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MULTICOLOURED TREES IN RANDOM GRAPHS

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1 INTRODUCTION

Let $G = (V, E)$ be a graph in which the edges are coloured. A set $S \subseteq E$ is said to be multicoloured if each edge of $S$ is a different colour. A spanning tree of $G$ is said to be multicoloured if its edge set is. In this paper we study

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the existence of a multicoloured spanning tree (MST) in a randomly coloured random graph.

In fact, our main result will concern a randomly coloured graph process. Here $e_1, e_2, \ldots, e_N$ is a random permutation of the edges of the complete graph $K_n$ and so $N = \binom{n}{2}$. Each edge $e$ independently chooses a random colour $c(e)$ from a given set of colours $W$, $|W| \geq n - 1$.

The graph process consists of the sequence of random graphs $G_m, m = 1, 2, \ldots, N$, where $G_m = ([n], E_m)$ and $E_m = \{e_1, e_2, \ldots, e_m\}$. We identify the following events:

$\mathcal{C}_m = \{G_m \text{ is connected}\}$.

$\mathcal{N}_m = \{|c(E_m)| \geq n - 1\}$, where $c(E_m)$ is the set of colours used by $E_m$.

$\mathcal{MT}_m = \{G_m \text{ has a multicoloured spanning tree}\}$.

Let $\mathcal{E}_m$ stand for one of the above three sequences of events and let

$$m_\mathcal{E} = \min \{m : \mathcal{E}_m \text{ occurs}\},$$

provided such an $m$ exists. Clearly, if $m_{\mathcal{MT}}$ is defined,

$$m_{\mathcal{MT}} \geq \max \{m_\mathcal{C}, m_\mathcal{N}\},$$

and the main result of the paper is

**Theorem 1** In almost every (a.e.) randomly coloured graph process

$$m_{\mathcal{MT}} = \max \{m_\mathcal{C}, m_\mathcal{N}\}.$$
To establish the existence of an MST we use a result of Edmonds [2] on the matroid intersection problem. In this scenario \( M_i, M_j \) are matroids over a common ground set \( E \) with rank functions \( r_i, r_j \) respectively. Edmonds' general theorem on this problem is

\[
\max(|/| : / \text{ is independent in both matroids}) = \min_{E_1 \cap E_2 = \emptyset} (r_i(E_i) + r_j(E_j))
\]

(1)

For us \( M_i \) is the cycle matroid of a graph \( G = G_m \) and \( M_2 \) is the partition matroid associated with the colours. Thus for a set of edges \( E \), \( r_i(E_i) = n - K(S) \) where \( K(S) \) is the number of components of the graph \( G = ([w], S) \) and \( r_j(E_j) \) is the number of distinct colours occurring in \( S \). If \( i \in W \) then \( C_i \) denotes the set of edges of colour \( i \) and for \( / \subseteq W \), \( C_j = \bigcup_{i \in I} C_i \). We will use Edmonds' theorem in the following form:

**Theorem 2** Suppose \( |W| = n - 1 \). Then a necessary and sufficient condition for the existence of an MST is that

\[
K(d) \leq n - |I|\]

for all \( I \subseteq W \).

(2)

[To see this, w.l.o.g. restrict attention in (1) to \( E_2 \) of the form \( C_j \) and then take / = \( W \setminus \text{Jim} \) in (2).]

2 Proof of Theorem 1

Observe first that if \( u = u(n) \to oo \) slowly, then in a.e. randomly coloured graph process

\[
m_c \geq m_0 = \left[ \frac{1}{2} n(\ln n - \omega) \right] \text{ and } m_N \leq m_1 = \left[ n(\ln n + \omega) \right]
\]
We will start by justifying a concentration on the case $|W| = n - 1$. We will describe a coupled process in which there are never more than $n - 1$ colours used: from $m_N$ onwards, the colours that have not yet been used are randomly changed to one of the $n - 1$ colours that have appeared so far. The relevant properties of this coupled process are

1. For each $m \in [m_0, m_1]$ the edges of $G_m$ are independently randomly coloured from a choice of $n - 1$ colours.

2. If $m_{MT} > \max\{m_C, m_N\}$ holds for the original process then it also holds for the coupled process.

Thus to prove our theorem we need only prove that

$$\Pr(m_{MT} > \max\{m_C, m_N\}) = o(1).$$

where $\Pr$ refers to the coupled process.

