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**Ginzburg-Landau Equation and
Motion by Mean Curvature, II:
Development of the Initial
Interface**

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Ginzburg-Landau equation and motion by mean
curvature, II : development of the initial interface

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1 Introduction

In an earlier paper [12], I have studied the asymptotic behavior of Ginzburg-Landau equation,

$$(1.1) \quad u_t^\epsilon - \Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon) = 0, \quad (0, \infty) \times \mathcal{R}^d,$$

$$(1.2) \quad u^\epsilon(0, x) = u_0^\epsilon(x), \quad x \in \mathcal{R}^d.$$

The nonlinearity f is the derivative of a bi-stable potential W :

$$(1.3) \quad W(u) = \frac{1}{2}(u^2 - 1)^2, \quad f(u) = W'(u) = 2u(u^2 - 1).$$

In [12], I proved that there are two open, disjoint subsets \mathcal{P}, \mathcal{N} of $(0, \infty) \times \mathcal{R}^d$ and a subsequence ϵ_n satisfying,

- a. $u^{\epsilon_n} \rightarrow 1$, uniformly on bounded subsets of \mathcal{P} ,
- b. $u^{\epsilon_n} \rightarrow -1$, uniformly on bounded subsets of \mathcal{N}
- c. $\Gamma = \text{complement of } (\mathcal{P} \cup \mathcal{N})$ has Hausdorff dimension d and it moves by mean curvature in the sense defined in [12], [1].

This convergence result generalizes the previous results of Rubinstein, Steinberg and Keller [10], DeMottoni and Schatzman [8], Chen [2], Evans, Soner and Souganidis [4], Barles, Soner and Souganidis [1] and Ilmanen [7]. For more information on the Ginzburg-Landau equation, the weak theories for the mean curvature flow and other related topics we refer the reader to the introduction of the companion paper [12] and the references therein.

The above result was proved under the assumption (c.f. (2.6) in [12]) that for every $\delta > 0$ there are positive constants K_δ and η such that for every continuous function φ ,

$$(A) \quad \sup\left\{ \int |\varphi(x)| \mu^\epsilon(dx; t) : \epsilon \in (0, 1), t \in \left[\delta, \frac{1}{\delta}\right] \right\} \\ \leq K_\delta \sup\{|\varphi(x)| e^{\eta|x|} : x \in \mathcal{R}^d\}$$

where

$$(1.4) \quad \mu^\epsilon(dx; t) = \left[\frac{\epsilon}{2} |Du^\epsilon(t, x)|^2 + \frac{1}{\epsilon} W(u^\epsilon(t, x)) \right] dx.$$

The main purpose of this paper is to verify (A) under some reasonable conditions on the initial data u_0^ϵ . This analysis requires a detailed description of $u^\epsilon(t, x)$ near the initial interface. Such an analysis have already been carried out by DeMottoni and Schatzman [9] and by Chen [2]. However, the condition (A) can not be directly obtained from the results of [2, 9].

There are two key estimates in the proof of (A). The first one is a detailed description of $u^\epsilon(t, x)$ near the initial interface; Theorem 4.1, below. This result is a sharper version of a result of DeMottoni and Schatzman [9] and its proof is similar to Lemma 4.1 in [5]. The description obtained in Theorem 4.1 is of independent interest. The second key step in the proof of (A) is a gradient estimate; Theorem 5.1, below.

The paper is organized as follows. In the next section the main result of this paper is described. In section 3, a result of DeMottoni and Schatzman is recalled and an easy gradient bound is proved. The behavior of $u^\epsilon(t, x)$ near the initial interface is analyzed in Section 4 and a second gradient estimate is obtained in Section 5. A proof of the main theorem is given in the last section.

2 Main Result

Multiply (1.1) by ϵu_t^ϵ , integrate and use integration by parts to obtain,

$$(2.1) \quad E^\epsilon(t_1) - E^\epsilon(t_2) = -\epsilon \int_{t_1}^{t_2} \int_{\mathcal{R}^d} (u_t^\epsilon)^2 dx dt, \quad t_1 > t_2,$$

where

$$E^\epsilon(t) = \mu^\epsilon(\mathcal{R}^d; t) = \int_{\mathcal{R}^d} \left[\frac{\epsilon}{2} |Du^\epsilon(t, x)|^2 + \frac{1}{\epsilon} W(u^\epsilon(t, x)) \right] dx.$$

