, 2.5], as well as of [14, Corollary 5], [17, Theorem 2.1 and Corollaries 2.2, 2.3], corresponding to equations $(V_{b,0,t})$ and $(V_{1,g,0})$ respectively.

Remark 3.2. Necessary or sufficient conditions for the boundedness of $u$ on $\mathbb{R}^+$ can readily be derived from Theorem 3.2 or Corollary 3.1. If the assumptions of Theorem 3.2 hold, the boundedness of $u$ necessarily implies $0 \in \mathbb{R}(A)$. On the other hand, if in addition to the assumptions of Corollary 3.1, $0 \in A$ and $F \in L^1(0, \infty ; X)$, then $u$ is bounded on $\mathbb{R}^+$. When $g \in L^1(\mathbb{R}^+)$, the condition $0 \in A$ can be weakened to $0 \in \mathbb{R}(A)$.

4. Proofs.

Proof of Lemma 3.1. Denote $\tilde{r}(t,u) = \int_0^t r(t,s)ds, \forall 0 \leq u \leq t < \infty$. Integrating (3.2) over $(u,t)$ and using Fubini's theorem, we get:

$$\tilde{r}(t,u) + \int_0^t a(t,\tau)\tilde{r}(\tau,u)d\tau = \tilde{r}(t,u).$$

In view of (3.3), this yields

$$R(t+u,u) + \int_u^t a(t+\tau)R(\tau+u)d\tau = 1. \quad (4.1)$$
Replacing $t$ by $t + u$ in (4.1) leads to

$$R(t+u,u) + \int_0^t s(t+u, \xi+u) R(\xi+u, u) d\xi = 1.$$  \hspace{1cm} (4.2)

Suppose $u > 0$ is fixed and denote $\tilde{R}(t) = R(t+u,u)$, $t \geq 0$. Then (4.2) can be rewritten as (cf. (3.1))

$$\tilde{R}(t) + \int_0^t [k(t-s) + g(s+u)] \tilde{R}(s) ds = 1.$$  \hspace{1cm} (4.3)

Clearly, (3.4) is equivalent to

$$0 \leq \tilde{R}(t) \leq 1, \ \forall \ t \in [0,\infty).$$  \hspace{1cm} (4.4)

Using the same approximation argument as Levin [18, Lemma 1.3], we see that it is sufficient to prove (4.4) for smooth $k$. Recalling (cf. Definition 2.2) that $k$ is nonnegative and nonincreasing, we confine ourselves to the case when:

$$k \in C^1[0,\infty) ; k \geq 0, k' \leq 0 \quad (0 \leq t < \infty).$$  \hspace{1cm} (4.5)

Then, from (4.3) it follows that $\tilde{R} \in C^1[0,\infty)$. We are going to show that

$$0 < \tilde{R}(t) \leq 1, \ \forall \ t \in [0,\infty).$$  \hspace{1cm} (4.6)

Assume that

$$0 < \tilde{R}(t) \quad (0 \leq t < \infty)$$

does not hold. Then, since $\tilde{R}(0) = 1$, there exists a unique $t_0 > 0$ such that

$$\tilde{R}(t_0) = 0, \ 0 < \tilde{R}(t) \quad for \ 0 \leq t < t_0.$$  \hspace{1cm} (4.7)

This implies

$$\tilde{R}'(t_0) \leq 0.$$  \hspace{1cm} (4.8)

Differentiating (4.3) and setting $t = t_0$, yields

$$\tilde{R}'(t_0) = - \int_0^{t_0} [k'(t_0-s) \tilde{R}(s)] ds.$$  \hspace{1cm} (4.9)

By (4.5), (4.7), (4.9), we conclude that $\tilde{R}'(t_0) > 0$, which contradicts (4.8), unless

$$k(t) \equiv k(0) \quad (0 \leq t \leq t_0).$$  \hspace{1cm} (4.10)
But (4.3) and (4.10) lead to
\[ \tilde{R}'(t) + (k(0) + g(t+u)) \tilde{R}(t) = 0 \quad (0 \leq t \leq t_0), \quad \tilde{R}(0) = 1. \]

It follows that \( \tilde{R}(t) = \exp(-\{(k(0)t + \int_0^t g(s+u)ds\}), \quad t \in [0,t_0], \) so that \( \tilde{R}(t_0) > 0. \) This contradicts (4.7), and consequently (4.6) is established. Since \( k \geq 0 \) and \( (H_g) \) holds, we have \( 0 < \tilde{R}(t) \leq 1 \) on \([0, +\infty)\). The proof is complete.