Fix some $m$ in the range $[m_0, m_1]$. We define the event

$$A_k = \{\exists I \subseteq W, |I| = k : \kappa(C_I) \geq n - |I| + 1\}.$$

We know that if $|W| = n - 1$, $G_m$ is connected and each colour is used at least once and there is no MST then $A_k$ occurs for some $k \in [3, n - 2]$ ($A_1 \cup A_2$ cannot occur if all $n - 1$ colours are used and $A_{n-1}$ cannot occur if $G_m$ is connected.) Take a minimal $k$, corresponding set $I$ and let $S = C_I$.

**Claim 1** $G_S$ has no bridges.

**Proof** If there is a bridge, remove it and all edges of the same colour. Clearly $A_{k-1}$ occurs, contradicting the minimality of $k$. □
With the notation of Claim 1 suppose then that $G_s$ has $i$ isolated vertices and $n - k + x - i$ non-trivial components, $x \geq 1$. Since non-trivial components without bridges have at least three vertices,

$$i + 3(n - k + x - i) \leq n$$  \hspace{1cm} (3)

or

$$i \rightarrow \frac{3}{2}k + \frac{3}{2}x \geq \frac{n}{2}k + \frac{3}{2}.$$  

So now let $B_k$ denote the event

$$\{3/ \subset W, \forall \lambda = k, T \subset \{n\} : t = \forall \lambda < 3(k - 1)/2, \text{all edges coloured with } / \text{ are contained in } T, \text{ there are } u \geq \max\{fc,i\} /-\text{coloured edges}\}.$$  

Here $T$ is the set of vertices in the non-trivial components of $G_C$. Thus if $|W| = n - 1$,

$$Mm n A_k \supset \bigcup_{t=3}^k J Bi \text{ for } k \geq 3.$$  \hspace{1cm} (4)

For $k > 9n/10$ we consider a slightly different event.

We first rephrase (2) as

$$K(C_W/J) \leq \forall \lambda + 1 \text{ for all } J \subset W.$$  \hspace{1cm} (5)

So if $|W| = n - 1$ and there is no MST then there exist $\ell \geq 1$ colours whose deletion produces $A \geq \ell + 2$ components of sizes $n_1, \ldots, n_k$.

**Claim 2** Some subsequence of the $n_i$'s sums to between $\ell + 1$ and $n/2$.  


\textbf{Proof} Assume $n \leq \ell^2 \leq \cdots \leq nth\ell$. If $n \geq \ell + 1$, one of $n_1, \ldots, n^{\ell-1}$ and 71A suffices.

Suppose then that $n_i \leq \ell$, $1 \leq i \leq A$.

Choose $r$ such that
\[
i H - h n_r \leq n/2, \quad nH - h n_{r+i} > n/2
\]
and then
\[
i H - h n_r > n/2 - n_{r+i} \
\geq n/2 - \ell \
\geq \ell.
\]
and we can take $n_i, \ldots, n_r$.

Note next that if $J$ is minimal in (5) then each colour in $J$ appears at least twice as an edge joining components of $G_{C \setminus J}$.

So if $G_m$ is connected and there is no MST and $Ak$ does not occur for $k \leq 9n/10$ then there is a set $L$ of $1 \leq \ell \leq n/10$ colours and a set $S$ of size $s$, $\ell + 1 \leq s \leq n/2$ such that (i) all $t = t(S) = \forall S : S \cap G_{C \setminus J} \geq 1$ edges are L-coloured, $(5* : 3)$ is the set of edges joining $5*$ and $S = V \setminus S$, (ii) the lexicographically first max$\{2^\ell - s, 0\}$ non-$\leftarrow$ edges joining up components (of the $\forall L$ coloured edges) are also L-coloured. Let $T > \ell$ denote this event. Then
\[
C_m \cap \left( \bigcup_{k=9n/10}^{n/2} A_k \right) \subseteq \ell^{n/10} \quad (5) \quad PT_m(V_\ell).
\]

It follows from (4) and (6) that
Pr(m_{MT} > \max\{m_N, m_C\}) \leq

\sum_{m=m_0}^{m_1} \left[ \sum_{k=3}^{9n/10} \Pr_m(B_k) + \sum_{t=2}^{n/10} \Pr_m(D_t) \right] + \Pr \left( \bigcup_{m=m_0}^{m_1} (C_m \cap A_{n-2}) \right). \tag{7}

Here \( Pr_m \) denotes probability w.r.t. \( G_m \) and the \( o(1) \) term is the probability that \( G_{m_0} \) is connected or that \( m_N > m_1 \). (Our calculations force us to separate out \( A_{n-2} \).)