Hence (A) holds with $\eta = 0$ provided that $E^\epsilon(0)$ is bounded in ϵ . In particular, an elementary computation shows that $E^\epsilon(0)$ is bounded in ϵ , if there are a function z_0^ϵ , a constant $\lambda \geq 1$ and a bounded open set Ω of finite perimeter (c.f. [3, 6]) satisfying,

$$u_0^\epsilon(x) = g\left(\frac{z_0^\epsilon(x)}{\epsilon}\right), \quad g(r) = \tanh(r),$$

$$|Dz_0^\epsilon| \leq \lambda, \quad \frac{1}{\lambda} d(x) \leq z_0^\epsilon(x) \leq \lambda d(x),$$

where $d(x)$ is the signed distance between x and the boundary of Ω .

When u_0^ϵ is independent of ϵ , we generally do not expect $E^\epsilon(0)$ to be bounded in ϵ . Indeed let $u_0^\epsilon \equiv \beta$ for some constant $\beta \neq \pm 1$. Then $u^\epsilon(t, x) = w^\epsilon(t)$ and $E^\epsilon(t) = +\infty$ for every $t \geq 0$ and $\epsilon > 0$. However, the condition (A) holds with any $\eta > 0$.

In the remainder of this paper, we assume that:

$$(2.2a) \quad u_0^\epsilon \text{ is independent of } \epsilon, \text{ i.e. } u_0^\epsilon = u_0,$$

$$(2.2b) \quad u_0 \in C_b^3(\mathcal{R}^d), \quad |u_0(x)| < 1,$$

$$(2.2c) \quad \Gamma_0 = \{x \in \mathcal{R}^d : u_0(x) = 0\} \text{ is bounded,}$$

$$(2.2d) \quad \inf_{\Gamma_0} |Du_0| > 0,$$

$$(2.2e) \quad \limsup_{R \rightarrow 0} \inf_{|x| \geq R} |u_0(x)| > 0,$$

where $C_b^3(\mathcal{R}^d)$ is the set of all bounded functions that are thrice continuously differentiable with bounded derivatives. Observe that (2.2b,c,d) imply that Γ_0 is a C^2 manifold. The main goal of this paper is to prove (A) under the above hypotheses, see Theorem 6.1 below.

3 Preliminaries

Let $d_0(x)$ be the signed distance between x and Γ_0 . Choose $\lambda > 0$ such that

$$(3.1) \quad d_0 \in C^2(\Omega_\lambda), \quad \Omega_\lambda = \{x \in \mathcal{R}^d : |d_0(x)| < 2\lambda\}.$$

We now recall a result of DeMottioni and Schatzman [9, Theorem 5].

Theorem 3.1 *For every $\delta, m > 0$ there are $C_1, C_2 > 0$ such that for every*

$$(3.2) \quad t \in I_\epsilon := [C_1\epsilon^2 \ln(\frac{1}{\epsilon}), C_2\epsilon^{\frac{1}{2}}],$$

we have,

$$(3.3) \quad |u^\epsilon(t, x) - q(\frac{d_0(x)}{\epsilon})| \leq \delta, \quad \text{if } |d_0(x)| \leq \lambda,$$

$$(3.4) \quad |u^\epsilon(t, x) - \text{sign}[u_0(x)]| \leq \epsilon^m, \quad \text{if } |d_0(x)| \geq \lambda.$$

Recall that $q(r) = \tanh(r)$. In the remainder of this paper C_1, C_2 denote the constants constructed in Theorem 3.1 with $m = 2$ and $\delta = 1/8$. Also set

$$(3.5) \quad C_3 = q^{-1}(7/8).$$

Fix $t \in I_\epsilon$. Then whenever $d(x) \in [\epsilon C_3, \lambda]$, (3.3) yields

$$u^\epsilon(t, x) \geq q^{-1}(\frac{d(x)}{\epsilon}) - \delta \geq \frac{3}{4}$$

Also if $d(x) \geq \lambda$, (3.4) implies the above inequality, provided that $\epsilon^2 < 1/4$. Hence

$$(3.6) \quad u^\epsilon(t, x) \geq 3/4, \quad \forall \epsilon \leq 1/2, \quad t \in I_\epsilon, \quad d(x) \geq \epsilon C_3.$$

Similarly,

$$(3.7) \quad u^\epsilon(t, x) \leq -3/4, \quad \forall \epsilon \leq 1/2, \quad t \in I_\epsilon, \quad d(x) \leq -\epsilon C_3.$$

We close this section with a simple gradient estimate.