**Proof of Theorem 3.1.** Let \( u_\lambda \) be the solution of (2.1), and let \( \tilde{u}_\lambda \) satisfy the same equation where \( f \) is replaced by \( \hat{f}. \) In view of Proposition 2.1(v) and Proposition 2.2, it clearly suffices to show that (3.5) holds with \( u_\lambda, \tilde{u}_\lambda \) in place of \( u, \hat{u}, \) respectively. Using the equivalent form (2.2) of (2.1), we deduce that \( u_\lambda - \tilde{u}_\lambda \) satisfies
\[
\frac{d}{dt}(u_\lambda - \tilde{u}_\lambda)(t) + \frac{d}{dt}(k(\lambda - \tilde{u}_\lambda))(t) + A_\lambda u_\lambda(t) - A_\lambda \hat{u}_\lambda(t) + g(t)(u_\lambda(t) - \tilde{u}_\lambda(t))
= k(t)(u_0 - \hat{u}_0) + \tilde{F}(t) - \hat{F}(t), \quad 0 < t < \infty,
\]
where \( u_0 = u_\lambda(0) = f(0), \quad \hat{u}_0 = \hat{u}_\lambda(0) = \hat{f}(0). \)

Recalling that \( A_\lambda \) is strictly accretive, and invoking Proposition 2.1, we infer from (4.11) that
\[
\frac{d}{dt}\| u_\lambda - \tilde{u}_\lambda \|((t) + \frac{d}{dt}(k(\lambda - \tilde{u}_\lambda))(t)) + g(t)\| u_\lambda - \tilde{u}_\lambda \|((t)
\leq k(t)\| u_0 - \hat{u}_0 \| + \| \tilde{F}(t) - \hat{F}(t), u_\lambda(t) - \tilde{u}_\lambda(t) \|.
\]
Applying Proposition 2.4 (the inequality (2.5)) then yields
\[
\frac{d}{dt}\| u_\lambda - \tilde{u}_\lambda \|((t) + \frac{d}{dt}(k\| u_\lambda - \hat{u}_\lambda \|)(t) + g(t)\| u_\lambda - \tilde{u}_\lambda \|((t)
\leq k(t)\| u_0 - \hat{u}_0 \| + \| \tilde{F}(t) - \hat{F}(t), u_\lambda(t) - \tilde{u}_\lambda(t) \|.
\]
Let (for a fixed \( \lambda > 0 \))
\[
\| u_\lambda - \hat{u}_\lambda \|((t) - \| u_0 - \hat{u}_0 \| = \varepsilon(t),
\]
\[
[\tilde{F}(t) - \hat{F}(t), u_\lambda(t) - \hat{u}_\lambda(t)] + g(t)\| u_0 - \hat{u}_0 \| = \varphi(t).
\]
Then (4.12) can be rewritten as
\[
\frac{d}{dt}(k\times)(t) + g(t)\times(t) \leq \varphi(t), \quad \text{a.e. } t > 0,
\]
x(0) = 0.
If we denote
\[ x(t) + k \cdot x(t) + \int_0^t g(s) x(s) ds = \psi(t), \ t \geq 0, \]  
(4.15)
we see that (4.14) implies \( \psi(0) = 0 \) and
\[ \psi'(t) \leq \varphi(t), \text{ a.e. } t > 0. \]  
(4.16)
Using (3.1), (3.2) we can solve (4.15) by means of the "variation of constants" formula [11, 20]:
\[ x(t) = \psi(t) - \int_0^t r(t,s) \psi(s) ds, \quad 0 \leq t < \infty. \]  
(4.17)
An integration by parts shows that (4.17) is equivalent to
\[ x(t) = \int_0^t R(t,s) \psi'(s) ds, \quad t \geq 0. \]  
(4.18)
where \( R \) is defined by (3.3). Since \( R(t,s) \geq 0 \) by Lemma 3.1, we deduce from (4.16) and (4.18) that
\[ x(t) \leq \int_0^t R(t,s) \varphi(s) ds, \quad t \geq 0. \]
On account of (4.13) this yields
\[ \| u_\lambda(t) - \hat{u}_\lambda(t) \| - \| u_\circ - \hat{u}_\circ \| (1 - \int_0^t R(t,s) g(s) ds) \]
\[ \leq \int_0^t R(t,s) [F(s) - \hat{F}(s), u_\lambda(s) - \hat{u}_\lambda(s)]_+ ds, \]
and (3.5) follows.

