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MULTICOLORED TREES IN

RANDOM GRAPHS

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MULTICOLOURED TREES IN RANDOM GRAPHS

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1 INTRODUCTION

Let G = (V, E) be a graph in which the edges are coloured. A set $S \subseteq E$ is said to be *multicoloured* if each edge of S is a different colour. A spanning tree of G is said to be multicoloured if its edge set is. In this paper we study

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the existence of a multicoloured spanning tree (MST) in a randomly coloured random graph.

In fact, our main result will concern a randomly coloured graph process. Here e_1, e_2, \ldots, e_N is a random permutation of the edges of the complete graph K_n and so $N = \binom{n}{2}$. Each edge *e* independently chooses a random colour c(e) from a given set of colours W, $|W| \ge n-1$.

The graph process consists of the sequence of random graphs $G_m, m = 1, 2, \ldots, N$, where $G_m = ([n], E_m)$ and $E_m = \{e_1, e_2, \ldots, e_m\}$. We identify the following events:

 $\mathcal{C}_m = \{G_m \text{ is connected }\}.$

 $\mathcal{N}_m = \{ |c(E_m)| \ge n-1 \}, \text{ where } c(E_m) \text{ is the set of colours used by } E_m.$ $\mathcal{MT}_m = \{ G_m \text{ has a multicoloured spanning tree } \}.$

Let \mathcal{E}_m stand for one of the above three sequences of events and let

$$m_{\mathcal{E}} = \min\{m : \mathcal{E}_m \text{ occurs}\},\$$

provided such an m exists. Clearly, if $m_{\mathcal{MT}}$ is defined,

$$m_{\mathcal{MT}} \geq \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\},\$$

and the main result of the paper is

Theorem 1 In almost every (a.e.) randomly coloured graph process

$$m_{\mathcal{MT}} = \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\}.$$

To establish the existence of an MST we use a result of Edmonds [2] on the matroid intersection problem. In this scenario Mi, M < i are matroids over a common ground set E with rank functions ri,r2 respectively. Edmonds' general theorem on this problem is

$$\max(|/| : / \text{ is independent in both matroids}) = \min_{\mathbf{E}_1 \cap \mathbf{E}_2 = \mathbf{\bar{\bullet}}} (ri(Ei) + r_2(\pounds?2))$$
(1)

For us *Mi* is the cycle matroid of a graph $G = G_m$ and M2 is the partition matroid associated with the colours. Thus for a set of edges \pounds^* , $r \setminus (S) = n - K(S)$ where K(5) is the number of components of the graph Gs = ([w], S) and r2(S) is the number of distinct colours occurring in 5. If *i* G W then C, denotes the set of edges of colour *i* and for $/ \underline{C} W$, $Cj = \setminus J_{iel} C_i$. We will use Edmonds' theorem in the following form:

Theorem 2 Suppose |W| = n - 1. Then a necessary and sufficient condition for the existence of an MST is that

$$K(d) \leq n - IJ \qquad for all I \subseteq W.$$
 (2)

[To see this, w.l.o.g. restrict attention in (1) to E2 of the form Cj and then take = W Jin (2).]

2 Proof of Theorem 1

Observe first that if $u = u(n) \longrightarrow$ oo slowly, then in a.e. randomly coloured graph process

$$m_{\mathcal{C}} \ge m_0 = \lfloor \frac{1}{2}n(\ln n - \omega) \rfloor$$
 and $m_{\mathcal{N}} \le m_1 = \lceil n(\ln n + \omega) \rceil$.

We will start by justifying a concentration on the case |W| = n - 1. We will describe a coupled process in which there are never more than n - 1 colours used: from m_N onwards, the colours that have not yet been used are randomly changed to one of the n-1 colours that have appeared so far. The relevant properties of this coupled process are

- 1. For each $m \in [m_0, m_1]$ the edges of G_m are independently randomly coloured from a choice of n-1 colours.
- 2. If $m_{\mathcal{MT}} > \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\}$ holds for the original process then it also holds for the coupled process.

Thus to prove our theorem we need only prove that

$$\Pr(m_{\mathcal{MT}} > \max\{m_{\mathcal{C}}, m_{\mathcal{N}}\}) = o(1).$$

where Pr refers to the coupled process.

