A NEUMANN PROBLEM WITH CRITICAL SOBOLEV EXPONENT

by

Myriam Comte

Laboratoire D'Analyse Numerique

Tour 55–65, 5^{e} Etage
Universite Pierre Ete Marie Curie
4 Place Jussieu, 75252 Paris Cedex 05, France

Gabriella Tarantello

Carnegie Mellon University
Department of Mathematics
Pittsburgh, PA 15213 U.S.A.

Research Report No. 91–131

August 1991

A NEUMANN PROBLEM WITH CRITICAL SOBOLEV EXPONENT

Myriam Comte
Laboratoire D'Analyse Numerique
Tour 55–65, 5^e Etage
Universite Pierre Et Marie Curie
4 Place Jussieu, 75252 Paris Cedex 05, France

Gabriella Tarantello*
Carnegie Mellon University
Department of Mathematics
Pittsburgh, PA 15213 USA

*Supported in part by NSF grant DMS–9003149

INTRODUCTION:

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 3$ with C^2 boundary $\partial\Omega$. Given $\lambda > 0$, we consider the following Neumann boundary value problem,

$$(1)\begin{cases} -\Delta u = |u|^{p-2}u - \lambda u & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

where n is the outward pointing normal on $\partial\Omega$ and $p = \frac{2N}{N-2}$ is the best exponent in the Sobolev embedding $H^1(\Omega) \longrightarrow L^p(\Omega)$. In this setting one easly checks that $u = 0$ and $u = \pm \lambda^{\frac{1}{p-2}}$ are solutions of (1). We shall refer to these as the *trivial* solutions.

In finding nontrivial solutions one has to deal with a lack of compactness. Using a variational approach in the same spirit of [B–N], results in this direction have been obtained in [A–M], [C–K] and [W]. There it is shown that, for a suitable constant $\lambda_* = \lambda_*(\Omega) > 0$, problem (1) admits a nontrivial *positive* solution, provided $\lambda > \lambda_*$.

Here we are concerned with changing sign solutions of (1). To this purpose, for given $u \neq 0$, denote by $(\mu_1(u), v_1(u))$ the first eigenpair for the eigenvalue problem:

$$\begin{cases} -\Delta v + \lambda v = \mu |u|^{p-2}v & \text{on } \Omega \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

$\mu \in \mathbb{R}$.

Since $\lambda > 0$, the variational characterization of the eigenvalues gives that $\mu_1(u) > 0$ and $v_1(u)$ cannot change sign in Ω.

We have:

Theorem 1: For $N \geq 5$ and $\lambda > 0$, there exists a nontrivial solution u of (1) satisfying:

$$\int_\Omega |u|^{p-2}u\, v_1(u) = 0,$$

in particular u changes sign in Ω.

□

Previous result on changing sign solutions of (1) have been established in [C–K,1] for domains with symmetries. See also [C–K] and [C].

Furthermore when $\lambda \le 0$, every solution of (1) must change sign. Existance in this situation has been established in [C–K].

We also point out that for Dirichlet boundary conditions the analogous of Theorem 1 has been established in [T], provided $N \ge 6$. See also [C–S–S] and [Z].

We follow [T] and first prove Theorem 1 in the sub–critical case where we replace $p = \frac{2N}{N-2}$ with $q \in (2, \frac{2N}{N-2})$.

That is, for given $2 < q < \frac{2N}{N-2}$ we show that the problem:

$$(1)_q \begin{cases} -\Delta u = |u|^{q-2}u - \lambda u & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

admits a solution $u = u_q$ satisfying the orthogonality condition, $\int_\Omega |u|^{q-2}uv_1(u) = 0$ where $v_1(u)$ is the first eigenfunction for the eigenvalue problem:

$$(*)_q \begin{cases} -\Delta v + \lambda v = \mu |u|^{q-2} v & \text{on } \Omega \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

This will be obtained applying the Ljusternik–Schnirelman theory to the even functional

$$I_q(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \lambda u^2 - \frac{1}{q}\int_\Omega |u|^q , \quad u \in H^1(\Omega)$$

whose critical points correspond to solutions of $(1)_q$.

To conclude, we then show that for a sequence $\epsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$, the given solution u_{ϵ_n} of $(1)_{p-\epsilon_n}$ converges (strongly) to a solution of (1) and the orthogonality condition is preserved at the limit.

Acknowledgement: The authors wish to express their gratitude to Prof. H. Brezis for his encouragement and support. They also thank respectively, the Department of Mathematics of Carnegie Mellon University and the Laboratoire d'Analyse Numerique of Paris VI for their kind hospitality during the preparation of this work.

1. Subcritical Case:

In this section we establish Theorem 1 in the subcritical case. To this purpose, let $q \in (2, \frac{2N}{N-2})$ be given.