Lemma 3.1 *There is a constant K , independent of ϵ , satisfying,*

$$(3.8) \quad |Du^\epsilon(t, x)| \leq \frac{K}{\epsilon}.$$

Proof: Since $|u_0| \leq 1$, $|u^\epsilon(t, x)| \leq 1$ for all (t, x) . Set

$$g(t, x) = \frac{1}{\epsilon^2} f(u^\epsilon(t, x)).$$

Then for all $0 \leq \tau \leq t$,

$$(3.9) \quad \begin{aligned} u^\epsilon(t, x) &= [G(t - \tau, \cdot) * u^\epsilon(\tau, \cdot)](x) \\ &+ \int_\tau^t [G(t - s - \tau, \cdot) * g(s, \cdot)](x) ds, \end{aligned}$$

where $*$ denotes the convolution and G is the heat kernel, i.e.,

$$G(\tau, y) = (4\pi\tau)^{-\frac{d}{2}} \exp\left(-\frac{|y|^2}{4\tau}\right).$$

Now, differentiate (3.9) with respect x_j and use the properties of the convolution and the heat kernel to obtain,

$$\begin{aligned} |u_{x_j}^\epsilon(t, x)| &\leq \|D_j G(t - \tau, \cdot)\|_{L^1} \|u^\epsilon(\tau, \cdot)\|_{L^\infty} + \int_\tau^t \|D_j(t - s - \tau, \cdot)\|_{L^1} \|g\|_{L^\infty} dx, \\ &\leq \frac{C}{\sqrt{t - \tau}} + \frac{C}{\epsilon^2} \sqrt{t - \tau}, \end{aligned}$$

where C is an appropriate constant. Choose $\tau = t - \epsilon^2$ to obtain (3.8). ■

4 Behaviour near the interface

In this section we prove a sharper version of (3.3), (3.4). Our approach is very similar to [5, Lemma 4.1]. Let λ be as in (3.1) and set

$$(4.1) \quad t_1 = C_1 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right).$$

Theorem 4.1 *There are $\mu, K > 0$ such that for sufficiently small $\epsilon > 0$,*

$$(4.2) \quad u^\epsilon(t, x) \geq W(t - t_1, d_0(x)), \quad \forall t \in I_\epsilon, d_0(x) \in [\epsilon C_3, \lambda],$$

$$(4.3) \quad u^\epsilon(t, x) \leq -W(t - t_1, |d_0(x)|), \quad \forall t \in I_\epsilon, d_0(x) \in [-\lambda, -\epsilon C_3],$$

where

$$W(t, d) = \max\left\{q\left(\frac{d - Kt}{\epsilon} - K\right) - K\epsilon - \frac{1}{4} \exp\left(-\frac{\mu t}{\epsilon}\right), \frac{3}{4}\right\}.$$

Proof: We will prove only (4.2). The proof of (4.3) is similar.

1. In view of (3.1) there is $d \in C_b^2(\mathcal{R}^d)$ satisfying

$$(4.4) \quad d(x) = d_0(x), \quad \text{if } |d_0(x)| \leq \lambda,$$

$$(4.5) \quad |d(x)| \geq \lambda, \quad \text{if } |d_0(x)| \geq \lambda,$$

$$(4.6) \quad |Dd(x)| \leq 1, \quad \forall x.$$

For $\xi(t), p(t) \geq 0$ (to be determined later) define

$$v(t, x) = q\left(\frac{d(x) - \epsilon C_3 - \xi\left(\frac{t}{\epsilon}\right)}{\epsilon}\right) - p\left(\frac{t}{\epsilon}\right),$$

where C_3 is as in (3.5).

We will show that for appropriately chosen $\xi(\cdot), p(\cdot)$ and a sufficiently small $\epsilon > 0$, v is a subsolution of (1.1) on $\{v \geq 0\}$. Indeed a direct computation shows that,

$$\begin{aligned}
I &:= v_t - \Delta v + \frac{1}{\epsilon^2} f(v), \\
&= \frac{1}{\epsilon} q'(\dots) \left[-\frac{1}{\epsilon} \xi'(\frac{t}{\epsilon}) - \Delta d(x) \right] - \frac{1}{\epsilon} p'(\frac{t}{\epsilon}), \\
&\quad + \frac{1}{\epsilon^2} [f(v) - q''(\dots) |Dd|^2],
\end{aligned}$$

where (\dots) denotes $[d(x) - \epsilon C_3 - \xi(\frac{t}{\epsilon})]/\epsilon$.

