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Universal bridge free graphs

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UNIVERSAL BRIDGE FREE GRAPHS

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ABSTRACT. We prove that there is no countable universal B_n-free graph for all n and that there is no countable universal graph in the class of graphs omitting all cycles of length at most $2k$ for $k \geq 2$.

§0 Introduction

Several papers have addressed the problem of existence of a universal element among all countable graphs omitting given finite subgraphs (see [KP] and the comprehensive bibliography there, and most recently [CK] and [KP1]).

Given a graph F we say that a graph G is F-free if F is not isomorphic to a subgraph of G. A countable F-free graph G^* is universal (strongly universal) in the class of all countable F-free graphs if every countable F-free graph is isomorphic to a subgraph (an induced subgraph) of G^*.

In [CK], Cherlin and Komjath raise the problem of determining for which finite trees T there exists a universal countable T-free graphs. In this paper we describe an infinite set of finite trees B_n, which we call bridges, and show that for no n is there a universal countable B_n-free graph.

In [CK] it is proved that for all $n \geq 4$ there is no universal countable C_n-free graph (where C_n is a cycle of length n). In [KMP] it is proved, on the other hand, that a strongly universal countable graph exists among all countable graphs that omit all odd cycles of length at most $2k+1$. What if we intersect some of those classes, say look at all graphs omitting C_3, C_4, C_5, C_6? We show here, using an idea of S. Mozes, that when all cycles of length at most $2k$ are to be excluded (for $k \geq 2$), then there is no universal countable graph.

0.1 Problem: Is there a countable universal graph in the class of graphs omitting all cycles of length at most $2k+1$ (for $k \geq 2$)? More generally, for what sets $F \subseteq N$ does the class of graphs omitting $\{C_n : n \in F\}$ have a countable universal element?

Following [KP] we make the following definition:

0.2 Definition: Let $\mathcal{G}$ be a class of graphs.

(i) The complexity $\mathrm{cp}(\mathcal{G})$ of $\mathcal{G}$ is the minimal cardinality κ of a set I of graphs in $\mathcal{G}$ with the property that every member in $\mathcal{G}$ is embedded as an induced subgraph into at least one of the members of I.

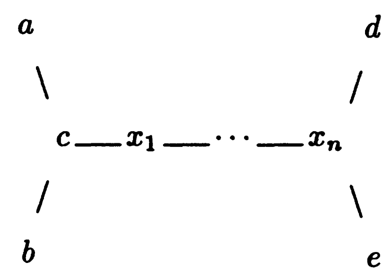
(ii) The weak complexity $\mathrm{wcp}(\mathcal{G})$ is defined by omitting the word "induced" from the definition of $\mathrm{cp}(\mathcal{G})$.

Notation: We denote the vertex degree of v in a graph G by $\deg_G(v)$. The *length* of a path is the number of edges in the path. For all m let K_m denote a complete graph with m vertices $k_1, \ldots, k_m$, let P_m be a simple path of length m with vertices $p_0, \ldots, p_m$ and edges (p_i, p_{i+1}) for $i \leq m$ and let C_m denote a cycle of length m. Let us call a simple path $h_0, h_1 \ldots, h_k$ in a graph G a *highway* iff $\deg_G(v_i) = 2$ for all $0 < i < k$.

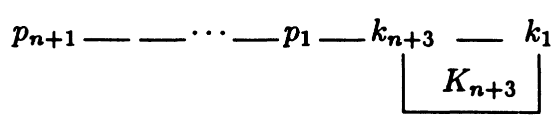
0.3 Advice: Drive carefully.

§1 Bridge-free graphs

1.1 Definition: A finite graph with $n+5$ vertices is called an n-bridge iff it is isomorphic to $B_n = \langle V, E\rangle$ where $V = \{a, b, c, x_1, x_2, \ldots, x_n, d, e\}$ and $E = \{(a,c), (b,c), (x_n, d), (x_n, e)\} \cup \{(x_i, x_{i+1}) : 1 \leq i < n\}$



1.2 Definition: Let us call a graph D a *dead end* if it is isomorphic to K_{n+3} to which a simple path P_{n+1} is freely adjoined by identifying k_{n+3} with p_0.