Fix some m in the range $[m_0, m_1]$. We define the event

$$\mathcal{A}_k = \{ \exists I \subseteq W, |I| = k : \kappa(C_I) \ge n - |I| + 1 \}.$$

We know that if |W| = n-1, G_m is connected and each colour is used at least once and there is no MST then \mathcal{A}_k occurs for some $k \in [3, n-2]$ $(\mathcal{A}_1 \cup \mathcal{A}_2$ cannot occur if all n-1 colours are used and \mathcal{A}_{n-1} cannot occur if G_m is connected.) Take a minimal k, corresponding set I and let $S = C_I$.

Claim 1 G_S has no bridges.

Proof If there is a bridge, remove it and all edges of the same colour. Clearly \mathcal{A}_{k-1} occurs, contradicting the minimality of k.



With the notation of Claim 1 suppose then that Gs has *i* isolated vertices and n-k+x-i non-trivial components, $x \ge 1$. Since non-trivial components without bridges have at least three vertices,

$$i + 3(n - k + x - i) \leq n \tag{3}$$

or

$$i \implies n_{2}^{3} k + 2x \\ \ge n^{-\frac{3}{2}k} + \frac{3}{2}.$$

So now let *Bk* denote the event

$$\{3/ \subseteq W, |I| = k, T \subseteq [n] : t = |T| < 3(k - 1)/2,$$

all edges coloured with / are contained in T,
there are $u \ge \max\{fc,i\}$ /-coloured edges}.

Here *T* is the set of vertices in the non-trivial components of $Gc_7 \bullet$ Thus if |W| = n - 1,

$$Mm n A_k C |J Bi \qquad \text{for } k \ge 3.$$
^k
_{t=3}
^k
(4)

For $k \ge 9n/10$ we consider a slightly different event.

We first rephrase (2) as

$$K(C_W/J) \le |J| + 1 \qquad \text{for all } JCW. \tag{5}$$

So if |W| = n - 1 and there is no MST then there exist $\pounds \ge 1$ colours whose deletion produces $A \ge \pounds + 2$ components of sizes ni, ..., n.

Claim 2 Some subsequence of the n- 's sums to between $\pounds + 1$ and n/2.

Proof Assume $n < \underline{ri2} < \underline{\bullet} \cdot \bullet < ^A$ -

If $n \ge f + 1$, one of ni,..., n^-i and 7lA suffices.

Suppose then that n-, $\leq f$, $1 \leq i \leq A$.

Choose r such that

ni H——h
$$n_r \le n/2$$
, nH——h $n_{r+i} > n/2$

and then

ni H—h
$$n_r > n/2$$
 — n_r+i
 $\geq n/2 - \ell$
 $\geq \ell$.

and we can take $ni, ..., n_r$.

Note next that if J is minimal in (5) then each colour in J appears at least twice as an edge joining components of $G_{C_{W \setminus J}}$.

So if G_m is connected and there is no MST and Ak does not occur for $k \le$ 9n/10 then there is a set L of $1 \le \pounds < n/10$ colours and a set S of size $s, \pounds + 1 \le s \le n/2$ such that (i) all $t = r/(5) = \langle (S : \overline{S}) \rangle \ge 1$ edges are L-coloured, $((5^* : \overline{S})$ is the set of edges joining 5^* and $\overline{S} = V \setminus S$, (ii) the lexicographically first max $\{2^{--}<,0\}$ non- $(5' : \overline{S})$ edges joining up components (of the $W \setminus L$ coloured edges) are also L-coloured. Let $T > \pounds$ denote this event. Then

$$\mathcal{C}_m \cap \left(\bigcup_{k=9n/10}^{n-2} \mathcal{A}_k\right) \subseteq \bigcup_{i=l}^{n/10} PT_m(V_{\pounds}).$$
(6)

It follows from (4) and (6) that

 $\Pr(m_{\mathcal{MT}} > \max\{m_{\mathcal{N}}, m_{\mathcal{C}}\}) \leq$

$$o(1) + \sum_{m=m_0}^{m_1} \left[\sum_{k=3}^{9n/10} \Pr_m(\mathcal{B}_k) + \sum_{\ell=2}^{n/10} \Pr_m(\mathcal{D}_\ell) \right] + \Pr\left(\bigcup_{m=m_0}^{m_1} (\mathcal{C}_m \cap \mathcal{A}_{n-2}) \right).$$
(7)

Here \Pr_m denotes probability w.r.t. G_m and the o(1) term is the probability that G_{m_0} is connected or that $m_N > m_1$. (Our calculations force us to separate out \mathcal{A}_{n-2} .)