We must now estimate the individual probabilities in (7). It is easier to work with the independent model \( G_p, p = m/N, \) where each edge occurs independently with probability \( p \) and is then randomly coloured. For any event \( E \) we have (see Bollobás [1] Chapter II) the simple bound

\[ \Pr_m(E) \leq 3\sqrt{n \ln n} \Pr_p(E). \tag{8} \]

where \( \Pr_p \) denotes probability w.r.t. the model \( G_p \).

Now, where \( p = \alpha \ln n/n, 1 - o(1) \leq \alpha \leq 2 + o(1), \)

\[ \Pr_p(B_k) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{k}{t}} \left( \begin{array}{c} n \\ t \end{array} \right) \left( \begin{array}{c} n-1 \\ k \end{array} \right) \left( \frac{\binom{k}{t}}{u} \right) \left(1 - \frac{kp}{n-1}\right)^{t-u} \left( \frac{kp}{n-1}\right)^u \]

\[ \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{k}{t}} \frac{n^t e^t}{t^t} \frac{n^k e^k}{k^k} \frac{t^2 e^2}{2u} \left( \frac{ak \ln n}{n^2} \right)^u. \tag{9} \]

Case 1: \( 3 \leq k \leq k_0 = n/(3 \ln n). \)

\[ \Pr_p(B_k) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{k}{t}} \left( \frac{e^3 n^{1-\alpha(1)\omega(1)}}{k} \right)^k \left( \frac{t}{n} \right)^{2u-t} \left( \frac{\alpha e^k \ln n}{2u} \right)^u \]

\[ = \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{k}{t}} \left( \frac{e^3 n^{1-\alpha(1)\omega(1)}}{k} \right)^k \left( \frac{t}{n} \right)^{u-t} \left( \frac{\alpha e^k \ln n}{2un} \right)^u \]
\[
\Pr_m(B_k) \leq \sum_{t=1}^{3(k-1)/2} \left( \frac{e^{3n^{1-\frac{1}{2}-\alpha(1)}}}{2kn} \right)^k \left( \frac{t}{n} \right)^{u-t} \left( \frac{\alpha e k \ln n}{2n} \right)^{u-k}
\]

\[
= O\left( \left( \frac{\ln n}{n^{2-\alpha(1)}} \right)^k \right).
\]

It follows from this and (8) that

\[
\sum_{m=m_0}^{m_1} \sum_{k=4}^{k_9} \Pr_m(B_k) = O((n \ln n)(\sqrt{n \ln n})((\ln n)^4/n^{2-\alpha(1)}))
\]

\[
= o(1).
\]

(10)

For \(k = 3\) we compute \(\Pr_m(B_3)\) directly, but since now \(u = t = k = 3\) is forced,

\[
\Pr_m(B_3) \leq \binom{n}{3}^2 \left( 1 - \frac{3}{n-1} \right)^{m-3} \left( \frac{3}{n-1} \right)^3 \binom{n-3}{m-3} \cdot \binom{N}{m}
\]

\[
= O(e^{3\omega(\ln n)^3} n^{-3/2})
\]

and so

\[
\sum_{m=m_0}^{m_1} \Pr_m(B_3) = o(1).
\]

(11)

Case 2: \(k_0 < k \leq n/2\).

