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Isoperimetrically Constrained  
Variational Problems**

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# On a Gap Phenomenon for Isoperimetrically Constrained Variational Problems

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**Abstract:** We consider functionals of the calculus of variations subjected to constraints of the form

$$\int_{\Omega} g(x, u) dx = 1.$$

We identify the relaxed problem and we show that, when a lack of compactness occurs, the constraint may relax to a gap term.

## 1. Introduction

The term *Lavrentiev phenomenon* refers to a surprising result first demonstrated by Lavrentiev in 1926 [29]. There he presented an example showing that it is possible for the variational integral of a two-point Lagrange problem which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, to possess an infimum on the dense subclass of  $C^1$  admissible functions that *strictly exceeds* its minimum value on the full admissible class. The global  $C^1$  regularity constraint on the admissible functions was thereby shown to incur an *infimum gap* in comparison with the relaxed problem in which this constraint has been removed. Since that time there have been additional works devoted to analyzing this gap phenomenon (see References), of which the paper [12] is most closely related to the work we present here.

The present article was stimulated by a result of Lezenina and Sobolevskii [30] in which a gap phenomenon having some analogy to the Lavrentiev phenomenon for free problems was encountered in an isoperimetric variational problem associated with a singular elliptic equation. We here demonstrate that for a large class of isoperimetrically constrained variational problems there is such a gap phenomenon, which has a natural interpretation as a relaxation effect, as was shown in Buttazzo and Mizel [12] to be also true of free problems for which the Lavrentiev phenomenon occurs. We prefer here to consider only the gap effect deriving from the *lack of compactness* and so, to avoid possible interactions with the gap due to regularity of admissible functions, we consider our problems as defined on the whole class of absolutely continuous functions. The result is that, after relaxation, the initial constraint becomes a kind of penalization term which, in several cases, can be explicitly computed.





## 2. The Relaxation Result

Let  $\Omega$  be the interval  $]0, 1[$ ; we consider  
 $W^{1,1}(\Omega)$  the space of all absolutely continuous functions on  $\Omega$ ;  
 $\mathcal{A}$  the class of all functions  $u \in W^{1,1}(\Omega)$  with  $u(0) = 0$ ;  
 $f(x, s, \zeta)$  a nonnegative Borel function from  $\Omega \times \mathbf{R} \times \mathbf{R}$  into  $\overline{\mathbf{R}}$  which is lower semicontinuous in  $(s, \zeta)$  and convex in  $\zeta$ ;  
 $g(x, s)$  a nonnegative Borel function from  $\Omega \times \mathbf{R}$  into  $\overline{\mathbf{R}}$  which is lower semicontinuous in  $s$ .  
 For every  $u \in \mathcal{A}$  we define

$$\begin{aligned} F(u) &= \int_{\Omega} f(x, u, u') dx \\ G(u) &= \int_{\Omega} g(x, u) dx \\ H(u) &= \begin{cases} F(u) & \text{if } G(u) = 1 \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

and we denote by  $\overline{H}$  the greatest functional on  $\mathcal{A}$  which is sequentially lower semicontinuous with respect to weak  $W_{loc}^{1,1}(\Omega)$  convergence and less than or equal to  $H$ . By the assumptions made on the integrand  $f$ , the functional  $F$  turns out to be sequentially lower semicontinuous, so that

$$\overline{H}(u) \geq F(u) \quad \forall u \in \mathcal{A}.$$

On the other hand, the equality  $\overline{H} = H$  is possible only if the constraint  $G(u) = 1$  is preserved in the relaxation, which does not occur in general when the integrand  $g(x, s)$  has singularities which prevent the compactness of embeddings. We want to write  $\overline{H}$  in the form

$$\overline{H}(u) = F(u) + L(u)$$

and to characterize the gap  $L$  explicitly. It is clear that, when the compactness condition

$$u_h \rightharpoonup u \text{ in } w - W_{loc}^{1,1}(\Omega), F(u_h) \leq c \Rightarrow G(u_h) \rightarrow G(u)$$

is fulfilled, then  $\overline{H} = H$ , that is

$$L(u) = \begin{cases} 0 & \text{if } G(u) = 1 \\ +\infty & \text{otherwise.} \end{cases}$$

In general, however, only the inequality  $G(u) \leq 1$  holds in the relaxed functional, by Fatou's lemma. Thus  $L(u) = +\infty$  whenever  $G(u) > 1$ ,  $L(u) = 0$  whenever  $G(u) = 1$ , but  $L(u)$  may take finite nonzero values on functions  $u \in \mathcal{A}$  such that  $G(u) < 1$ .