For $u \in L^q(\Omega)$, $u \neq 0$ denote by $(\mu_1(u), v_1(u))$ the *first* eigenpair for the eigenvalue problem:

$$(*)_q \begin{cases} -\Delta v + \lambda v = \mu |u|^{q-2} v & \text{on } \Omega \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

$\mu \in \mathbb{R}$.

Namely,

$$\mu_1(u) = \inf \left\{ \frac{\|\nabla w\|_2^2 + \lambda \|w\|_2^2}{\int_\Omega |u|^{q-2} w^2} \quad w \in H^1(\Omega) \quad w \neq 0 \right\}$$

and $v_1(u) \in H^1(\Omega)$ satisfies:

$$\mu_1(u) = \frac{\|\nabla v_1(u)\|_2^2 + \lambda \|v_1(u)\|_2^2}{\int_\Omega |u|^{q-2} v_1^2(u)}.$$

The eigenfunction $v_1(u)$ is uniquely determined under the normalization:

$$\int_\Omega |u|^{q-2} v_1^2(u) = 1 \quad \text{and} \quad v_1(u) > 0 \quad \text{on } \Omega.$$

This allows to establish the following:

Lemma 1.1: For $q \in (2, \frac{2N}{N-2}]$ the map:

$$L^q(\Omega) \longrightarrow H^1(\Omega)$$

$$u \longrightarrow v_1(u)$$

is continuous.

□

We omit the tedious details.

Remark 1.1: Continuity also holds with respect to the parameter q. That is if $q_n \longrightarrow q$ then $v_{1,q_n} \longmapsto v_{1,q}$ in $H^1(\Omega)$. Here, v_{1,q_n} and $v_{1,q}$ denote the first eigenfunction of $(1)_{q_n}$ and $(1)_q$ respectively, normalized as above.

Set $H = H^1(\Omega)$ and denote by $\| \ \|$ and $(\cdot,\cdot)$ the corresponding norm and scalar product.

We have already noticed that the solutions of $(1)_q$ correspond to the critical point of the functional,

$$I_q(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \lambda u^2 - \frac{1}{q}\int_\Omega |u|^q \; ; \; u \in H.$$

To find the desired critical point of I_q we use the Ljusternik–Schnirelman theory for even functional.

To this purpose, let $A \subset H$ be a closed and symmetric set (i.e. $u \in A \Longrightarrow -u \in A$), and denote by $i(A) \in \mathbb{N}$ the Krosnoselski's genus of A (see [R]).

For $k \in \mathbb{N}$ define,

$\mathcal{F}_k = \{A \subset H$ closed, symmetric: $i(A \cap h(S)) \geq k \quad \forall h \in \mathcal{H}\}$,

where $\mathcal{H} = \Big\{h : H \longrightarrow H$ even homeomorphism$\Big\}$ and $S = \Big\{u \in H : \|u\| = 1\Big\}$.

Set,

$$c_k = \inf_{A \in \mathcal{F}_k} \sup_A I_q \quad .$$

It is not difficult to check that $-\infty < c_1 \leq c_2 \leq c_3 \leq \ldots$

We have:

Theorem 1': For $q \in \left[2, \frac{2N}{N-2}\right]$ and $\lambda > 0$, there exists a nontrivial solution u of $(1)_q$ satisfying:

$$(1)\ I(u) = c_2 \; ; \quad (2) \int_\Omega |u|^{q-2} u \, v_1(u) = 0 \; ;$$

where $v_1(u)$ is the first eigenfunction for the eigenvalue problem $(*)_q$.

Remark 1.2: Using well known arguments (cf [R]) and the fact that I_q satisfies the P.S. condition (see below) it follows that the c_k's are critical values for I_q for $k = 1, 2, 3, \ldots$. However for $k \geq 3$, we are unable to establish condition (2). This is due to the fact that lemma 1.1, which relies on the simplicity of the first eigenvalue, becomes difficult to obtain when $k \geq 2$.

Proof: We follow [T]. This approach has been inspired by an argument of Coffman (cf [Co]) in connection with the nodal properties of an eigenvalue problem of O.D.E.

Since $q < \frac{2N}{N-2}$, it is a simple exercise to check that the functional I_q satisfies the Palais–Smale (P.S.) condition.

That is, every sequence $\{u_n\} \subset H$ satisfying:

$$\text{i) } I_q(u_n) \longrightarrow c, \quad c \in \mathbb{R}$$

$$\text{ii) } \|I_q'(u_n)\| \longrightarrow 0 \text{ in } H$$

admits a *convergent* subsequence.

We shall refer to these sequences as (P.S.) sequences.

Fact 1: For every $A \in \mathcal{F}_2$ there exists $u \in A \cap \Lambda_q$:

$$\int_\Omega |u|^{q-2} u \, v_1(u) = 0$$

where

$$\Lambda_q = \left\{ u \in H : u \neq 0 \text{ and } < I_q'(u), u > = 0 \right\}$$

To see this, notice that the map:

$$h: \Lambda_q \longrightarrow S$$
$$u \longrightarrow \frac{u}{\|u\|}$$

defines an even homeomophism; (it is essential here that $\lambda > 0$).