2. Since $q(\dots) = v + p$ and $p \geq 0$, $q(\dots) \geq 0$ whenever $v(t, x) \geq 0$. Therefore on $\{v \geq 0\}$, $q''(\dots) \leq 0$ and (4.6) yields

$$q''(\dots) |Dd|^2 \geq q''(\dots) = f(q(\dots)).$$

So on $\{v \geq 0\}$ we have,

$$(4.7) \quad I \leq -\frac{1}{\epsilon^2} q'(\dots) \xi'(\frac{t}{\epsilon}) - \frac{1}{\epsilon} p'(\frac{t}{\epsilon}) + \frac{1}{\epsilon^2} [f(v) - f(q(\dots))] + \frac{\beta}{\epsilon},$$

where $\beta := \|q'\|_\infty \|\Delta d\|_\infty$.

3. Set

$$(4.8) \quad \mu = f'(\frac{5}{8}) = \min\{f'(u) : u \geq \frac{5}{8}\} > 0,$$

and

$$(4.9) \quad p(\tau) = \frac{\epsilon\beta}{\mu} + \left(\frac{1}{4} - \frac{\epsilon\beta}{\mu}\right) \exp(-\frac{\mu\tau}{\epsilon}), \quad \tau \geq 0.$$

We will choose $\xi \geq 0$ in step 5 satisfying,

$$(4.10) \quad \xi' \geq 0.$$

4. Suppose that

$$(4.11) \quad q(\dots) \in [\frac{7}{8}, 1]$$

The case $q(\dots) \leq \frac{7}{8}$ will be analyzed in the next step. Since $|p(\tau)| \leq \frac{1}{4}$, (4.11) implies that

$$v(t, x) = q(\dots) - p\left(\frac{t}{\epsilon}\right) \geq \frac{5}{8}.$$

Since $v = q(\dots) - p \leq q(\dots)$, (4.8) yields

$$f(v(t, x)) - f(q(\dots)) \leq -\mu p\left(\frac{t}{\epsilon}\right).$$

Use (4.9), (4.10) and the above inequality in (4.7) to obtain,

$$I \leq \frac{\beta}{\epsilon} - \frac{1}{\epsilon} p'\left(\frac{t}{\epsilon}\right) - \frac{\mu}{\epsilon^2} p\left(\frac{t}{\epsilon}\right) = 0,$$

on $\{v \geq 0\}$.

5. Suppose that (4.11) does not hold, i.e.,

$$q(\dots) \leq \frac{7}{8}.$$

Then on $\{v \geq 0\}$, $q(\dots) \in [0, \frac{7}{8}]$ and

$$(4.12) \quad q'(\dots) = (1 - q(\dots))^2 \geq (1 - (\frac{7}{8}))^2 := \gamma.$$

Set

$$\alpha := \max\{|f'(u)| : u \in [0, 1]\}.$$

Since $v \leq 1$, on $\{v \geq 0\}$ we have,

$$f(v) - f(q(\dots)) \leq \alpha |v(t, x) - q(\dots)| = \alpha p\left(\frac{t}{\epsilon}\right).$$

Use the above inequality and (4.12) in (4.7) to obtain,

$$I \leq -\frac{\gamma}{\epsilon^2} \xi'\left(\frac{t}{\epsilon}\right) - \frac{1}{\epsilon} p'\left(\frac{t}{\epsilon}\right) + \frac{\alpha}{\epsilon^2} p\left(\frac{t}{\epsilon}\right) + \frac{\beta}{\epsilon}.$$

We now choose $\xi(\cdot)$ satisfying $\xi(0) = 0$ and

$$\xi'(\tau) = \frac{1}{\gamma} \{\beta\epsilon + \alpha p(\tau) - \epsilon p'(\tau)\} = \frac{\alpha + \mu}{\gamma} p(\tau), \quad \tau \geq 0.$$

Using (4.9) we integrate the above equation,

$$\xi(\tau) = \frac{\epsilon}{\gamma} \left(1 + \frac{\alpha}{\mu}\right) \left[\beta\tau + \left(\frac{1}{4} - \frac{\epsilon\beta}{\mu}\right) \left(1 - \exp\left(-\frac{\mu\tau}{\epsilon}\right)\right)\right].$$

Observe that this choice of ξ satisfies (4.10).