In order to characterize explicitly the gap functional  $L$  we introduce the following notations:

$$\begin{aligned} F_x(u) &= \int_0^x f(t, u, u') dt \\ G_x(u) &= z_u(x) = \int_0^x g(t, u) dt \\ V(x, s, z) &= \inf \{ F_x(u) : u \in \mathcal{A}, u(x) = s, G_x(u) = z \} \\ W(x, s, z) &= \liminf_{(\xi, \eta) \rightarrow (x, z)} V(x, \xi, \eta). \end{aligned}$$

The representation result for  $L$  is then the following.

**Theorem 2.1.** Assume the following conditions are fulfilled:

(2.1) for every  $\delta > 0$  there exists a function  $a_\delta \in L^1(\Omega)$  and a function  $\theta_\delta : \mathbf{R} \rightarrow \mathbf{R}$  with  $\theta_\delta(r)/r \rightarrow +\infty$  as  $r \rightarrow +\infty$  such that

$$f(x, s, \zeta) \geq \theta_\delta(|\zeta|) - a_\delta(x) \quad \forall x \in ]\delta, 1[, \forall s \in \mathbf{R}, \forall \zeta \in \mathbf{R};$$

(2.2)  $g(x, s)$  is continuous in  $s$ , and for every  $\delta > 0$  there exists a function  $\gamma_\delta(x, t)$  increasing in  $t$  and integrable in  $x$  such that

$$g(x, s) \leq \gamma_\delta(x, |s|) \quad \forall x \in ]\delta, 1[, \forall s \in \mathbf{R}.$$

Then for every  $u \in \mathcal{A}$  with  $G(u) \leq 1$  we have

$$(2.3) \quad L(u) \geq \limsup_{x \rightarrow 0} W \left( x, u(x), 1 - \int_x^1 g(t, u) dt \right)$$

$$(2.4) \quad L(u) \leq \liminf_{x \rightarrow 0} V \left( x, u(x), 1 - \int_x^1 g(t, u) dt \right).$$

Hence

$$L(u) = \lim_{x \rightarrow 0} V \left( x, u(x), 1 - \int_x^1 g(t, u) dt \right)$$

if  $V(x, \cdot, \cdot)$  is lower semicontinuous on  $\mathbf{R} \times \mathbf{R}$ .

**Proof.** Consider any sequence  $(u_h)$  in  $\mathcal{A}$  with  $G(u_h) = 1$  and  $u_h \rightarrow u$  weakly in  $W_{loc}^{1,1}(\Omega)$ ; then for every  $\delta > 0$  we have

$$\begin{aligned} F(u_h) &= \int_0^\delta f(x, u_h, u_h') dx + \int_\delta^1 f(x, u_h, u_h') dx \geq \\ &\geq V \left( \delta, u_h(\delta), 1 - \int_\delta^1 g(x, u_h) dx \right) + \int_\delta^1 f(x, u_h, u_h') dx \geq \\ &\geq W \left( \delta, u_h(\delta), 1 - \int_\delta^1 g(x, u_h) dx \right) + \int_\delta^1 f(x, u_h, u_h') dx. \end{aligned}$$

Passing to the limit as  $h \rightarrow +\infty$  gives, by using (2.1) and (2.2)

$$\bar{H}(u) \geq W \left( \delta, u(\delta), 1 - \int_\delta^1 g(x, u) dx \right) + \int_\delta^1 f(x, u, u') dx$$

and, as  $\delta \rightarrow 0$ ,

$$\bar{H}(u) \geq \limsup_{\delta \rightarrow 0} W \left( \delta, u(\delta), 1 - \int_\delta^1 g(x, u) dx \right) + F(u)$$

so that inequality (2.3) is proved.