We must now estimate the individual probabilities in (7). It is easier to work with the independent model G_p , p = m/N, where each edge occurs independently with probability p and is then randomly coloured. For any event \mathcal{E} we have (see Bollobás [1] Chapter II) the simple bound

$$\Pr_{m}(\mathcal{E}) \leq 3\sqrt{n\ln n} \Pr_{p}(\mathcal{E}).$$
(8)

where \Pr_p denotes probability w.r.t. the model G_p .

Now, where $p = \alpha \ln n/n$, $1 - o(1) \le \alpha \le 2 + o(1)$,

$$\Pr_{p}(\mathcal{B}_{k}) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \binom{n}{t} \binom{n-1}{k} \binom{\binom{t}{2}}{u} \left(1 - \frac{kp}{n-1}\right)^{\binom{n}{2}-u} \left(\frac{kp}{n-1}\right)^{u} \\ \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \frac{n^{t}e^{t}}{t^{t}} \frac{n^{k}e^{k}}{k^{k}} \left(\frac{t^{2}e}{2u}\right)^{u} n^{-k\alpha(\frac{1}{2}-o(1))} \left(\frac{\alpha k \ln n}{n^{2}}\right)^{u}.$$
(9)

Case 1: $3 \le k \le k_0 = n/(3 \ln n)$.

$$\Pr_{p}(\mathcal{B}_{k}) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^{3}n^{1-\alpha(\frac{1}{2}-o(1))}}{k}\right)^{k} \left(\frac{t}{n}\right)^{2u-t} \left(\frac{\alpha ek \ln n}{2u}\right)^{u}$$
$$= \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^{3}n^{1-\alpha(\frac{1}{2}-o(1))}}{k}\right)^{k} \left(\frac{t}{n}\right)^{u-t} \left(\frac{\alpha ek t \ln n}{2un}\right)^{u}$$

$$\leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{2}-o(1))} \alpha e k \ln n}{2kn} \right)^k \left(\frac{t}{n} \right)^{u-t} \left(\frac{\alpha e k \ln n}{2n} \right)^{u-k}$$
$$= O\left(\left(\frac{\ln n}{n^{\frac{1}{2}-o(1)}} \right)^k \right).$$

It follows from this and (8) that

$$\sum_{m=m_0}^{m_1} \sum_{k=4}^{k_0} \Pr_m(\mathcal{B}_k) = O((n \ln n)(\sqrt{n \ln n})((\ln n)^4/n^{2-o(1)}))$$

= $o(1).$ (10)

For k = 3 we compute $Pr_m(\mathcal{B}_3)$ directly, but since now u = t = k = 3 is forced,

$$\Pr_{m}(\mathcal{B}_{3}) \leq {\binom{n}{3}}^{2} \left(1 - \frac{3}{n-1}\right)^{m-3} \left(\frac{3}{n-1}\right)^{3} \frac{\binom{N-3}{m-3}}{\binom{N}{m}} \\ = O(e^{3\omega} (\ln n)^{3} n^{-3/2})$$

and so

$$\sum_{m=m_0}^{m_1} \Pr_m(\mathcal{B}_3) = o(1).$$
(11)

Case 2: $k_0 < k \le n/2$.