We now write (9) as

\[
\Pr_p(B_k) \leq \sum_{t=1}^{3(k-1)/2} \left( \frac{e^{3n^{1-\frac{1}{2}-\alpha(1)}}}{k} \right)^k \left( \frac{t}{n} \right)^{u-t} \left( \frac{\alpha e t \ln n}{2un} \right)^u
\]

\[
\leq \sum_{t=1}^{3(k-1)/2} \left( \frac{e^{3n^{1-\frac{1}{2}-\alpha(1)}}}{k} \right)^k \left( \frac{t}{n} \right)^{u-t} \frac{\alpha t k}{n^{2n}}
\]

8
(after maximising the last term over $u$)

\[
\begin{align*}
&= \sum_{i=1}^{3(k-1)/2} \sum_{u=\max\{i,k\}}^{n} \frac{e^{3n^{1-\frac{A}{2}(1-\frac{1}{4}-\alpha(1))} \cdot k^{t \cdot u-1}}{k} \\
&\leq \sum_{i=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{n} \frac{e^{3n^{1-\alpha(\frac{3}{4}-\alpha(1))} \cdot k^{t \cdot n-1}}{k} \\
\end{align*}
\]

(12) and (8) clearly imply

\[
\sum_{m=m_0}^{m_{i/2}} \sum_{k=k_0}^{n/2} \Pr_m(B_k) = o(1).
\]

Case 3: $n/2 < A: \leq 9n/10$

Claim 3 Choose any constant $A > 0$. Then, in a.e. process, simultaneously for each $m \in [m_0,m_i]$, the sets of $s \leq A$ vertices of $G_m$ which span at least $s$ edges together contain at most $(\ln n)^{A+1}$ vertices.

Proof We need only prove this for $G_{m_i}$ and since the property is monotone decreasing we need only prove it for $G_{P_I}$, $P_\lambda = m/n$ ([1], Chapter II.)

But

\[
E_{P_I} (number\ of\ vertices) \leq \sum_{k=3}^{n} \binom{n}{k} P^{k}_{+} P^{k+1}_{-} = 0(e^{2A(\ln n)^A}).
\]

Now use the Markov inequality. Q

It follows that we may rewrite (3) as

\[
i + 3(\ln n)^{A+1} + (A + 1)(n - k + x - i) \leq n
\]
and so

$$i \geq n - \frac{A+1}{A} k - O((\ln n)^{A+1})$$

$$\geq n - \frac{A}{A-1} k.$$ 

By making $A$ sufficiently large we see that if $k \leq 9n/10$ then $t \leq 19n/20$ in (12) and consequently

$$\mathop{\sum_{m=m_0 \leq k \leq n/2}}_{m_{=\text{mi} \leq 9n/10}} \mathbb{P}(B_k) = o(i). \quad (15)$$

Case 4: $A; \geq 9n/10$

$$\Pr_p(D_t) \leq \mathop{\sum_{n/2}}_{(s_1 \neq s_2 \neq \ldots \neq s_{10n/20})} \frac{\ell}{\ell} \frac{n+1}{n-1} s(n-s)/s(n) \ell \mathbb{P}$$

Let $u(s, t, t)$ denote the summand in the above and let $p = a In n/n$ and note that $a \in [1 - \frac{\ln n}{10}]$.

Case 4.1: $i \leq 2\ell$

It will generally be convenient to split $s$ into two ranges:

Case 4.1.1: $s \leq n^{1/10}$

$$u(s, \ell, t) = \left(\frac{n}{s}\right)\left(\frac{n-1}{\ell}\right)\left(\frac{t(n-s)}{s(n)}\right) p^{r(1-p)^{s(n-s)-\ell}} \left(\frac{\ell}{n-1}\right)^{2\ell}$$

$$\leq \left(\frac{ne}{\ell}\right)^{s} \left(\frac{(n-1)e}{\ell}\right)^{\ell} \left(\frac{s(n-s)e^{1+p\ln n}}{tn}\right)^{t} n^{-\alpha(s(n-s)/n} \left(\frac{\ell}{n-1}\right)^{2\ell}$$

$$\leq \left(\frac{n^{1-\alpha+\alpha s/n}}{s}\right)^{s} \left(\frac{\ell e}{n-1}\right) \left(\frac{e^{s(n-s)/2(\ln n)^2}}{n^3 \ell}\right)^{\ell} \left(\frac{\ell e}{n-1}\right)^{t} \left(\frac{e^{s(n-s)/2(\ln n)^2}}{n^3 \ell}\right)^{t} \left(\frac{\ell}{n-1}\right)^{2\ell}$$

$$\leq \left(\frac{n^{1-\alpha+\alpha s/n}}{s}\right)^{s} \left(\frac{e^{s(n-s)/2(\ln n)^2}}{n^3 \ell}\right)^{t} \left(\frac{\ell e}{n-1}\right)^{t} \left(\frac{e^{s(n-s)/2(\ln n)^2}}{n^3 \ell}\right)^{t} \left(\frac{\ell}{n-1}\right)^{2\ell}.$$
Now

$$n^{t-a+\alpha s/n} \leq \left( \frac{n}{s} \right)^{\alpha} + o(1)e^{\omega}$$

(17)

where $a > 1$ — $u/n$ and $u \rightarrow \infty$ slowly.