In order to prove inequality (2.4) let  $x_h \rightarrow 0$  be such that

$$(2.5) \quad \liminf_{x \rightarrow 0} V \left( x, u(x), 1 - \int_x^1 g(t, u) dt \right) = \lim_{h \rightarrow +\infty} V \left( x_h, u(x_h), 1 - \int_{x_h}^1 g(t, u) dt \right) < +\infty$$

and let  $v_h \in \mathcal{A}$  be such that

$$(2.6) \quad v_h(x_h) = u(x_h), \quad \int_0^{x_h} g(t, v_h) dt = 1 - \int_{x_h}^1 g(t, u) dt$$

$$(2.7) \quad \int_0^{x_h} f(t, v_h, v_h') dt \leq \frac{1}{h} + V \left( x_h, u(x_h), 1 - \int_{x_h}^1 g(t, u) dt \right).$$

Define

$$u_h(x) = \begin{cases} v_h(x) & \text{if } x \in ]0, x_h[ \\ u(x) & \text{if } x \in [x_h, 1[; \end{cases}$$

we have  $u_h \in \mathcal{A}$ ,  $G(u_h) = 1$  by (2.6), and  $u_h \rightharpoonup u$  weakly in  $W_{loc}^{1,1}(\Omega)$  by (2.1). Furthermore, by (2.5) and (2.7)

$$\begin{aligned} \overline{H}(u) &\leq \liminf_{h \rightarrow +\infty} F(u_h) = \\ &= \liminf_{h \rightarrow +\infty} \left[ \int_0^{x_h} f(t, v_h, v'_h) dt + \int_{x_h}^1 f(t, u, u') dt \right] \leq \\ &\leq \liminf_{h \rightarrow +\infty} \left[ \frac{1}{h} + V \left( x_h, u(x_h), 1 - \int_{x_h}^1 g(t, u) dt \right) + F(u) \right] = \\ &= \liminf_{x \rightarrow 0} V \left( x, u(x), 1 - \int_x^1 g(t, u) dt \right) + F(u). \end{aligned}$$

Hence inequality (2.4) is also proved. ■

**Remark 2.2.** Similar results with similar proofs hold if we replace the class  $\mathcal{A}$  with  $W^{1,1}(\Omega)$  or  $W_0^{1,1}(\Omega)$ .

**Remark 2.3.** The assumptions of Theorem 2.1 ensure that the lack of compactness may occur only at the origin. In a similar way we can treat problems in which the lack of compactness occurs at a finite number of points, but we do not know the form of the relaxed functional  $\overline{H}$  when the function  $g(x, s)$  may allow more general "singular sets".

### 3. Some Examples

In order to provide a source of isoperimetrically constrained one-dimensional variational problems possessing a gap it will be useful to present a dynamic programming type of result by studying the link between the value function  $V(x, s, z)$  and the solutions of the associated Hamilton-Jacobi equation. In the following we denote by  $f^*(x, s, \cdot)$  the Fenchel conjugate of  $f(x, s, \cdot)$ . We will also suppose that

$$(3.1) \quad g(x, s) > 0 \quad \text{for a. e. } x \in \Omega \text{ and for all } s \neq 0;$$

this implies that for every  $x \in \Omega$  and  $s \neq 0$

$$V(x, s, 0^+) = +\infty.$$

**Theorem 3.1.** Suppose that  $U(t, s, z)$  is a positive  $C^1$  solution on  $D = \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$  of the Hamilton-Jacobi equation

$$(3.2) \quad \begin{cases} U_t + g(t, s)U_z + f^*(t, s, U_s) = 0 \\ U(t, s, 0^+) = +\infty \end{cases}$$

such that

$$(3.3) \quad \lim_{\substack{(t,s,z) \rightarrow (0,0,0) \\ (t,s,z) \in D_j}} U(t, s, z) = 0 \quad \forall j \in \mathbf{N}$$

where  $(D_j)$  is an expanding sequence of bounded subdomains of  $D$  such that  $(0, 0, 0) \in \overline{D_j}$ ,  $U$  is  $C^1$ -bounded on  $D_j$ , and  $\cup_j D_j = D$ .