Hence for every $A \in \mathcal{F}_2$ we have:

$$i(A \cap \Lambda_q) \geq 2 \tag{1.1}$$

Furthermore, the map $\psi: A \cap \Lambda_q \longrightarrow \mathbb{R}$ given by:

$$\psi(u) = \int_\Omega |u|^{q-2} v_1(u)\, u$$

is odd and continuous (see lemma 1.1). In virtue of (1.1) it must vanish somewhere.

Fact 2: If $u \in \Lambda_q$ and $\int_\Omega |u|^{q-2} v_1(u)\, u = 0$, then $I(u) \geq c_2$.

To establish this, denote by $(\mu_2(u), v_2(u))$ the *second* eigenpair for $(*)_q$. Hence,

$$\mu_2(u) = \inf \left\{ \frac{\|\nabla w\|_2^2 + \lambda \|w\|_2^2}{\int_\Omega |u|^{q-2} w^2},\ w \in H \setminus \{0\} \text{ and } \int_\Omega |u|^{q-2} v_1(u)\, w = 0 \right\}$$

Therefore,

$$\mu_2(u) \le \frac{\|\nabla u\|_2^2 + \lambda\|u\|_2^2}{\int_\Omega |u|^q} = 1$$

Thus, if we let $A = \text{span}\left\{v_1(u), v_2(u)\right\}$ we derive that,

$$A \in \mathcal{F}_2 \text{ and } \frac{\|\nabla w\|_2^2 + \lambda\|w\|_2^2}{\int_\Omega |u|^{q-2} w} \le 1 \quad \forall\, w \in A \text{ and } w \ne 0.$$

In turn,

$$\sup_A I_q \ge c_2 \tag{1.2}$$

Moreover the structure of I_q guarantees that the suprimum in (1.2) is obtained at some point $\omega_0 \in A$, which in particular satisfies,

$$< I_q' (\omega_0), \omega_0 > = 0.$$

This yields:

$$1 \ge \frac{\|\nabla \omega_0\|_2^2 + \lambda\|\omega_0\|_2^2}{\int_\Omega |u|^{q-2}\omega_0^2} \ge \frac{\|\nabla \omega_0\|_2^2 + \lambda\|\omega_0\|_2^2}{\|u\|_q^{q-2}\;\|\omega_0\|_q^2} = \frac{\|\omega_0\|_q^{q-2}}{\|u\|_q^{q-2}}$$

That is,

$$I_q(u) = \frac{1}{N}\|u\|_q^q \ge \frac{1}{N}\|\omega_0\|_q^q = I(\omega_0) \ge c_2.$$

Since I_q satisfies the (P.S.) condition, we will conclude the proof by using a suitable form of the deformation lemma as obtained in [B–N,1]. For $k \in \mathbb{N}$ fixed, there exist a deformation $y = y(t,u)$, $t \in [0,1]$ $u \in H$, and a constant $0 < \delta_k < \frac{1}{k}$ satisfying:

$$\text{(I)}\quad y(t, \cdot) \text{ is an homeomophism and } I_q(y(t, u)) \leq I_q(u) \quad \forall t;$$

$$\text{(II)}\quad \text{if } I_q(u) < c_2 + \delta_k \text{ and } I_q(y(1, u)) > c_2 - \delta_k \Longrightarrow \| I_q'(y(t, u)) \| < \frac{1}{k} \quad \forall t \in [0, 1];$$

(see [B–N, 1 Corollary 4].

In addition, the oddness of I_q' will allow to take $y(t, u)$ to be odd in u.

Set $h(u) = y(1, u)$, so $h \in \mathscr{H}$.

By definition of c_2, there exists $\tilde{A}_k \in \mathscr{F}_2$:

$$I_q(u) \leq c_2 + \delta_k \qquad \forall\, u \in \tilde{A}_k.$$

Let

$$A_k = h(\tilde{A}_k), \text{ hence } A_k \in \mathscr{F}_2 \text{ and}$$

$$I_q(u) \leq c_2 + \delta_k, \quad u \in A_k.$$

As derived above, we can find $u_k \in A_k$ such that,

$$u_k \in \Lambda_q \text{ and } \int_\Omega |u_k|^{q-2} u_k\, v_1(u_k) = 0;$$

moreover,

$$I_q(u_k) \geq c_2. \tag{1.3}$$

Let $v_k \in \tilde{A}_k$ with $h(v_k) = u_k$. In particular,

$$I_q(v_k) \le c_2 + \delta_k$$

and from (1.3) we can apply (II) to conclude:

$$\| I_q'(u_k) \| \le \frac{1}{k}, \quad c_2 \le I_q(u_k) \le c_2 + \frac{1}{k}.$$

Thus, $\left\{u_k\right\}$ defines a (P.S.) sequence for I_q and we can extract a subsequence converging to a function u with the desired properties.