**Proof of Corollary 3.1.** If \( z \in A y \), we obviously have
\[ \frac{dy}{dt} + \frac{d}{dt} (k^*y)(t) + z + g(t)y = k(t)y + z + g(t)y. \]
Applying (3.15) with \( \hat{u}(t) \equiv y, \hat{F}(t) = g(t)y + z \), we get (3.6). Next take
\[ y = J_t u_\circ, z = A_t u_\circ (u_\circ = u(0)) \] in (3.6) and notice that
The inequality (3.7) now follows easily.

Proof of Lemma 3.2. Let \( p(t) = \int_0^t R(t,s)ds, \ t \geq 0. \) Integrating (4.1) over \((0,t)\) yields

\[
\begin{align*}
p(t) + \int_0^t (k(t-s) + g(s))p(s)ds &= t. \quad (4.19)
\end{align*}
\]

From (4.19) we conclude (cf.e.g. [10, Lemma 3.4]) that \( p \in A_{\text{loc}}(R^+; R) \); hence

\[
\begin{align*}
dp(t) + g(t) p(t) + d\left(k^*p(t)\right) &= 1, \ a.e. \ t > 0. \quad (4.20)
\end{align*}
\]

Recall now (see (3.9)) that \( h' = hg \geq 0; \) also, \( k \) is nonincreasing and \( p \geq 0. \) Consequently,

\[
\begin{align*}
h(t) \frac{d}{dt}(k*p)(t) &= h(t)(k(0)p(t) + \int_0^t p(s)ds) \\
&\leq \frac{d}{dt}(k^*(hp))(t), \ a.e. \ t > 0. \quad (4.21)
\end{align*}
\]

Multiplying (4.20) by \( h(t) \) and invoking (4.21) gives

\[
\begin{align*}
\frac{d}{dt}(hp)(t) + \frac{d}{dt}(k^*(hp))(t) &\geq h(t), \ a.e. \ on \ (0, +\infty). \quad (4.22)
\end{align*}
\]

Since \( p(0) = 0 \) and \( h(0) = 1, \) we may rewrite (4.22) as

\[
\begin{align*}
(hp)'(t) + k^*(hp)'(t) &\geq h(t), \ t > 0. \quad (4.23)
\end{align*}
\]

Take the convolution of (4.23) with \( b, \) and use (2.3) and \( b \geq 0 \) to obtain

\[
\begin{align*}
h(t) p(t) &\geq b^* h(t), \ 0 \leq t < \infty. \quad (4.24)
\end{align*}
\]

On the other hand, we notice that \( b^*h \) is nondecreasing (since \( (b^*h)'(t) = \)

\[
\begin{align*}
b(t) h(0) + \int_0^t (h'(t-s)b(s)ds \geq 0); \ this \ implies \ (k \geq 0)
\end{align*}
\]

\[
\begin{align*}
(b^*h) * k(t) &\leq (b^*h)(t) \int_0^t k(s)ds, \ t \geq 0. \quad (4.25)
\end{align*}
\]
If we now take the convolution of (2.3) with \( h(t) \), we get on account of (4.25),

\[
(b * h)(t) (1 + \int_{0}^{t} k(s)ds) \geq H(t).
\]  