Putting

$$\mathcal{B} = \left\{ u \in \mathcal{A} : \liminf_{x \rightarrow 0} U(x, u(x), z_u(x)) = 0 \right\},$$

one has the following inequality for every  $u \in \mathcal{B}$

$$(3.4) \quad F_x(u) \geq U(x, u(x), z_u(x)) \quad \forall x \in [0, 1].$$

Furthermore, if for some  $(x, s, z) \in D$  there is a  $u \in \mathcal{B}$  such that  $u(x) = s$ ,  $z_u(x) = z$ , and (3.4) holds with equality, then  $u$  is a minimizer for the problem

$$(\mathcal{P}_{x,s,z}) \quad \min \{ F_x(u) : u \in \mathcal{B}, u(x) = s, z_u(x) = z \}.$$

**Proof.** Fix  $u \in \mathcal{B}$ ; then for every  $x, y$  with  $0 < y < x \leq 1$  the graph  $(t, u(t), z_u(t))$ ,  $t \in [y, x]$  lies in  $D_j$  for some  $j \in \mathbb{N}$ . Then by the chain rule for the function  $U$  on  $D_j \cap ([y, x] \times \mathbb{R}^+ \times \mathbb{R}^+)$  we get

$$U(x, u(x), z_u(x)) - U(y, u(y), z_u(y)) = \int_y^x [U_t + g(t, u(t))U_z + u'(t)U_s] dt.$$

Hence, by approaching the liminf as  $y \rightarrow 0^+$  and using the definition of  $\mathcal{B}$  and property (3.3) we can write

$$U(x, u(x), z_u(x)) - F_x(u) = \int_0^x [U_t + g(t, u)U_z + u'U_s - f(t, u, u')] dt.$$

Now, by the Hamilton-Jacobi equation (3.2), the integrand is nonpositive everywhere on  $]0, x]$ , whence (3.4) follows. The conclusion that  $u$  is a minimizer of problem  $(\mathcal{P}_{x,s,z})$  when equality holds in (3.4) is now immediate. ■

**Remark 3.2.** When problem (3.2) has multiple solutions the above result applies to the maximal solution, in particular, as the only candidate for the value function of the problem.

Our next result demonstrates a homogeneity property of the value functions  $V$  of  $(\mathcal{P}_{x,s,z})$  and  $V$  of  $(\hat{\mathcal{P}}_{x,s,z})$ , the analogue of  $(\mathcal{P}_{x,s,z})$  on the full admissible class  $\mathcal{A}$ .

**Theorem 3.3.** Suppose that the integrands  $f$  and  $g$  satisfying (2.1) and (2.2) have the form

$$\begin{aligned} f(t, s, \zeta) &= t^\alpha |s|^\gamma h\left(\frac{t\zeta}{s}\right) \quad \forall s \neq 0 \\ g(t, s) &= t^\beta |s|^\delta \end{aligned}$$

with  $\alpha, \beta \in \mathbb{R}$  and  $\gamma, \delta > 0$ . Then the value function  $V$  associated with the problems  $(\hat{\mathcal{P}}_{x,s,z})$  satisfies

$$(3.5) \quad V(x, s, z) = x^{\alpha+1} |s|^\gamma V(1, 1, zx^{-\beta-1} |s|^{-\delta}) \quad \forall s \neq 0.$$

Likewise, the value function  $Y$  associated with  $(\mathcal{P}_{x,s,z})$  satisfies

$$Y(x, s, z) = x^{\alpha+1} |s|^\gamma Y(1, 1, zx^{-\beta-1} |s|^{-\delta}) \quad \forall s \neq 0.$$

**Proof.** Given  $u \in \mathcal{A}$  with  $u(x) = s$  and  $z_u(x) = z$  define  $v(t) = u(tx)$  so that

$$v'(t) = xu'(tx), \quad z_v(1) = x^{-\beta-1} z_u(x), \quad F_1(v) = x^{-\alpha-1} F_x(u).$$

It follows from these relations that

$$(3.6) \quad V(x, s, z) = x^{\alpha+1} V(1, s, zx^{-\beta-1}).$$

On the other hand, given  $\lambda \in \mathbb{R}$  define  $w(t) = \lambda u(t)$  so that

$$w'(t) = \lambda u'(t), \quad z_w(x) = |\lambda|^\delta z_u(x), \quad F_x(w) = |\lambda|^\gamma F_x(u).$$

These relations yield

$$(3.7) \quad V(x, s, z) = |\lambda|^{-\gamma} V(x, \lambda s, z|\lambda|^\delta).$$

The validity of (3.5) now follows from (3.6) and (3.7). The proof for  $Y$  is similar. ■

As a consequence of Theorems 3.1 and 3.3 we obtain the following result.