1.3 Claim: A dead end is B_n-free and if a dead end D with vertices $k_1, \ldots, k_{n+3} = p_0, x_1, \ldots p_{n+1}$ is a subgraph of a B_n-free graph G then $\deg_D(v) = \deg_G(v)$ for all vertices $v \in D$ except maybe $v = p_{n+1}$.

Proof: Suppose first that for some $i \leq n$ there is an edge (p_i, y) in G which is not an edge of D. If $y \notin D$, by labeling y as a, labeling p_i as c and p_{i+1} as b it is possible to label vertices of D as x_i $(i \leq n)$ and as d, e to produce a copy of B_n.

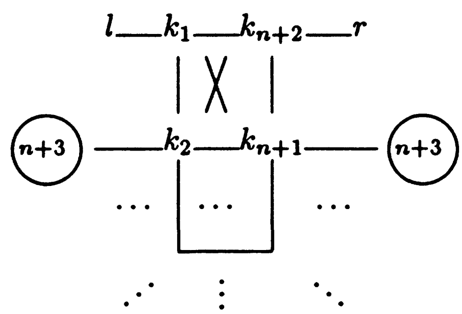
Suppose, then, that (p_i, p_j) is an edge not among the edges of D. Without loss of generality, $j \geq i + 2$. Now label p_j as a, label P_i as c and p_{i+1} as b. Again, a copy of B_n is easily found.

The remaining possibility for an edge of the form (p_i, y) is that $y = k_j$ for some $j \leq n+3$. Here we distinguish two subcases. First, $j = 1$ (and, of course, $i > 1$). Labeling p_1 as a, k_1 as c and p_1 as b, the remaining $n+2$ vertices of K_{n+3} complete the three labeled ones to make a copy of B_n.

Second, $j \neq 1$. In this case if $i > 1$ label k_j as c, label k_1 as x_n and label p_1 as e. The remaining vertices of K_{n+3} serve as x_i for $1 \leq i < n$ and as b. If, however, $i = 1$, label p_1 as c, label p_2 as a and label k_j as b. Again, a copy of B_n is found.

We show next that the $\deg_D(k_i)$ is preserved. Suppose that (k_i, y) is an edge in G which is not an edge in D. We already proved that $y \neq p_j$ for all $j \leq n$. Therefore, either $y \notin D$ or $y = p_{n+1}$. If $i = 1$ label p_1 as a, label y as b and k_1 as c; otherwise label k_i as c, y as a, p_1 as e and k_1 as x_n. In both cases a copy of B_n results ☺ 1.3

1.4 Definition: Let us call a graph T a *drive through* if it is isomorphic to the graph obtained as follows: Let $k_1, \ldots, k_{n+2}$ be the vertices of a copy of K_{n+2}. For $1 < i < n+2$ adjoin freely to k_i a copy of a dead end by identifying p_{n+1} in that copy with k_i. To k_1 connect a vertex l by an edge and to k_{n+2} connect a vertex r by an edge. Call l the *left exit* of T and call r the *right exit* of T.



1.5 Claim: A drive through T is B_n-free and if T is a subgraph of a B_n-free graph G then $\deg_T(v) = \deg_G(v)$ for all vertices $v \in T$ except l and r.

Proof. : Suppose to the contrary that B_n is a subgraph of a drive through T. As $\deg_{B_n}(c) = \deg_{B_n}(x_n) = 3$, both a and x_n are either in a copy of K_{n+3} or in the copy of K_{n+2}. Both cannot be in the same copy of K_{n+3} because the minimum of distances of x_i to c and x_n is smaller than $n+1$, and all the points satisfying this would be in the same dead end as x_n and c, contrary to claim 1.3. Similarly, c and x_n are not both in the copy of K_{n+2}.