This conclude the proof of Theorem 1′.

2. The Critical Case

In this section we carry out the limiting process.

For $\epsilon > 0$ small, let $p_\epsilon = p - \epsilon$. Set

$$c_{2,\epsilon} = I_{p_\epsilon}(u_\epsilon)$$

where u_ϵ is the solution obtained by Theorem 1′.

In particular, u_ϵ is a critical point of I_{p_ϵ} and $\int_\Omega |u_\epsilon|^{p_\epsilon - 2} u_\epsilon v_\epsilon = 0$

(we have set $v_\epsilon = v_1(u_\epsilon)$).

To shorten notation, set $I_\epsilon = I_{p_\epsilon}$ and $I = I_{\epsilon = 0}$.

Associated to I_ϵ and I are respectively the manifolds:

$$\Lambda_\epsilon = \left\{u \ne 0 : (I_\epsilon'(u), u) = 0\right\}$$

and

$$\Lambda = \left\{ u \neq 0 : (I'(u), u) = 0 \right\}.$$

One easily checks that I_ϵ and I are bounded below on Λ_ϵ and Λ respectively.
Furthermore, the minimization problem:

$$\inf_{\Lambda_\epsilon} = c_{1,\epsilon} \text{ and } \inf_{\Lambda} I = c_1 \tag{1.4}$$

obtain their infimum respectively at some point $u_{1,\epsilon} \in \Lambda_\epsilon$ and $u_1 \in \Lambda$. This follows easily for I_ϵ since it satisfies the (P.S.) condition (see [L–N–T]), while it is more delicate for I and it has been established in [W] (see also [A–M] and [C–K]).
Using these facts, it follows:

Lemma 2.1:

$$c_{1,\epsilon} \longrightarrow c_1 \quad \text{as } \epsilon \longrightarrow 0 \tag{1.5}$$

□

The proof can be obtained as in [T] with the obvious modifications.

To carry out our compactness argument we need a crucial estimate on the value $c_{2,\epsilon}$.
This will be the content of next section.

(2.1) Estimates for $c_{2,\epsilon}$.

Let $S = S(N)$ be the best constant in the Sobolev embedding: $H_0^1 \longrightarrow L^{\frac{2N}{N-2}}$
(see [Ta] for the precise value of S).
We have:

Proposition 2.1: Let $N \geq 5$ and $\lambda > 0$. There exist $\sigma > 0$ and $\epsilon^* > 0$ such that

$$c_{2,\epsilon} < c_{1,\epsilon} + \frac{S^{N/2}}{2N} - \sigma \tag{2.1}$$

for every $\epsilon \in [0, \epsilon^*]$.

Proof: Since the bondary of Ω is of class C^2, there exists a point $x_0 \in \partial\Omega$ where the scalar curvature of $\partial\Omega$ is strictly positive. Without loss of generality we can take $x_0 = 0$. Thus, in local coordinates near $x_0 = 0$ we can write $\partial\Omega$ as:

$$x_N = \sum_{i=1}^{N-1} c_i x_i^2 + 0(|x|^3) \text{ with } c_i > 0 \; i = 1,\dots, N-1$$

and

$$\Omega \subset \left\{x = (x_1,\dots, x_N) : x_N > 0\right\}.$$

We follow Comte–Knaap [C–K] and choose $0 < R_1 < R_2$ such that for sufficiently small $\rho > 0$ we have:

$$\tilde{B}_{R_1} \cap B_\rho \subset \Omega \cap B_\rho \subset \tilde{B}_{R_2} \cap B_\rho$$

where $\tilde{B}_R$ is the ball of radius R and center $(0,\dots, 0, R)$ and $B_\rho = \left\{x \in \mathbb{R}^N : |x| < \rho\right\}$.

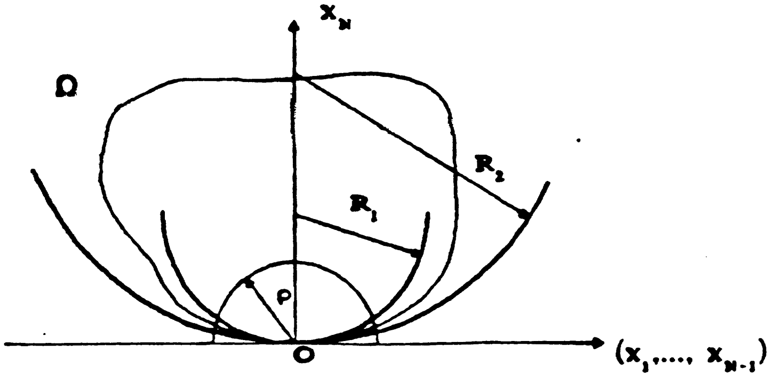

Fig. 1

Set

$$U_\delta(x) = \frac{(N(N-2)\,\delta)^{\frac{N-2}{4}}}{(\delta + |x|^2)^{\frac{N-2}{2}}} \qquad \delta > 0\,, x \in \mathbb{R}^N.$$

the extremal functions for the Sobolev inequality.