(4.26)

Inasmuch as \( b(\omega) > 0 \) by hypothesis, we deduce from (4.26) (in view of Remark 2.1 and Proposition 2.3) that

\[
b(h(t)) \geq b(\omega) H(t), \quad 0 \leq t < \omega.
\]  

(4.27)

From (4.24) and (4.27) it follows that

\[
p(t) \geq b(\omega) \frac{H(t)}{h(t)} > 0, \quad 0 < t < \infty,
\]  

(4.28)

which implies (3.13). (In case when (3.10) is satisfied, \( h(\omega) < +\omega \) and \( H(\omega) \geq t \); if (3.11) holds, then \( \lim_{t \to \infty} \frac{H(t)}{h(t)} = +\omega \).) To prove (3.14) when (3.10) is fulfilled, we simply remark (cf. (3.4)) that

\[
0 \leq \frac{\int_{0}^{t} R(t,s)g(s)ds}{\int_{0}^{t} R(t,s)ds} \leq \frac{\int_{0}^{\infty} g(s)ds}{p(t)}.
\]

In case when (3.11) holds, it is easily verified that (3.14) is a consequence of (3.4), (3.13) and \( g(\omega) = 0 \).

**Proof of Theorem 3.2.** By (3.6) and Proposition 2.1(i), we have

\[
\|u(t) - y\| \leq \|u(0) - y\| + \int_{0}^{t} R(t,\tau) \| F(\tau) \| d\tau + (\int_{0}^{t} R(t,\tau)g(\tau)d\tau)\|y\|
\]  

\[
+ (\int_{0}^{t} R(t,\tau)d\tau)\|z\|
\]  

(4.29)

for any \([y,z] \in A\). Taking into account (3.4), (3.13), (3.14), (4.28) and assumption (3.8), we infer from (4.29) that
\[ \lim_{t \to \infty} \sup_{s \leq t} \| u(t) \| / \int_0^t R(t,s)ds \leq d(0, R(A)). \quad (4.30) \]

If \( d(0, R(A)) = 0 \), then (3.12) holds for any \( \Theta \in S(X^*) \), so that we consider the case when
\( d(0, R(A)) > 0 \). Following [16, Theorem 2.4] (cf. also [17, Theorem 2.1]), we choose for each
\( t > 0 \), an element \( \Theta_t \in S(X^*) \) with the property that \( \langle J_t u_0 - u_0, \Theta_t \rangle = \| J_t u_0 - u_0 \| \). This
together with (3.7) implies (recall that \( p(s) = \int R(s,r)dr > 0, \forall s > 0 \), cf. (4.28))
\[ \langle u(s) - u_0, \Theta_t \rangle / \int_0^s R(s,\tau)d\tau \geq \frac{1}{t} \| u_0 - J_t u_0 \| - \frac{1}{s} \int_0^s \frac{R(s,\tau)\| F(\tau) \| d\tau}{R(s,\tau)} \]
\[ \int_0^s \frac{R(s,\tau)\| F(\tau) \| d\tau}{R(s,\tau)} - \frac{2}{s} \int_0^s \frac{R(s,\tau)\| u(\tau) - u_0 \| d\tau}{R(s,\tau)} \quad (0 < s < t < \infty), \]
which, by (3.4), (4.28) leads to
\[ \langle u(s) - u_0, \Theta_t \rangle / \int_0^s R(s,\tau)d\tau \geq \frac{1}{t} \| u_0 - J_t u_0 \| - \frac{1}{s} \int_0^s \frac{h(s)\| F(\tau) \| d\tau}{b(s) H(s)} \]
\[ \frac{1}{s} \int_0^s \frac{R(s,\tau)\| F(\tau) \| d\tau}{R(s,\tau)} - \frac{2}{s} \int_0^s \frac{h(s)\| u(\tau) - u_0 \| d\tau}{b(s) H(s)} \quad (0 < s < t < \infty), \]