So if $5 \leq 3e^n$ then (16) implies that

$$u(s, \ell, t) \leq n^{-(1-\alpha(1))t},$$

and if $s > 3e^n$

$$U(SJ_A) \leq \left( \frac{e^{1+6e^{-\ell}} - s}{nH} \right)^{\ell}$$

$$= O \left( \left( \frac{s}{n^{1-\alpha(1)}} \right)^{\ell} \right).$$

**Case 4.1.2:** $5 > n^{1/10}$.

Claim 4 /n a.e. process, every $G_{m, m} \in \{m_0, m_{i5} \}$ such that $T \{S\} \geq y \|S\| \ln n$

/or all $n^{1/10} \leq |S| \leq n/2$; where $y > 0$ is some absolute constant.

**Proof** (outline) For $|S| \geq n^{2/3}$ one can use the Chernoff bounds on the tails of the binomial $r_j(S)$. If $\|S\| \leq n^{2/3}$ we use the fact that with high probability (i) $G_{m_0}$ has $n^{e^\ell}$ vertices of degree $\leq 6Inn$ where $e^\ell = ^\ell(e) \rightarrow 0$

with 6, and (ii) in $G_{m_1}$ no set $S$ of size $\leq n/(1nn)$ contains $3151$ edges.

So if $s \geq n^{1/10}$ then we can take $t \geq 7sInn > 2\ell$ for some constant $y > 0$

and this case is vacuous.

**Case 4.2 :** $t > 2\ell$.

$$u(s, \ell, t) \leq \left( \frac{ne}{s} \right)^{s} \left( \frac{(n-1)e}{\ell} \right)^{t} \left( \frac{s(n-s)e^{1+\ell\alpha} \ln n}{n(n-1)} \right)^{t} n^{-\alpha s(n-s)/n}$$
\[
\begin{align*}
(18) & \quad \frac{F^{\alpha} f(n-l) e^{\alpha} f_s(n-s) e^{\alpha} + e^{\alpha n}}{V^*} \\
& \quad \text{Case 4.2.1: } t < 2n \text{ and so } ((n - \sqrt{t})/t)^{1/2} \ll (2n/e)^{1/2}.
\end{align*}
\]

\[
u(s, t, t) \leq \left( \frac{n^{1-\alpha+\alpha s/2}}{s} \right) \left( \frac{20\sqrt{\ln n}}{t^{3/2}n^{1/2}} \right).
\]

Case 4.2.1: \( s < n^{1/10} \). Now (17) gives
\[
\left( \frac{n^{1-\alpha+\alpha s/2}}{s} \right)^s \leq \left( \frac{(1 + \alpha(1))e^{\omega+1}}{s} \right)^s \leq e^{(1+\alpha(1))e^\omega} = e^{\omega}, \text{ say,}
\]
and so (19) implies
\[
u(s, t, t) \leq \left( \frac{n^{1-\alpha+\alpha s/2}}{s} \right)^s \end{equation}
\]

Case 4.2.1.2: \( s > n^{1/10} \).

Using Claim 4 and (19),
\[
u(s, t, t) \leq n^{-s/11} \left( \frac{1}{n^{2-\alpha(1)\sqrt{s}}} \right)^s.
\]

Case 4.2.2: \( t \geq 2n \) and so \((ne/t)^t \leq e^n \leq e^{t/2} \).