They satisfy,

$$-\Delta U_\delta = U_\delta^{p-1} \text{ in } \mathbb{R}^N.$$

Let u_1 be the solution of (1) such that $I(u_1) = c_1$ (see (1.4)).

Define

$$A_\delta = \text{span}\left\{u_1\,, U_\delta\right\} \in \mathcal{F}_2.$$

Thus,

$$c_{2,\epsilon} \leq \sup_{A_\delta} I_\epsilon\,, \qquad \delta > 0.$$

To estimate $\sup I_\epsilon$ we recall the following:

Calculus Lemma

For $1 < q < +\infty$, there exists a constant $C > 0$ (depending on q only) such that for $\alpha, \beta \in \mathbb{R}$ we have:

$$\left|\, |\alpha + \beta|^q - |\alpha|^q - |\beta|^q \right| \leq C\,(|\alpha|^{q-1}\,|\beta| + |\alpha|\,|\beta|^{q-1}).$$

(cf. [B–L]).

In virtue of the calculus lemma we have:

$$I_\epsilon(su_1 + tU_\delta) \le \frac{s^2}{2}\|u_1\|_p^p - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} + \frac{t^2}{2}(\|\nabla U_\delta\|_2^2 + \lambda\|U_\delta\|_2^2)$$

$$- \frac{|t|^{p_\epsilon}}{p_\epsilon}\|U_\delta\|_{p_\epsilon}^{p_\epsilon} + st\int_\Omega u_1^{p_\epsilon - 1} U_\delta$$

$$+ C\left\{\int_\Omega |su_1|\,|tU_\delta|^{p_\epsilon - 1} + \int_\Omega |su_1|^{p_\epsilon - 1}|tU_\delta|\right\}$$

Using well known estimates on the function U_δ (see[B–N]) and the fact that $u_1 \in L^\infty(\Omega)$, for $\epsilon > 0$ small, it is not difficult to derive the following:

(i) $$|st\int_\Omega u_1^{p_\epsilon - 1} U_\delta| \le k_1 (s^2 + t^2)\,\delta^{\frac{N-2}{4}}$$

(ii) $$|st|\int_\Omega |u_1|\,|U_\delta|\,(|su_1|^{p_\epsilon - 2} + |t\,U_\delta|^{p_\epsilon - 2})| \le k_1 (|s|^{p_\epsilon} + |t|^{p_\epsilon})\,\delta^{\frac{N-2}{4}}$$

for a suitable constant $k_1 > 0$.

Notice in fact that,

$$\|U_\delta\|_1 = 0\,(\delta^{\frac{N-2}{4}}) \text{ and } \|U_\delta\|_{p_\epsilon - 1}^{p_\epsilon - 1} = 0\,(\delta^{\frac{N-2}{4}}).$$

Substituting in the above expression we obtain:

$$I_\epsilon(su_1 + tU_\delta) \le \frac{s^2}{2}\|u_1\|_p^p - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} + \frac{t^2}{2}(\|\nabla U_\delta\|_2^2 + \lambda\|U_\delta\|_2^2)$$

$$-\frac{|t|^{p_\epsilon}}{p_\epsilon}\|U_\delta\|_{p_\epsilon}^{p_\epsilon} + k_1(s^2+t^2)\,\delta^{\frac{N-2}{4}} + k_2\,(|s|^{p_\epsilon} + |t|^{p_\epsilon})\,\delta^{\frac{N-2}{4}}$$

$k_2 > 0$.

Since $\|U_\delta\|_{p_\epsilon}^{p_\epsilon} \ge A\ \delta^{\epsilon(\frac{N-2}{4})} - a\ \delta^{1/2 + \epsilon\,(\frac{N-2}{4})}$ for suitable A , $a > 0$, we can find positive constants M and θ (independent of ϵ and δ) such that,

$$I_\epsilon(su_1 + tU_\delta) \le 0$$

for $s^2 + t^2 \ge M\ \delta^{-\theta\,\epsilon}$.

On the other hand, for $s^2 + t^2 \le M\ \delta^{-\theta\,\epsilon}$ we have:

$$I_\epsilon(su_1 + tU_\delta) \le \frac{s^2}{2}\|u_1\|_p^p - \frac{|s|^{p_\epsilon}}{p_\epsilon}\|u_1\|_{p_\epsilon}^{p_\epsilon} + \frac{t^2}{2}(\|\nabla U_\delta\|_2^2 + \lambda\|U_\delta\|_2^2)$$

$$-\frac{|t|^{p_\epsilon}}{p_\epsilon}\|U_\delta\|_{p_\epsilon}^{p_\epsilon} + k_3\,\delta^{\frac{N-2}{4} - \theta_1\epsilon}$$

with $k_3 > 0$ and $\theta_1 > 0$.