We now write (9) as

$$\Pr_{p}(\mathcal{B}_{k}) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^{3}n^{1-\alpha(\frac{1}{2}-o(1))}}{k}\right)^{k} \left(\frac{t}{n}\right)^{u-t} \left(\frac{\alpha ekt \ln n}{2un}\right)^{u}$$
$$\leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^{3}n^{1-\alpha(\frac{1}{2}-o(1))}}{k}\right)^{k} \left(\frac{t}{n}\right)^{u-t} n^{\frac{\alpha tk}{2n}}$$

(after maximising the last term over *u*)

$$= \sum_{\substack{i=1\\t=l}\\t=l}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{i}{2}} \left(\frac{e^3 n^{1-\frac{\alpha}{2}(1-\frac{i}{n}-o(1))}}{k}\right)^k \left(\frac{t}{n}\right)^{u-t}$$
(12)

$$\leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{\binom{t}{2}} \left(\frac{e^3 n^{1-\alpha(\frac{1}{8}-o(1))}}{k}\right)^k$$
(13)

since $t \le 3(\text{fc} - 1)/2 \le 3n/4$.

(13) and (8) clearly imply

$$\sum_{m=m_0}^{mi} \sum_{k=k_0}^{n/2} \Pr_m(\mathcal{B}_k) = o(1).$$
(14)

Q

Case 3: $n/2 < A \le 9n/10$

Claim 3 Choose any constant A > 0. Then, in a.e. process, simultaneously for each m G [mo,mi], the sets of $s \le A$ vertices of G_m which span at least s edges together contain at most $(\ln n)^{A+l}$ vertices.

Proof We need only prove this for G_{mi} and since the property is monotone decreasing we need only prove it for G_{Pl} , p = m j N ([1], Chapter II.) But

$$E_{Pl} \text{ (number of vertices)} \leq \bigwedge_{k=3}^{A} \begin{pmatrix} U \\ k \end{pmatrix} \begin{pmatrix} k \\ k \end{pmatrix} \begin{pmatrix} k \\ k \end{pmatrix} P_{i}^{k}$$
$$= 0(e^{2A}(lnn)^{A}).$$

Now use the Markov inequality.

It follows that we may rewrite (3) as

$$i + 3(\ln n)^{A+1} + (A+1)(n - k + x - i) \le n$$

and so

$$i \geq n - \frac{A+1}{j}k - O((\ln n)^{A+1})$$
$$\geq n - \frac{A}{A-1}k.$$

By making A sufficiently large we see that if $k \le 9n/10$ then $t \le 19n/20$ in (12) and consequently

$$\begin{array}{rcl}
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Case 4: A; $\geq 9n/10$

$$\Pr_{p}(\mathcal{D}_{\ell}) \leq \frac{n/2}{\sum} (n \setminus (n-1))^{s(n-s)} (s(n-s)) (e_{P} \bigvee (n \setminus n-1))^{s(n-s)} (s(n-s)) (e_{P} \bigvee (n \setminus n-1))^{max\{2\ell-t,0\}}$$

Let u(s, t, t) denote the summand in the above and let $p = a \ln n/n$ and note that $a \in [1 - UJ/\ln n, 2 + u > /\ln n]$.

Case 4.1: $i \leq 2$ £

It will generally be convenient to split *s* into two ranges:

Case 4.1.1:
$$s \leq n^{1/10}$$

$$u(s,\ell,t) = \binom{n}{s} \binom{n-1}{\ell} \binom{s(n-s)}{t} p^{t}(1-p)^{s(n-s)-t} \left(\frac{\ell}{n-1}\right)^{2\ell}$$

$$\ll \binom{ne}{T}^{s} \left(\frac{(n-1)e}{\ell}\right)^{\ell} \left(\frac{s(n-s)e^{1+p}\alpha \ln n}{tn}\right)^{t} n^{-\alpha s(n-s)/n} \left(\frac{\ell}{n-1}\right)^{2\ell}$$

$$\leq \left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^{s} \left(\frac{\ell e}{n-1}\right)^{k} \left(\frac{e^{2}s(n-s)\ln n}{Tn}\right)^{t}$$

$$\leq \left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^{s} \left(\frac{e^{4}s^{2}(n-s)^{2}(\ln n)^{2}}{n^{3\ell}}\right)^{\ell}.$$
(16)

Now

$$\sum_{N'}^{n} \frac{l - a + as/n}{\Delta a_{XX}} \leq (1) e^{\omega}$$
(17)

where $a > 1 - u > / \ln n$ and u > - > oo slowly.