6. By the previous two steps,

$$I \leq 0 \quad \text{on} \quad \{v \geq 0\}.$$

Also by (3.6)

$$u^\epsilon(t, x) \geq \frac{3}{4}, \quad \forall t \in I_\epsilon, \quad d_0(x) \geq \epsilon C_3.$$

In particular,

$$v(0, x) = q(\dots) - \frac{1}{4} \leq \frac{3}{4} \leq u^\epsilon(t_1, x), \quad \forall d_0(x) \geq \epsilon C_3,$$

and since $p, \xi > 0$,

$$v(t - t_1, x) \leq q(0) = 0 \leq u^\epsilon(t, x), \quad \forall t \in I_\epsilon, \quad \forall d_0(x) = \epsilon C_3.$$

Since $u^\epsilon(t, x) \geq 0$ for all $t \in I_\epsilon$ and $d_0(x) \geq \epsilon C_3$, the maximum principle yields,

$$(4.13) \quad u^\epsilon(t, x) \geq v(t - t_1, x), \quad \forall t \in I_\epsilon, \quad d_0(x) \geq \epsilon C_3.$$

Now (4.2) follows from (4.13), (4.4), (3.6) and the definitions of p and ξ . ■

6 Conclusion

Theorem 6.1 *Assume (2.2). Then (A) holds.*

Proof: Let A be a Borel subset of \mathcal{R}^d with a finite Lebesgue measure. Set

$$\Omega_1 = \{x \in \mathcal{R}^d : |d_0(x)| \leq \epsilon C_3\},$$

$$\Omega_2 = \{x \in \mathcal{R}^d : |d_0(x)| \in [\epsilon C_3, \lambda]\},$$

$$\Omega_3 = \{x \in \mathcal{R}^d : |d_0(x)| \geq \lambda\},$$

$$A_i = A \cap \Omega_i, i = 1, 2, 3,$$

$$I_i(t) = \int_{A_i} \frac{\epsilon}{2} |Du^\epsilon(t, x)|^2 dx, \quad i = 1, 2, 3, \quad t \geq 0,$$

$$J_i(t) = \int_{A_i} \frac{1}{\epsilon} W(u^\epsilon(t, x)) dx, \quad i = 1, 2, 3, \quad t \geq 0,$$

where λ, C_3 are as in (3.1) and (3.5), respectively. In the following steps we will estimate I_i and J_i 's separately.

1. By Lemma 3.1,

$$I_1(t) + J_1(t) \leq \int_{A_1} \frac{\epsilon K^2}{2 \epsilon^2} + \frac{1}{\epsilon} = \left(\frac{K^2}{2} + 1\right) \frac{|\Omega_1|}{\epsilon}.$$

Since Γ_0 is smooth and bounded, for sufficiently small $\epsilon > 0$, $|\Omega_1| \leq \epsilon \hat{C}$ for an appropriate constant \hat{C} . Hence

$$I_1(t) + J_1(t) \leq \hat{C} \left(1 + \frac{K^2}{2}\right), \quad \forall t \geq 0.$$

2. Set

$$C_4 = C_1 + \frac{1}{\delta}, \quad t_4 = C_4 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right),$$

where $\delta > 0$ is the constant appearing in (5.1) and C_1 is as in Theorem 3.1. Then for all $t \in I_\epsilon \cap [t_4, \infty)$, by (5.1) we have,

$$|Du^\epsilon(t, x)|^2 \leq \frac{K^2}{\epsilon^2} \left[\epsilon + \exp\left(-\frac{\alpha}{\epsilon} (|d(x)| - \epsilon C_3)\right) \right].$$

Therefore,

$$I_2(t) \leq \frac{K^2}{2} |A_2| + \frac{K^2}{2\epsilon} \int_{\Omega_2} \exp\left(-\frac{\alpha}{\epsilon} (|d(x)| - \epsilon C_3)\right) dx.$$

By (4.4), $d_0 = d$ on A_2 . In the above integral we use local orthogonal coordinates w , with $w_1 = d_0(x)$. Since d_0 is smooth in Ω_2 , there is a constant C , depending on the $(d-1)$ dimensional measure of Γ_0 , such that

$$\begin{aligned} I_2(t) &\leq \frac{K^2}{2} |A_2| + \frac{K^2}{2\epsilon} C \int_{\epsilon C_3}^{\lambda} e^{-\frac{\alpha}{\epsilon} (w_1 - \epsilon C_3)} dw_1 \\ &\leq \frac{K^2}{2} (|A_2| + \hat{C}), \quad \forall t \in I_\epsilon \cap [t_4, \infty), \end{aligned}$$

where \hat{C} is an appropriate constant, possibly different than the constant appearing in the first step.