Also, c and x_n cannot be in different copies of K_{n+3}, or in a copy of K_{n+3} and in the copy of K_{n+2} because the distance between x_n and c would be greater than n. We conclude that T is B_n-free.

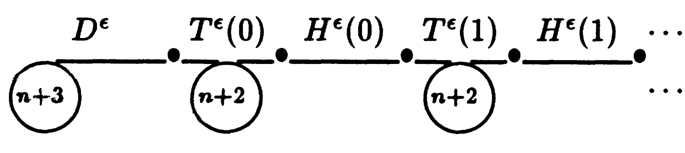
Suppose that T is a subgraph of a B_n free graph G. By claim 1.3 we know that $\deg_T(v) = \deg_G(v)$ for all vertices v in the dead ends except those which are also in the copy of K_{n+2}. Suppose that for some vertex k_i in the copy of K_{n+2} there is an edge (k_i, y) in G which is not an edge of T. Label y as a. If $i = 1$ or $i = n+2$ label l or r respectively as b. Label k_i as c. Label n of the remaining k_i as $x_1, \ldots, x_n$. Label the last remaining k_i as d. If this i is 1 or $n+1$ label l or r respectively as e. Otherwise label as e the vertex p_n in the dead end adjoined to k_i. This yields a copy of B_n. ☺ 1.5

For every $\epsilon \in {}^{\omega}2$ we construct a connected B_n-free graph G_ϵ as follows.

Let $T^\epsilon(m)$ for $m \in N$ and $\epsilon \in {}^{\omega}2$ be disjoint copies of a drive through. Let $l^\epsilon(m)$ and $r^\epsilon(m)$ be the left and right exits of $T^\epsilon(m)$. Let D^ϵ be a copy of D with vertices $k_1^\epsilon, \ldots, k_{n+3}^\epsilon = p_0^\epsilon, \ldots p_{n+1}^\epsilon$. Let $H^\epsilon(m)$ be a simple path of length $n - 1 + \epsilon(m)$ with vertices $h_0^\epsilon(m), \ldots, h_{n-1+\epsilon(m)}^\epsilon(m)$.

Adjoin D^ϵ to $l^\epsilon(0)$ by setting $p_{n+1}^\epsilon = l^\epsilon(0)$. Connect $r^\epsilon(m)$ to $l^\epsilon(M+1)$ by $H^\epsilon(m)$ by setting $r^\epsilon(m) = h^\epsilon(0)$ and $l^\epsilon(+1) = h_{n-1+\epsilon(m)}^\epsilon(m)$. (If $n = 1$ then when $\epsilon(m) = 0$ we identify $r^\epsilon(m)$ with $l^\epsilon(m+1)$.)

Let $G_\epsilon = D^\epsilon \cup \bigcup T^\epsilon(m) \cup \bigcup H^\epsilon(m)$.



Let us observe that all highways in G_ϵ are either of length $n+1$ or of length $n+2$. All highways that have an end of degree $n+3$ are of length $n+1$ except a unique highway — the one containing $l^\epsilon(0)$ — which is of length $n+2$. Let us denote this highway by $H(\epsilon)$.

1.6 Claim: The graph G_ϵ is B_n-free and if G_ϵ is a subgraph of a B_n-free graph G then the vertex degree of every vertex $v \in G_\epsilon$ in G_ϵ equals the degree of v in G.

Proof: : A similar argument to that in 1.5 shows that G_ϵ is B_n-free. Suppose now that $G_\epsilon \subseteq G$ and that G is B_n-free. By 1.5 we already know for $\deg_{G_\epsilon}(v) = \deg_G(v)$ for each vertex $v \in T^\epsilon(m)$ except $l^\epsilon(m), r^\epsilon(m)$. If, however, $\deg_G(v) > \deg_{G_\epsilon}(v)$ when v is on one of the highways of G_ϵ, there must be some $y \in G \setminus G_\epsilon$ such that (v, y) is an edge of G and a copy of B_n is easily produced. ☺ 1.6

1.7 Corollary: For every $\epsilon \in {}^\omega 2$ and every connected B_n-free graph G, if $G_\epsilon \subseteq G$ then $G_\epsilon = G$.