In other words,

$$I_\epsilon(su_1 + tU_\delta) \le \left[\frac{1}{2} - \frac{1}{p_\epsilon}\right]\left[\frac{\|u_1\|_p^p}{\|u_1\|_{p_\epsilon}^2}\right]^{\frac{p_\epsilon}{p_\epsilon - 2}} +$$

$$+\left[\frac{1}{2}-\frac{1}{p_\epsilon}\right]\left[\frac{\|\nabla U_\delta\|_2^2+\lambda\|U_\delta\|_2^2}{\|U_\delta\|_p^2}\right]^{\frac{p_\epsilon}{p_\epsilon-2}}\left[\frac{\|U_\delta\|_p}{\|U_\delta\|_{p_\epsilon}}\right]^{\frac{2p\epsilon}{p_\epsilon-2}}+k_3\,\delta^{\frac{N-2}{4}-\theta_1\epsilon}\qquad\forall\, s,t\in\mathbb{R}.$$

For $N\geq 5$, our choice of U_δ allows a sharp estimate of the following type:

$$\frac{\|\nabla U_\delta\|_2^2+\lambda\|U_\delta\|_2^2}{\|U_\delta\|_p^2}\leq\frac{S}{2^{2/N}}-c\,\delta^{1/2}+o(\delta^{1/2})\tag{2.2}$$

for suitable $c>0$.

The proof of (2.2) can be found in [A–M], [C–K] and [W].

From (2.2) we derive:

$$I_\epsilon(su_1+tU_\delta)\leq\frac{1}{N}\|u_1\|_p^p+\frac{S^{N/2}}{2N}\delta^{-\theta_2\epsilon}-c_1\,\delta^{\frac{1}{2}-\theta_2\epsilon}+k_3\,\delta^{\frac{N-2}{4}-\theta_1\epsilon}$$

$$+\,0(\epsilon)+o\,(\delta^{\frac{1}{2}-\theta_2\epsilon}).$$

with $\theta_2, c_1>0$.

Since $N\geq 5$, we can fix $\epsilon_0>0$ such that

$$\frac{N-2}{4}-\theta_1\,\epsilon_0>\frac{1}{2}.$$

Consequently, choosing $\delta_0>0$ sufficiently small we have:

$$-c_1\,\delta_0^{\frac{1}{2}-\theta_2\epsilon}+k_5\,\delta_0^{\frac{N-2}{4}-\theta_1\epsilon}+o\,(\delta^{\frac{1}{2}-\theta_2\epsilon})\leq-\frac{c_1}{2}\,\delta_0^{\frac{1}{2}}:=-2\sigma$$

for all $\delta \in [0\,,\,\delta_0]$ and $\epsilon \in [0, \epsilon_0]$.

In conclusion;

$$\sup_{A_{\delta_0}} I_\epsilon \le c_{1,\epsilon} + \frac{1}{2N} S^{N/2} + \left[\left[1 - \delta_0^{-\theta_2 \epsilon}\right] \frac{S^{N/2}}{2N} + c_1 - c_{1,\epsilon} + 0(\epsilon) \right] - 2\sigma$$

Since the term in the square bracket tends to zero as $\epsilon \longrightarrow 0$, we derive (2.1) by taking $\epsilon > 0$ sufficiently small.

(2.2) The Existence of the Solution:

Theorem 1 will be established as soon as we obtain the following result for the given solution u_ϵ of $(1)_{p-\epsilon}$.

Proposition 2.2:

There exists a sequence $\epsilon_n \longrightarrow 0$ and $u \in H^1(\Omega)$ such that,

$$u_{\epsilon_n} \longrightarrow u \text{ strongly in } H^1(\Omega),$$

$$\int_\Omega |u|^{p-2} u\, v_1(u) = 0.$$

In particular u satisfies (1).

Proof:

Since we have seen that $c_{2,\epsilon}$ is bounded uniformly in ϵ and u_ϵ satisfies $(1)_{p_\epsilon}$, it is not difficult to check that,

$$\| \nabla u_\epsilon^\pm \|_2 \le K\,, \quad (\epsilon > 0 \text{ small, } K > 0)$$

where,

$$u_\epsilon^+ = \max \left\{ u_\epsilon\,, 0 \right\} \in H^1(\Omega) \setminus \{0\}$$

and

$$u_\epsilon^- = \max\left\{-u_\epsilon, 0\right\} \in H^1(\Omega) \setminus \{0\}.$$

Thus, for a sequence $\epsilon_n \longrightarrow 0$ we can find $u^+, u^- \in H^1(\Omega)$ such that,

$$u_{\epsilon_n}^\pm \longrightarrow u^\pm \text{ weakly in } H^1(\Omega).$$

To shorten notation, set $u_n^\pm = u_{\epsilon_n}^\pm$, $c_{i,n} = c_{i,\epsilon_n}$ $i = 1, 2$, $p_n = p_{\epsilon_n}$, $I_n = I_{\epsilon_n}$ and $\Lambda_n = \Lambda_{\epsilon_n}$.