From (18),
\[
u(s, t, t) \leq \left( \frac{(1 + \alpha(1))e^{\omega+1}}{s} \right)^s \left( \frac{20\sqrt{\ln n}}{t^{3/2}n^{1/2}} \right)^t.
\]

Case 4.2.2.1: \( s < n^{1/10} \).
Arguing as in (20),

$$u(s, \ell, t) \leq \left( \frac{s}{(n^{1-\alpha(1)})} \right)^{t}.$$ 

**Case 4.2.2: s \geq n^{1/10}.

From Claim 4

$$u(s, \ell, t) \leq \left( \frac{(1 + o(1))e^{s+1}}{s} \right)^{s} \left( \frac{A\ell}{n} \right)^{t}.$$ 

for some constant $A > 0$. Now this clearly implies

$$u(s, \ell, t) = O(2^{-n})$$

for $\ell \leq n/(3A)$. For $\ell > n/(3A)$ we have $s \geq \ell$ and

$$u(s, \ell, t) \leq n^{-s/2}A^{n}$$

and so (21) holds here also.

Summarising,

$$\Pr(D_{t}) = O \left( \sum_{t=1}^{2\ell} \sum_{s=\ell+1}^{n^{1/10}} \left( \frac{s}{n^{1-\alpha(1)}} \right)^{t} + \sum_{t=2\ell+1}^{2n} \sum_{s=\ell+1}^{n^{1/10}} \left( \frac{s}{n^{1/2-\alpha(1)}} \right)^{t} \right)$$

$$+ \sum_{t=2\ell+1}^{2n} \sum_{s=n^{1/10}}^{n/2} \left( \frac{s}{n^{1/2-\alpha(1)}} \right)^{t} + \sum_{s=1}^{n/2} \sum_{t=2n+1}^{n^{1/10}} \left( \frac{s}{n^{1/2-\alpha(1)}} \right)^{t}$$

$$+ \sum_{s=n^{1/10}}^{n/2} \sum_{t=2n+1}^{n^{1/10}} 2^{-n}$$

$$= O(\ell n^{-(\gamma - \alpha(1))}t).$$

where the double summations correspond to the five cases enumerated above.

Thus, we see that

$$\sum_{m=m_{0}}^{m_{1}} \sum_{t=2}^{n/10} \Pr_{m}(D_{t}) = O((\ln n)(\sqrt{n \ln n})n^{-1.7})$$

$$= o(1).$$

(22)
We are thus left with $\Pr\left(\bigcup_{m=m_0}^{m_1} (C_m \cap \mathcal{A}_{n-2})\right)$.

We consider $G_{m_0}$. We know that a.e. $G_{m_0}$ consists of a giant connected component $C$ plus $O(e^\omega)$ isolated vertices $T$. If $\bigcup_{m=m_0}^{m_1} (C_m \cap \mathcal{A}_{n-2})$ occurs at some time during the process then either

(i) there exist $u, v \in T$ such that the first edges of the process that are incident with each of $u$ and $v$ are the same colour,

OR

(ii) there exists a colour $c$ and a set $S$, $2 \leq |S| \leq n/2$ such that in $G_{m_0}$ the $t > 2 (S : S)$ edges are all of colour $c$.

(Suppose that deleting the edges of colour $c$ from $G_m$ produces at least three components. If colour $k$ has not occurred by time $m_0$ then two of these components must be vertices from $T$, contradicting (i). If $G_{m_0}$ has edges of colour $c$ then deleting these edges must break $C$ into at least three pieces.)

Clearly

$$\Pr((i)) = o(1) + O(e^{2\omega}/n) = o(1).$$

Furthermore

$$\Pr_p((ii)) \leq \sum_{s=2}^{n/2} \binom{n}{s} \sum_{t=2}^{s(n-s)} \binom{s(n-s)-t}{t} \left(\frac{p}{n}\right)^t \left(1 - p\right)^{s(n-s)-t}$$

$$\leq 2 \sum_{s=2}^{n/2} \binom{n}{s} \sum_{t=2}^{s(n-s)} \frac{\ln n}{t!} \left(\frac{\alpha \ln n}{n^2}\right)^t n^{-\alpha s}$$

$$\leq n \sum_{s=2}^{n/2} \left(\frac{n-1-\alpha}{s}\right) \sum_{t=2}^{s \ln n} \left(\frac{s \alpha n}{n}\right)^t$$

$$= O(n^{-(1-\alpha(1))}).$$
The upper bound is good enough to apply (8) and so $\Pr_{\nu_0}((ii)) = o(1)$. Thus

$$\Pr \left( \bigcup_{m=m_0}^{m_1} (C_m \cap A_{n-2}) \right) = o(1).$$

(23)

Our theorem now follows from (7), (10), (11), (14), (15), (22) and (23).

**References**


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