Proof : Suppose that $y \in G \setminus G_\epsilon$. By connectedness of G we may assume that y is connected by an edge to a vertex of G_ϵ. This contradicts 1.6

1.8 Claim: If $\epsilon \neq \nu$ are two members of ${}^\omega 2$ then G_ϵ and G_ν are not isomorphic.

Proof : Suppose that $f : G_\epsilon \to G_\nu$ is an isomorphism. We show that $\epsilon = \nu$. Clearly, f maps every highway in G_ϵ onto some highway in G_ν.

The highway $H(\epsilon)$ has to be mapped by f onto $H(\nu)$, both being the unique highways in their respective graphs of length $n+2$ with an end of degree $n+3$. As $l^\epsilon(0)$ is connected by an edge to the end of $H(\epsilon)$ that has degree $n+2$, we conclude that $f(l^\epsilon(0)) = l^\nu(0)$. We argue by induction on m that $H^\epsilon(m)$ is mapped by f onto the $H^\nu(m)$ and that $f(l^\epsilon(m+1)) = l^\nu(m+1)$.

If $m = 0$, we already showed that $f(l^\epsilon(0)) = l^\nu(0)$. Therefore $f(r^\epsilon(0)) \neq l^\nu(0)$. Also, $f(r^\epsilon(0)$ cannot lie on any of the highways in $T^\nu(0)$ which are part of a dead end, because both ends of $H^\epsilon(0)$ have degree $n+2$. Therefore necessarily $f(r^\epsilon(0)) = r^\nu(0)$ and consequently $H^\epsilon(0)$ is mapped by f onto the $H^\nu(0)$, with $f(l^\epsilon(1)) = l^\nu(1)$.

Similarly, if f maps $H^\epsilon(m)$ onto $h^\nu(m)$ with $f(l^\epsilon(m+1)) = l^\nu(m+1)$, it follows that f maps $H^\epsilon(m+1)$ onto $h^\nu(m+1)$ with $f(l^\epsilon(m+2)) = l^\nu(m+2)$.

As for all m we have established that $n-1+\epsilon(m) = n-1+\nu(m)$, we have shown that $\epsilon = \nu$. ☺ 1.8

1.9 Theorem: There is no universal B_n-free graph. In fact, the weak complexity of the class of countable B_n-free graphs equals $2^{\aleph_0}$.

Proof Suppose that $\{G_\alpha : \alpha \in I\}$ is a collection of less than $2^{\aleph_0}$ many countable B_n free graphs. By splitting each graph to its connected components we assume that each G_α is connected. Suppose that for every $\epsilon \in {}^\omega 2$ the graph G_ϵ constructed above is isomorphic to a subgraph of G_α for some $\alpha \in I$. By corollary 1.7 and the assumption just made, each G_ϵ is isomorphic to G_α for some $\alpha \in I$. By the pigeon hole principle there is a single G_α which is isomorphic to uncountably many G_ϵ. This contradicts claim1.8 ☺ 1.9

§2 Graphs without short cycles

In this section we show that the class of all graphs omitting all cycles of length at most $2k$ ($k \geq 2$) has no countable universal element.

2.1 Definition: Let S_k be the following graph: For five vertices $\{x_i : i \in Z_5\}$ indexed cyclically connect x_i to x_{i+1} by a simple path $x_i, y_{i,1} \ldots, y_{i,k-1}, x_{i+1}$.

2.2 Claim: If f_1 and f_2 are two embeddings of S_k into a graph G omitting all cycles of length at most $2k$ ($k \geq 2$) and $f_1(x_i) = f_2(x_i)$ for $i \in Z_5$ then $f_1 = f_2$.