We claim that $u^+ \neq 0$ and $u^- \neq 0$.

To see this, notice that $u_n^\pm \in \Lambda_n$; thus:

$$I_n(u_n^\pm) \geq c_{1,n} \tag{2.3}$$

On the other hand, for n large we have:

$$I_n(u_n^+) + I_n(u_n^-) = I_n(u_n) = c_{2,n} < c_{1,n} + \frac{S^{N/2}}{2N} - \sigma$$

That is,

$$I_n(u_n^\pm) < \frac{S^{N/2}}{2N} - \sigma \tag{2.4}$$

for n large.

Moreover, a well known inequality of Cherrier (see [Ch]) gives that, $\forall\, \tau > 0$ then exists a constant $M_\tau > 0$ such that,

$$\left(\frac{S}{2^{2/N}} - \tau\right) \| u \|_p^2 \leq \| \nabla u \|_2^2 + M_\tau \| u \|_2^2 \tag{2.5}$$

$\forall\, u \in H^1(\Omega)$.

We derive,

$$\| \nabla u_n^{\pm} \|_2^2 + \lambda \| u_n^{\pm} \|_2^2 = \| u_n^{\pm} \|_{p_n}^{p_n} \le K_1 \left(\| \nabla u_n^{\pm} \|_2^2 + \lambda \| u_n^{\pm} \|_2^2 \right)^{p_n/2}$$

$K_1 > 0$, and therefore:

$$\| u_n^{\pm} \|_p \ge k_2 > 0 \quad \text{for large } n. \tag{2.6}$$

Arguing by contradiction, assume for example that $u^+ \equiv 0$.

This implies,

$$\frac{1}{2} \| \nabla u_n^+ \|_2^2 - \frac{1}{p_n} \| u_n^+ \|_{p_n}^{p_n} \le \frac{S^{N/2}}{2N} - \sigma + o(1) \tag{2.7}$$

and

$$\| \nabla u_n^+ \|_2^2 - \| u_n^+ \|_{p_n}^{p_n} = o(1) \tag{2.8}$$

Consequently,

$$\left[\frac{1}{2} - \frac{1}{p_n} \right] \| u_n^+ \|_{p_n}^{p_n} \le \frac{1}{2N} S^{N/2} - \sigma + o(1)$$

Now fix $\tau_0 > 0$ in (2.5) such that

$$\left[\frac{S}{2^{2/N}} - \tau_0 \right]^{N/2} > \frac{S^{N/2}}{2} - \frac{\sigma}{2} \tag{2.9}$$

From (2.8) we obtain,

$$|\Omega| \, \frac{2\,(p - p_n)}{p_n p} \| u_n^+ \|_{p_n}^{p_n - 2} \| u_n^+ \|_p^2 \ge \| u_n^+ \|_{p_n}^{p_n} = \| \nabla u_n^+ \|_2^2 + o(1) \ge$$

$$\left[\frac{S}{2^{N/2}} - \tau_0\right] \| u_n^+ \|_p^2 - M_0 \| u_n^+ \|_2^2 + o(1)$$

($|\Omega|$ = Lebesgue measure of Ω).

But $\| u_n^+ \|_p$ is bounded below away from zero, (see (2.6)). So

$$\| u_n^+ \|_{p_n}^{p_n-2} \geq \frac{S}{2^{N/2}} - \tau_0 + o(1).$$

That is,

$$\frac{1}{2} \| \nabla u_n^+ \|_2^2 - \frac{1}{p_n} \| u_n^+ \|_{p_n}^{p_n} = \frac{1}{N} \| u_n^+ \|_{p_n}^{p_n} + o(1) \geq \frac{1}{N} \left[\frac{S}{2^{N/2}} - \tau_0\right]^{N/2} + o(1)$$

which is impossible in virtue of (2.9).

Similarly one sees that $u^- \neq 0$.

Set $u = u^+ - u^- \neq 0$. Clearly $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ and u is a (changing sign) solution for (1). We claim that (a subsequence of) u_n converges strongly to u in $H^1(\Omega)$. This can be seen easly by setting $u_n = u + w_n$ with $w_n \rightharpoonup 0$ weakly in $H^1(\Omega)$.