3. For $t \in I_\epsilon \cap [t_4, \infty)$ and $|d_0(x)| \geq \lambda$, (5.1) and (4.5) yield,

$$|Du^\epsilon(t, x)|^2 \leq \frac{K^2}{\epsilon^2} [\epsilon + e^{-\frac{\alpha}{\epsilon} (\lambda - \epsilon C_3)}].$$

Therefore for sufficiently small $\epsilon \geq 0$,

$$I_3(t) \leq \frac{K^2}{2} (|A_3| + \hat{C}), \quad \forall t \in I_\epsilon \cap [t_4, \infty),$$

for an appropriate constant \hat{C} , again possibly different than the constant appearing in the previous steps.

4. Recall that we have chosen C_1, C_2 satisfying (3.4) with $m = 2$. Hence for all $|d_0(x)| \geq \lambda$, and $t \in I_3$,

$$\begin{aligned} W(u^\epsilon) &= \frac{1}{2} (1 - u^\epsilon)^2 (1 + u^\epsilon)^2 \\ &\leq 2(u^\epsilon - \text{sign}(u_0))^2 \leq 2\epsilon^4. \end{aligned}$$

Therefore,

$$J_3(t) \leq 2\epsilon^2 |A_3|, \quad \forall t \in I_\epsilon.$$

5. Set

$$C_\epsilon = C_1 + \frac{1}{\mu}, \quad t_\epsilon = C_\epsilon \epsilon^2 \ln\left(\frac{1}{\epsilon}\right),$$

where μ is the constant appearing in Theorem 4.1 and C_1 is as in Theorem 3.1. Then (4.2) and (4.3) imply that for all $t \in I_\epsilon \cap [t_\epsilon, \infty)$, $|d_0(x)| \in [\epsilon C_3, \lambda]$,

$$|u^\epsilon(t, x)| \geq \left[q\left(\frac{|d_0(x)| - Kt}{\epsilon} - K\right) - K\epsilon - \frac{1}{4}\epsilon \right]^+,$$

where $(a)^+ = \max\{a, 0\}$. Since $|W'(u)| \leq 1$ for $|u| \leq 1$, for sufficiently small $\epsilon > 0$ we have,

$$\begin{aligned} J_2(t) &\leq \int_{A_2} \frac{1}{\epsilon} W\left(\left[q\left(\frac{|d_0(x)| - Kt}{\epsilon} - K\right) - \epsilon\left(K + \frac{1}{4}\right)\right]^+\right) dx \\ &\leq \int_{A_2} \frac{1}{\epsilon} W\left(\left[q\left(\frac{|d_0(x)| - Kt}{\epsilon} - K\right)\right]^+\right) dx + \left(K + \frac{1}{4}\right) |A_2| \\ &\leq \int_{\Omega_2} \frac{1}{\epsilon} W\left(\left[q\left(\frac{|d_0(x)| - 2K\epsilon}{\epsilon}\right)\right]^+\right) dx + \left(K + \frac{1}{4}\right) |A_2|, \end{aligned}$$

for all $t \in I_\epsilon \cap [t_\epsilon, \epsilon]$. Now using the same change of variables used in step 2 we obtain,

$$\begin{aligned} J_2(t) &\leq \frac{C}{\epsilon} \int_{\epsilon C_3}^\lambda W\left(\left[q\left(\frac{w_1 - 2K}{\epsilon}\right)\right]^+\right) dw_1 + \left(K + \frac{1}{4}\right) |A_2| \\ &\leq \frac{C}{\epsilon} \int_{\epsilon C_3}^{2\epsilon K} W(0) dw_1 + \left(K + \frac{1}{4}\right) |A_2| \\ &+ C \int_0^{\lambda/\epsilon - 2K} W(q(r)) dr. \end{aligned}$$

Since

$$W(q(r)) = \frac{(q'(r))^2}{2} = \frac{8e^{4r}}{(e^{2r} + 1)^4},$$

$$J_2(t) \leq \hat{C}(|A_2| + 1).$$

6. Combining the previous steps we conclude that

$$(6.1) \quad \begin{aligned} \mu^\epsilon(A; t) &= \sum_{i=1}^3 (I_i(t) + J_i(t)) \\ &\leq C(|A| + 1), \end{aligned}$$

for all $t \geq 0$ satisfying,

$$(6.2) \quad t \in I_\epsilon, \quad t \geq t_4, \quad t \geq t_5, \quad t \leq \epsilon,$$

and sufficiently small $\epsilon \geq 0$.