**Theorem 3.4.** *Let  $f$  and  $g$  be as in Theorem 3.3. Then every solution  $U$  of (3.2) having the form*

$$U(t, s, z) = t^{\alpha+1} s^\gamma R(zt^{-\beta-1} s^{-\delta})$$

for some positive function  $R \in C^1(\mathbf{R}^+)$ , is such that  $R$  is a positive solution of the ordinary differential equation

$$(3.8) \quad \begin{cases} (\alpha + 1)R(w) - (\beta + 1)wR'(w) + R'(w) + h^*(\gamma R(w) - \delta wR'(w)) = 0 \\ R(0^+) = +\infty. \end{cases}$$

In particular, for every  $u \in \mathcal{B}$  satisfying  $u(x) = s$  and  $z_u(x) = z$  one has

$$F_x(u) \geq x^{\alpha+1} |s|^\gamma R(zx^{-\beta-1} |s|^{-\delta}),$$

and equality ensures that  $u$  is a minimizer for  $(\mathcal{P}_{x,s,z})$ .

**Corollary 3.5.** *Let  $f$  and  $g$  be as in Theorem 3.3 with  $\alpha = \beta$ ,  $\gamma = \delta$  and assume that  $h \in C^1(\mathbf{R})$ , is strictly convex, and satisfies  $h(\zeta)/\zeta \rightarrow +\infty$  as  $|\zeta| \rightarrow +\infty$ . Then equation (3.8) becomes*

$$(3.9) \quad \begin{cases} h^*(\gamma(R(w) - wR'(w))) + (\alpha + 1)(R(w) - wR'(w)) + R'(w) = 0 \\ R(0^+) = +\infty. \end{cases}$$

The maximal solution of (3.9) on  $]0, +\infty[$  is given by the convex function

$$R(w) = wh \left( \frac{1}{\gamma w} - \frac{\alpha + 1}{\gamma} \right).$$

corresponding to the  $C^1$  solution of (3.2)

$$(3.10) \quad U(x, s, z) = zh \left( \frac{s^\gamma x^{\alpha+1}}{\gamma z} - \frac{\alpha + 1}{\gamma} \right).$$

Moreover, the unique  $\mathcal{B}$ -optimal trajectory  $(u, z_u)$  for  $(\mathcal{P}_{1,s,z})$  corresponding to the above solution is given by

$$(3.11) \quad u(t) = st^{(|s|^\gamma/z - \alpha - 1)/\gamma}, \quad z_u(t) = zt^{|s|^\gamma/z}.$$

Finally, if for some  $a > 0$

$$(3.12) \quad \frac{1}{th^{-1}(t)} \in L^1(a, +\infty)$$

where  $h^{-1}$  denotes the inverse of the restriction of  $h$  to some interval  $[A, +\infty[$  on which it is strictly monotone, then

$$V(x, s, z) = Y(x, s, z) = U(x, s, z)$$

and the trajectory in (3.11) is  $(\mathcal{P}_{1,s,z})$  optimal.