Since $I(u) \geq c_1$, we obtain:

$$c_{1,n} + \frac{S^{N/2}}{2N} - \sigma \geq I_n(u + w_n) = I(u) + \frac{1}{2} \| \nabla w_n \|_2^2 - \frac{1}{p_n} \| w_n \|_{p_n}^{p_n} + o(1)$$

$$\geq c_1 + \frac{1}{2} \| \nabla w_n \|_2^2 - \frac{1}{p_n} \| w_n \|_{p_n}^{p_n} + o(1);$$

that is,

$$\frac{1}{2}\| \nabla w_n \|_2^2 - \frac{1}{p_n}\| w_n \|_{p_n}^{p_n} \leq \frac{S^{N/2}}{2N} - \sigma + o(1) \tag{2.10}$$

Furthermore,

$$0 = \left[I_n'(u_n), u_n\right] = \left[I'(u), u\right] + \| \nabla w_n \|_2^2 - \| w_n \|_{p_n}^{p_n} + o(1)$$

or,

$$\| \nabla w_n\|_2^2 - \| w_n \|_{p_n}^{p_n} = o(1) \tag{2.11}$$

As above, one sees that conditions (2.10) and (2.11) can hold simultaneously only if $\lim_{n \to +\infty} \| \nabla w_n \|_2 = 0$.

Moreover (for a subsequence of u_n) we have:

$$0 = \int_\Omega |u_n|^{p_n - 1} u_n v_1(u_n) \longrightarrow \int_\Omega |u|^{p-1} u \, v_1(u)$$

This concludes the proof.

REFERENCES

[A.M] Adimurthi and G. Mancini, The Neumann Problem for Elliptic Equations with Critical Nonlinearity. To appear.

[B.L] H. Brezis and E. Lieb, A relation between pointwise convergence of functionals. Proc. Amer. Math. Soc. **88** (1983), 486–490.

[B.N.1] H. Brezis and L. Nirenberg, Remarks on finding Critical points. To appear.

[B.N.2] H. Brezis and L. Nirenberg, A. Minimization Problem with Critical Exponent and Non Zero Data. In "Symmetry in Nature" Scuola Norm. Sup. Pisa, (1989) 129–140.

[C.S.S] G. Cerami – S. Solimini and M. Struwe, Some Existence result for Nonlinear Elliptic Problem Involving Critical Sobolev Exponent, J. of Funct. Anal. **69** (1986), 289–309

[Ch] P. Cherrier, Meilleures constants dans les inígalités relatives aux espaces de Sobolev, Bull. Sc. Math., 2^{e} série, **108** (1984), 225–262.

[Co] C. V. Coffman, Ljusternik – Schnirelman Theory: Complimentary Principles and Morse Index. Nonlinear Analysis T.M.A. **12,5** (1988), 507–529.

[C] M. Comte, Existence de solutions d'une équation elliptique avrc exposant de Sobolev critique. Thèse de Doctorat de l'Université Paris 6, 1991.

[C.K.1] M. Comte and M.C. Knaap, Solutions of elliptic equations involving critical Sobolev exponents with Neumann boundary condition. Manuscripta Math., **69** (1990), 63–70.

[C.K.] M. Comte and M. C. Knaap, Existence of solutions of elliptic equations involving critical Sobolev exponents with Neumann boundary condition in general domains. To appear in Differential and Integral Equations.

[L.N.T] C.S. Lin, W. M. Ni and I. Tagaki, Large amplitude stationary solutions to a chemotaxis system. J. Differential Equations., **72** (1988), 1–27.

[R] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Regional Conference Series Math. **65** A.M.S. Providence (1986).

[Ta] G. Talenti, Best Constant in Sobolev Inequality. Ann. Math. Pure Appl., 110 (1976), 353–372.

[T] G. Tarantello, Nodal solutions of semilinear elliptic equation with critical exponent. To appear in Differential and Integral Equations.

[W] X. J. Wang, Neumann problem of semilinear elliptic equations involving critical Sobolev exponents. To appear in J. Differential Equations.

[Z] D. Zhang, On multiple solutions of $\Delta u + \lambda u + |u|^{\frac{4}{N-2}} u = 0$, Nonlinear Analysis T.M.A. **13** (1989), 353–372.

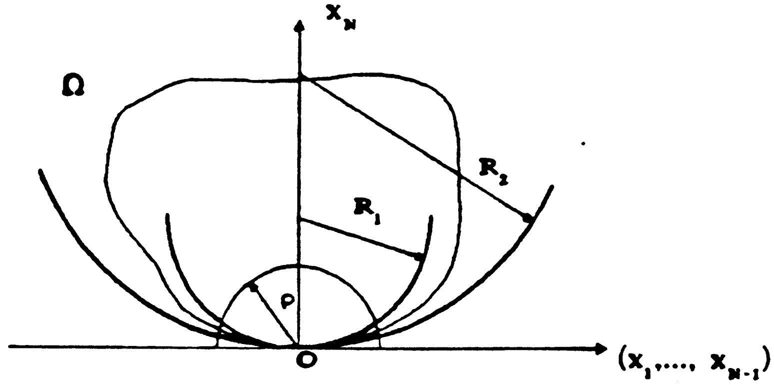

Fig. 1