7. Let Ψ be a smooth positive function decaying exponentially as $|x| \rightarrow \infty$. Then using 1.1 we obtain,

$$\begin{aligned} \frac{d}{dt} \int \Psi(x) \mu^\epsilon(dx; t) &= -\epsilon \int \Psi(-\Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon))^2 dx \\ &+ \epsilon \int D\Psi \cdot Du^\epsilon (-\Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon)) dx \\ &\leq -\epsilon \int \Psi(-\Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon) - \frac{D\Psi \cdot Du^\epsilon}{2\Psi})^2 dx \\ &+ \epsilon \int |Du^\epsilon|^2 \frac{|D\Psi|^2}{4\Psi} dx \\ &\leq \epsilon \int |Du^\epsilon|^2 \frac{|D\Psi|^2}{4\Psi} dx. \end{aligned}$$

Let

$$\hat{\Psi}(x) = \exp(-\sqrt{1+|x|^2}).$$

Then $|D\hat{\Psi}| \leq \hat{\Psi}$ and

$$\begin{aligned} \frac{d}{dt} \int \hat{\Psi}(x) \mu^\epsilon(dx; t) &\leq \frac{1}{2} \int \frac{\epsilon}{2} |Du^\epsilon|^2 \hat{\Psi} dx \\ &\leq \frac{1}{2} \int \hat{\Psi}(x) \mu^\epsilon(dx; t) \end{aligned}$$

Therefore for any $t \geq t_0 \geq 0$,

$$(6.3) \quad \int \hat{\Psi}(x) \mu^\epsilon(dx; t) \leq \int \hat{\Psi}(x) \mu^\epsilon(dx; t_0) e^{\frac{t-t_0}{\epsilon}}$$

8. Let t_0 be a point satisfying (6.2). Then (6.1) yields,

$$\begin{aligned} \int \hat{\Psi}(x) \mu^\epsilon(dx; t_0) &\leq \sum_{i=1}^{\infty} e^{-i} \mu^\epsilon(\{|x| \in [i-1, i)\}; t_0) \\ &\leq \bar{C} w_d \sum_{i=1}^{\infty} e^{-i} (1 + (i)^d - (i-1)^d) \\ &\leq \bar{C}, \end{aligned}$$

where w_d is the volume of the unit space in \mathcal{R}^d and \bar{C} is an appropriate constant. Then by (6.3)

$$\int \hat{\Psi}(x) \mu^\epsilon(dx; t) \leq \bar{C} e^{\frac{t}{\epsilon}},$$

for every sufficiently small ϵ and

$$(6.4) \quad t \geq \max\{C_1, C_4, C_5\} \epsilon^2 \ln\left(\frac{1}{\epsilon}\right).$$

9. Now let ϕ be any continuous function satisfying,

$$\Lambda := \sup\{|\phi(x)| e^{\sqrt{2}(1+|x|)} : x \in \mathcal{R}^d\} < \infty.$$

Then $|\phi(x)| \leq \Lambda \hat{\Psi}(x)$, and

$$(6.5) \quad \int |\phi(x)| \mu^\epsilon(dx; t) \leq \bar{C} \Lambda e^{\frac{t}{\epsilon}},$$

for all t satisfying (6.4), and sufficiently small $\epsilon > 0$. Since for every $\epsilon > 0$, by (3.9)

$$\mu^\epsilon(dx; t) \leq \frac{1}{\epsilon} \left(\frac{K^2}{2} + 1 \right) dx.$$

Hence for every $t \geq 0$,

$$(6.6) \quad \int |\phi(x)|\mu^\epsilon(dx; t) \leq \frac{\Lambda}{\epsilon} \left(\frac{K^2}{2} + 1 \right) \int \hat{\Psi}(x) dx.$$

Now (A) follows from (6.5) and (6.6) with $\eta = \sqrt{2}$. ■

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