Proof: Suppose for simplicity that f_1 is the inclusion, and suppose that $f_2 \neq f_1$. Let i be the least such that among $\{y_{i,j} : j < k\}$ there is a vertex v for which $v \neq f_2(v)$ and let $j(0)$ be the least such that $y_{i,j} \neq f_2(y_{i,j(0)})$. Let $j(1)$ be the the least $j > j(0)$ such that $x_{i,j(1)} = f_2(x_{i,j(1)}$. Now $x_{i,j(0)-1}, x_{i,j(0)}, \ldots, x_{i,j(1)}, f_2(x_{i,j(0)}), \ldots, f_2(x_{i,j(1)-1})$ forms a cycle of length $\leq 2k$ in G, contrary to the assumption. ☺

Let us define an infinite graph U by induction. For every natural m let $S(m)$ be a copy of S_k with vertices $x_i^m, y_{i,j}^m$ ($i \in Z_5, 1 \leq j < k$).

Let $U(0) = S(0)$. Suppose that $U(m)$ is defined and $S(m) \subseteq U(m)$. To obtain $U(m+1)$ adjoin freely $S(m+1)$ of to $S(m)$ by identifying x_i^{m+1} with with $y_{2i,[(k+1)/2]}^m$. Let $U = \bigcup U(m)$

2.3 Claim: (i) The graph U contains no cycles of length $\leq 2k$, and $\deg_U(v) \leq 3$ for all $v \in U$. (ii) If f_1 and f_2 are two embeddings of U into a graph G which contains no cycles of length at most $2k$ and $f_1(x_i^0) = f_2(x_i^0)$ for $i \in Z_5$ then $f_1 = f_2$.

Proof: (i) is clear. Suppose that f_1, f_2 are as stated. Using claim 2.2 inductively one sees that $f_1 = f_2$ ☺ 2.3

Let us choose, by induction on m, vertices v_m in U such that the distance in U between v_m and v_{m+1} is at least $2k+1$. For every $\epsilon \in {}^\omega 2$ let us construct a graph U_ϵ as follows:

let $u(m)$ be distinct vertices not in U. Connect $v(m)$ to $v(m+1)$ by an edge if $\epsilon(m) = 1$ and connect $u(m)$ by edges to $v(m), v(m+1)$ otherwise. Let $U_\epsilon = U \cup \{u(m) : m \in M\}$.

It is not hard to verify that each U_ϵ omits all cycles of length at most $2k$.

2.4 Theorem: For all $k \geq 2$ there is no universal countable graph in the class of all graphs omitting all cycles of length at most $2k$. In fact, the weak complexity of the class of all such countable graphs is $2^{\aleph_0}$.

Proof: Suppose to the contrary that $\{G_\alpha : \alpha \in I\}$ is a set of countable graphs, each omitting all cycles of length at most $2k$, with the property that every countable graph omitting all cycles of length at most $2k$ is isomorphic to a subgraph of G_α for at least one $\alpha \in I$, and assume that $|I| < 2^{\aleph_0}$. Fix an embedding f_ϵ of U_ϵ to some $G_{\alpha(\epsilon)}$. By the pigeon hole principle there is a single $\alpha \in I$ which equals $\alpha(\epsilon)$ for all $\epsilon \in A$ for some uncountable set $A \subseteq {}^{\omega}2$. For each $\epsilon \in A$ let $\eta(\epsilon) = \langle f_\epsilon(x_i^0) : i \in Z_5 \rangle$. As there are only countably many finite sequences of vertices in G_α, there are different $\epsilon, \nu \in A$ with $\eta(\epsilon) = \eta(\nu)$. But then it follows by claim 2.3 that $f_\epsilon \restriction U = f_\nu \restriction U$. Let m be such that $\epsilon(m) \neq \nu(m)$. The vertices $f_\epsilon(v(m)), f_\epsilon(u(m)), f_\epsilon(v(m+1))$ span a copy of C_3 in G_α, contrary to the assumption that G_α contains no cycles of length at most $2k$. ☺ 2.4

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