**Proof.** By virtue of superlinearity and strict convexity of  $h$ , we have for every  $\zeta$

$$(3.13) \quad h^*(h'(\zeta)) = \zeta h'(\zeta) - h(\zeta) \quad \text{and} \quad (h^*)'(\zeta) = (h')^{-1}(\zeta)$$

so that the supremum yielding  $h^*(\gamma(R(w) - wR'(w)))$  in (3.9) is attained at (assuming existence of a minimizer  $u$  for problem  $(\mathcal{P}_{1,s,z})$ )

$$(3.14) \quad \frac{tu'(t)}{u(t)} = (h')^{-1}(\gamma(R(w) - wR'(w))) = (h^*)'(\gamma(R(w) - wR'(w))).$$

On the other hand, by formally differentiating the implicit Clairaut type equation (3.9) one obtains (we recall that, by Rockafellar [36], Theorem 26.5,  $h^*$  is  $C^1$ )

$$R''(w) \left[ 1 - (\alpha + 1)w - \gamma w (h^*)'(\gamma(R(w) - wR'(w))) \right] = 0.$$

Since  $R'' = 0$  is inconsistent with  $R(0^+) = +\infty$ , we have

$$(3.15) \quad (h^*)'(\gamma(R(w) - wR'(w))) = \frac{1}{\gamma w} - \frac{\alpha + 1}{\gamma}.$$

Then, from (3.14) and (3.15) we obtain

$$(3.16) \quad u'(t) = \frac{u(t)}{\gamma t} \left( \frac{1}{w} - \alpha - 1 \right).$$

Moreover, inserting (3.15) into (3.9), and taking into account (3.13) gives as singular solution the convex function

$$R(w) = wh \left( \frac{1}{\gamma w} - \frac{\alpha + 1}{\gamma} \right)$$

which proves (3.10) by Theorem 3.4. Now, recalling that  $w = zt^{-1-\alpha}|s|^{-\gamma}$ , while  $z'_u(t) = g(t, u(t)) = t^\alpha |u(t)|^\gamma$ , we obtain by (3.16)

$$\begin{aligned} w'(t) &= z'_u t^{-\alpha-1} |u|^{-\gamma} - (\alpha + 1) z_u t^{-\alpha-2} |u|^{-\gamma} - \gamma z_u t^{-\alpha-1} |u|^{-\gamma-1} \operatorname{sgn}(u) u' = \\ &= \frac{1}{t} - \frac{\alpha + 1}{t} w - \frac{\gamma w}{t} \left( \frac{1}{\gamma w} - \frac{\alpha + 1}{\gamma} \right) = 0. \end{aligned}$$

That is,  $w$  is constant along an optimal trajectory. In particular,  $w(t) = w(1) = z|s|^{-\gamma}$ , so that (3.11) follows from (3.16).

Finally, we show that when (3.12) is satisfied, then  $Y = V$  by proving that for each  $u \in \mathcal{A} \setminus \mathcal{B}$  we have  $z_u(x) = +\infty$  for every  $x \in \Omega$ . By definition of  $\mathcal{B}$  it follows that for some positive  $b, \sigma$  one has

$$U(x, u(x), z_u(x)) = z_u(x) h \left( \frac{x^{\alpha+1} |u(x)|^\gamma}{\gamma z_u(x)} - \frac{\alpha + 1}{\gamma} \right) \geq b \quad \forall x \in ]0, \sigma[.$$

If we suppose  $z_u(1) < +\infty$ , then by taking  $\sigma$  sufficiently small, we can ensure that the argument of  $h$  lies in the semi-axis  $[A, +\infty[$  where  $h$  is invertible. Thus, since  $z'_u(x) = x^\alpha |u(x)|^\gamma$ , we have the relation

$$\frac{x z'_u(x)}{z_u(x)} \geq \gamma h^{-1} \left( \frac{b}{z_u(x)} \right) + \alpha + 1 \quad \forall x \in ]0, \sigma[.$$

Since  $z_u(0^+) = 0$  and  $h$  is superlinear, we may deduce from this (by decreasing  $\sigma$ ) that

$$(3.17) \quad \frac{z'_u(x)}{z_u(x) h^{-1}(b/z_u(x))} \geq \frac{\gamma}{2x} \quad \forall x \in ]0, \sigma[.$$

Denoting by  $k(z_u)$  the coefficient of  $z_u'$  in (3.17) we have by (3.12) that  $k \in L^1(0, \varepsilon)$  for  $\varepsilon$  sufficiently small. Letting  $K(t) = \int_0^t k(s) ds$ , it follows from the local absolute continuity of  $z_u$  by the chain rule (cf. e.g. [34]) and from (3.17) that

$$K(z_u(x)) - K(z_u(y)) \geq \frac{\gamma}{2} \log(x/y) \quad 0 < y < x < \sigma$$

which gives, as  $y \rightarrow 0^+$ ,  $z_u(x) = +\infty$  for every  $x$  small enough. ■

We now examine a special subclass of the integrands discussed in Corollary 3.5 for which an explicit estimate is available.