So if $5 \leq 3e''$ then (16) implies that

$$u(s,\ell,t) \leq n^{-(1-o(1))\ell},$$

and if s > 3e''

$$U(SJA), \leq \left(e^{l} L_{\frac{s}{n(1-o(1))}}^{+5} (1-n)^{2} (\ln n)^{2} \right)^{\ell}$$
$$= O\left(\left(\left(\frac{s}{n^{1-o(1)}} \right)^{\ell} \right).$$

Case 4.1.2: $5 > n^{1/10}$.

Claim 4 /n a.e. process, every G_m , m E $[m_0, mj \text{ i5 5wc/i} that TJ(S) \ge -y |S|$ In n /or a// $n^{1/10} \le |5| \le n/2$; where 7 > 0 is some absolute constant.

Proof (outline) For $|S^*| \ge n^{2/3}$ one can use the Chernoff bounds on the tails of the binomial rj(S). If $|S| \le n^{2/3}$ we use the fact that with high probability (i) G_{mo} has $n^{e>}$ vertices of degree ≤ 6 Inn where $e^1 = \wedge(e) \longrightarrow 0$ with 6, and (ii) in G_{mi} no set S of size $\le n/(1nn)^2$ contains 3151 edges.

So if $s \ge n^{1/10}$ then we can take $t \ge 7 s \ln n > 2 \pounds$ for some constant 7 > 0 and this case is vacuous.

Case $4.2 : t > 2 \pounds$.

$$u(s,\ell,t) \leq \left(\frac{ne}{s}\right)^{s} \left(\frac{(n-1)e}{\ell}\right)^{\ell} \left(\frac{s(n-s)e^{1+p}\alpha\ell\ln n}{tn(n-1)}\right)^{t} n^{-\alpha s(n-s)/n}$$

$$= \left(\frac{n^{1-\alpha+\alpha s/\prime\prime}e}{\sqrt{\ast}}\right)' \frac{f(n-l)e}{\sqrt{1-\varepsilon}}' \frac{fs(n-s)e'+annn}{\sqrt{1-\varepsilon}}$$
(18)

Case 4.2.1: $t \leq 2n$ and so $((n - 1)e/t)^{l} \leq (2ne/\ll)^{1/2}$ -

$$u(s,\ell,t) \le \left(\frac{n^{1-\alpha+\alpha s/n} \mathrm{e}}{s}\right) \left(\frac{20^* \mathrm{\pounds lnn}}{t^{3/2} n^{1/2}}\right).$$
(19)

Case 4.2.1.1: $s < n^{1/10}$. Now (17) gives

$$\left(\frac{n^{1-\alpha+\alpha s/n}e}{s}\right)^s \leq \left(\frac{(1+o(1))e^{\omega+1}}{s}\right)^s \\ \leq e^{(1+o(1))e^{\omega}} \\ = e^{\hat{\omega}}, \text{ say,}$$

and so (19) implies

$$u(s,\pounds,t) \leq \left(\begin{array}{c} -\mathbf{r} & \mathbf{r} \\ -\mathbf{r} & \mathbf{r} \\ \hline \mathbf{r} & \mathbf{r} \end{array} \right)^{t} \bullet$$
(20)

Case 4.2.1.2: $s \ge n^{1/10}$.

Using Claim 4 and (19),

$$u(s, \ell, t) \leq n^{-s/11} \left(\frac{1}{n^{\frac{1}{2}-o(1)}\sqrt{s}}\right)^t.$$

Case 4.2.2: $t \ge 2n$ and so $(ne/\pounds)^e \le e^n \le e^{1/2}$.

From (18),

$$u(sj,t) < \underbrace{(\frac{(1+o(1))e^{\omega+1}}{V}^{s}}_{1-10} \underbrace{(\frac{20s\ell \ln n}{tn})^{t}}_{1-10}.$$

Case 4.2.2.1: $s < n^{1/10}$.

Arguing as in (20),

$$u(s,\ell,t) \leq \left(\frac{s}{n^{1-o(1)}}\right)^t.$$

Case 4.2.2.2: $s \ge n^{1/10}$.