On the other hand, for \( 0 < s < t \), we have
\[ \langle J_s u_0 - u_0, \Theta_t \rangle / s \geq \| J_t u_0 - u_0 \| / t. \quad (4.32) \]
Also recall [22, Lemma 2.1] that
\[ \lim_{t \to \infty} \| J_t u_0 \| / t = d(0, R(A)) \cdot (4.33) \]
Let \( \Theta \in X^* \) be a weak-star cluster point of \( \{\Theta_t\} \), \( \ast \to \ast \). Then from (4.31) – (4.33), we
obtain
\[
\langle u(t) - u_0, \Theta \rangle \leq \int_0^t R(s,\tau) d\tau \geq d(0(R(A)) - \frac{1}{b(\infty)} \cdot \frac{h(s)}{H(s)} \int_0^s \| F(\tau) \| d\tau
\]

\[
-\frac{1}{s} \int_0^s \left( \int R(s,\tau) g(\tau) d\tau \right) \| u_0 \|
\]

\[
\langle J_\infty u_0 - u_0, \Theta \rangle /s \geq d(0(R(A)).
\]

Letting \( s \to \infty \) in (4.34) yields (in view of (3.8), (3.13), (3.14))

\[
\lim \inf_{s \to \infty} \langle u(s), \Theta \rangle /s \geq d(0(R(A)),
\]

while (4.35) implies

\[
\lim \inf_{s \to \infty} \langle J_s u_0, \Theta \rangle /s \geq d(0(R(A)).
\]

The conclusion of Theorem 3.2 now follows from (4.30), (4.33), (4.36) and (4.37).

5. **An Example.** In this section we suggest a special heat flow model to which our previous theory applies. Consider a homogeneous bar of unit length of a material with memory. Let \( u(t,x), e(t,x), q(t,x) \) and \( \mu(t,x) \) denote, respectively, the temperature, internal energy, heat flux, and external heat supply at time \( t \) and position \( x \) \((-\infty < t < \infty, 0 \leq x \leq 1)\). Let the ends of the bar at \( x = 0 \) and \( x = 1 \) be maintained at zero temperature, and for simplicity, let the history of \( u \) be prescribed as zero when \( t < 0 \) and \( 0 \leq x \leq 1 \). According to the theory developed by e.g. Gurtin and Pipkin [12] and Nunziato [21] for heat flow in materials of fading memory type, we may assume that

\[
e(t,x) = u(t,x) + \int_0^t \beta(t-s) u(s,x) ds + \int_0^t \alpha(t-s) g(s) u(s,x) ds,
\]
\[ q(t,x) = -\sigma(u_x(t,x)) + \int_0^t \gamma(t-s) \sigma(u_x(s,x))ds, \quad (5.2) \]

for \( t \geq 0 \) and \( 0 < x < 1 \). Here \( \beta, \gamma : [0,\infty) \to \mathbb{R} \) are sufficiently smooth functions, \( \alpha(t) = 1 - \int_0^t \gamma(s)ds, \ g \in C(\mathbb{R}^+, \mathbb{R}^+) \), and \( \sigma \) is a real function satisfying
\[ \sigma \in C^1(\mathbb{R}), \ \sigma(0) = 0, \ \sigma'(\xi) \geq c_0 > 0 \ (\xi \in \mathbb{R}), \ \text{for some} \ c_0 > 0. \quad (5.3) \]

The balance of heat requires that the equation \( e_t = -q + \mu \) should hold. If also \( u(0,x) = u_0(x) \ (0 < x < 1) \) is the initial temperature distribution, we obtain in view of (5.1), (5.2) and the assumption that the temperature at the ends of the rod is zero:
\[ u(t) + \beta(u)(t,x) + (\alpha^*g)(t,x) = \sigma(u_x(t,x)) - (\gamma^*\sigma(u_x))(t,x) + \mu(t,x), \quad 0 < t < \infty, \ 0 < x < 1, \quad (5.4) \]