**Corollary 3.6.** Let  $p > n$  and let, with the notation of Corollary 3.5,  $h(\zeta) = |\zeta|^p$ ,  $\alpha = n - 1 - p$ ,  $\gamma = p$ , so that

$$f(t, \zeta) = t^{n-1} |\zeta|^p, \quad g(t, s) = t^{n-p-1} |s|^p.$$

Consider the variational problem

$$(\mathcal{P}_{1,s,z}) \quad \min \{ F_1(u) : u \in \mathcal{B}, u(1) = s, z_u(1) = z \}.$$

Then an optimal trajectory is given by

$$u(t) = st^{(|s|^p/z - n + p)/p}, \quad z_u(t) = zt^{|s|^p/z}$$

corresponding to a minimal cost given by

$$V(x, s, z) = Y(x, s, z) = \frac{z}{p^p} \left| \frac{|s|^p}{zx^{p-n}} - n + p \right|^p, \quad (x, s, z) \in D.$$

In particular,

$$V(1, 0^+, z) = Y(1, 0^+, z) = \left( 1 - \frac{n}{p} \right)^p z.$$

**Corollary 3.7.** Let  $d(t) = \text{dist}(t, \partial\Omega)$  and consider the problem

$$(\mathcal{P}_{1,z}) \quad \min \left\{ \int_0^1 |u'|^p dt : u \in W_0^{1,1}(0, 1), \int_0^1 \frac{|u(t)|^p}{|d(t)|^p} dt = z \right\}.$$

Then, if  $V$  is the value function of Corollary 3.6 with  $n = 1$ , the infimum  $m$  is given by  $m = 2V(1/2, 0^+, z/2)$  and there is a minimizing sequence which converges to  $u_0 \equiv 0$ .

**Proof.** By the symmetry in  $t$  the infimum  $m$  can be obtained as

$$\begin{aligned} m &= \inf_{s, u, v} \left\{ \int_0^{1/2} |u'|^p dt + \int_0^{1/2} |v'|^p dt : u, v \in \Gamma_s \right\} = \\ &= \inf_{s, z_1, z_2} \{ V(1/2, s, z_1) + V(1/2, s, z_2) : z_1 + z_2 = z \} \end{aligned}$$

where  $\Gamma_s$  is the class of functions such that  $u(0) = v(0) = 0$ ,  $u(1/2) = v(1/2) = s$ ,  $z_u(1/2) + z_v(1/2) = z$ . By the convexity of  $V(x, s, \cdot)$  we get  $z_1 = z_2 = z/2$  so that

$$m = \inf_s 2V(1/2, s, z/2) = 2V(1/2, 0^+, z/2) = (1 - 1/p)^p z$$

and a minimizing sequence (as  $s \rightarrow 0^+$ ) is

$$u_s(x) = \begin{cases} s(2x)^{(2^{-p}s^p/z+p-1)/p} & \text{if } 0 \leq x \leq 1/2 \\ s(2-2x)^{(2^{-p}s^p/z+p-1)/p} & \text{if } 1/2 \leq x \leq 1. \quad \blacksquare \end{cases}$$



**Remark 3.8.** In accordance with Theorem 2.1 we have in the example of Corollary 3.7  $F(u_0) = 0 < m$  and  $G(u_0) = 0 < z$ . For  $p = 2$  this is the one-dimensional case of the gap phenomenon noted by Lezenina and Sobolevskii in [30]. An analogous problem in dimension  $n > 1$  and for  $\Omega$  the punctured unit ball in  $\mathbb{R}^n$  is

$$\min \left\{ \int_{\Omega} |Du|^p dx : u \in W_0^{1,1}(\Omega), \int_{\Omega} |x|^{-p} |u|^p dx = z \right\}.$$

We conjecture that there is a radially symmetric minimizer for the relaxed problem and thus the gap is as described in Corollary 3.6.

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