From Claim 4

$$u(s,\ell,t) \le \left(\frac{(1+o(1))e^{\omega+1}}{s}\right)^s \left(\frac{A\ell}{n}\right)^t.$$

for some constant A > 0. Now this clearly implies

$$u(s,\ell,t) = O(2^{-n})$$
(21)

for $\ell \leq n/(3A)$. For $\ell > n/(3A)$ we have $s \geq \ell$ and

$$u(s,\ell,t) \le n^{-s/2} A^n$$

and so (21) holds here also.

Summarising,

$$\Pr(\mathcal{D}_{\ell}) = O\left(\sum_{t=1}^{2\ell} \sum_{s=\ell+1}^{n^{1/10}} \left(\frac{s}{n^{1-o(1)}}\right)^{\ell} + \sum_{t=2\ell+1}^{2n} \sum_{s=\ell+1}^{n^{1/10}} \left(\frac{s}{n^{\frac{1}{2}-o(1)}}\right)^{t} + \sum_{t=2\ell+1}^{2n} \sum_{s=n^{1/10}}^{n/2} \left(\frac{s}{n^{\frac{1}{2}-o(1)}\sqrt{s}}\right)^{t} + \sum_{s=1}^{n^{1/10}} \sum_{t=2n+1}^{s(n-s)} \left(\frac{s}{n^{\frac{1}{2}-o(1)}}\right)^{t} + \sum_{s=n^{1/10}}^{n/2} \sum_{t=2n+1}^{s(n-s)} 2^{-n}\right)$$
$$= O(\ell n^{-(.9-o(1))\ell}).$$

where the double summations correspond to the five cases enumerated above.

Thus, we see that

$$\sum_{m=m_0}^{m_1} \sum_{\ell=2}^{n/10} \Pr_m(\mathcal{D}_\ell) = O((n \ln n)(\sqrt{n \ln n})n^{-1.7})$$

= $o(1).$ (22)

We are thus left with $\Pr\left(\bigcup_{m=m_0}^{m_1}(\mathcal{C}_m \cap \mathcal{A}_{n-2})\right)$.

We consider G_{m_0} . We know that a.e. G_{m_0} consists of a giant connected component C plus $O(e^{\omega})$ isolated vertices T. If $\bigcup_{m=m_0}^{m_1} (\mathcal{C}_m \cap \mathcal{A}_{n-2})$ occurs at some time during the process then either

(i) there exist $u, v \in T$ such that the first edges of the process that are incident with each of u and v are the same colour,

OR

(ii) there exists a colour c and a set $S, 2 \leq |S| \leq n/2$ such that in G_{m_0} the $t \geq 2$ $(S : \overline{S})$ edges are all of colour c.

(Suppose that deleting the edges of colour c from G_m produces at least three components. If colour k has not occurred by time m_0 then two of these components must be vertices from T, contradicting (i). If G_{m_0} has edges of colour c then deleting these edges must beak C into at least three pieces.)

Clearly

$$\Pr((i)) = o(1) + O(e^{2\omega}/n) = o(1).$$

Furthermore

$$\begin{aligned} \Pr_{p}((ii)) &\leq \sum_{s=2}^{n/2} \binom{n}{s} n \sum_{t=2}^{s(n-s)} \binom{s(n-s)}{t} \left(\frac{p}{n}\right)^{t} (1-p)^{s(n-s)-t} \\ &\leq 2 \sum_{s=2}^{n/2} \binom{n}{s} n \sum_{t=2}^{10\ln n} \frac{(s(n-s))^{t}}{t!} \left(\frac{\alpha \ln n}{n^{2}}\right)^{t} n^{-\alpha s} \\ &\leq n \sum_{s=2}^{n/2} \left(\frac{n^{1-\alpha}}{s}\right)^{s} \sum_{t=2}^{10\ln n} \left(\frac{s\alpha \ln n}{n}\right)^{t} \\ &= O(n^{-(1-o(1)}). \end{aligned}$$

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The upper bound is good enough to apply (8) and so $\Pr_{m_0}((ii)) = o(1)$. Thus

$$\Pr\left(\bigcup_{m=m_0}^{m_1} (\mathcal{C}_m \cap \mathcal{A}_{n-2})\right) = o(1).$$
(23)

Our theorem now follows from (7),(10),(11),(14),(15),(22) and (23).

References

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