Following [5, Section 4] we transform the initial—boundary value problem (5.4) to a Volterra integral equation in the space \( X = L^2(0,1) \). Let
\[ G(t,x) = u_0(x) + \int_0^t \mu(s,x)ds, \ 0 \leq t < \infty, \ 0 < x < 1 \quad (5.5) \]

and remark that
\[ \sigma(u_x)_x - \gamma^* \sigma(u_x) = \frac{\partial}{\partial t}(\alpha^* \sigma(u_x)_x). \]

Then (5.4) leads to the equation
\[ u + \beta^* u + \alpha^*(Au + gu) = G, \ 0 \leq t < \infty, \ 0 < x < 1, \quad (5.6) \]

where \( A : D(A) \subset X \to X \) is defined by \( Au = -\sigma(u_x)_x \), with
\[ D(A) = \{ u \in H^1_0(0,1) : \sigma(u_x)_x \in X \} \]. By (5.3), it is easily verified that \( A \) is maximal monotone (equivalently, \( m \)-accretive, cf[3]) in \( X \), with \( 0 \in \mathcal{R}(A) \). If \( r(\beta) \) denotes the resolvent
kernel of $f$ (i.e. $f$ satisfies $r(f) + f \cdot r(f) = f; r(f) \in L^1_{\text{loc}}([0,\infty))$ if $f \in L^1_{\text{loc}}([0,\infty))$, and

\[b = a - r(f) \cdot a, \quad (5.7)\]

\[f = G - r(f) \cdot G, \quad (5.8)\]

then the variation of constants formula shows that (5.6) is equivalent to

\[u + b^*(Au+gu) = f, \quad (5.9)\]

i.e. an equation of the standard form $(V_{b,g,t})$ in $X$. The next result is essentially [5, Lemma 4.2]:

**Lemma 5.1.** Let $f$ be bounded, nonnegative, nonincreasing and convex on $[0,\infty)$. Let $\gamma$ be positive, nonincreasing, log convex, and bounded on $[0,\infty)$. Suppose that

\[
\alpha(\omega) = 1 - \int_0^\infty \gamma(s)ds > 0, \text{ and } f'(t) + \gamma(0) f(t) < 0, \text{ a.e. } t > 0.
\]

Then $b$ (given by (5.7)) satisfies $(H_b)$ and is completely positive, with $b(\omega) > 0$.

We can now apply the theory developed in § 3 and 4 to discuss the asymptotic behavior of the generalized solution of equation (5.9) (equivalent to the heat flow problem (5.4)). We assume that $u_0 \in L^2(0,1)$ and the forcing function $\mu \in L^1_{\text{loc}}([0,\infty); L^2(0,1))$. Then, by (5.5), (5.8) and $r(f) \in L^1_{\text{loc}}([0,\infty))$ (at least) it is easily seen that $f \in W^{1,1}_{\text{loc}}([0,\infty); L^2(0,1))$. Also remark that $D(A)$ is dense in $X$, so that all of $(H_f)$ is satisfied. As soon as $(H_b)$, $(H_\mu)$ hold, Proposition 2.2 implies that (5.9) has a unique generalized solution $u$ on $[0,\infty)$. A direct application of Corollary 3.2, combined with Lemma 5.1 now yields

**Theorem 5.1.** Let the assumptions of Lemma 5.1 be satisfied. Let $u_0 \in L^2(0,1), \mu \in L^1_{\text{loc}}([0,\infty); L^2(0,1))$, and $b,k,f,F$ be defined by (5.7), (2.3), (5.8), (2.4), respectively. If also $g$ satisfies $(H_g)$ and (3.11), $R(t,s)$ is given by (3.1)-(3.3), and (3.8), (3.9) hold ($\| \cdot \|$ stands for the norm in $L^2(0,1)$), then equation (5.9) has a unique generalized solution $u$, such that:
\[ \lim_{t \to \infty} \frac{u(t)}{\int_0^t R(t,s) \, ds} = 0, \text{ strongly in } L^2(0,1). \]

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