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Polychromatic Hamilton cycles

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POLYCHROMATIC HAMILTON CYCLES

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Abstract

The edges of the complete graph $K_n$ are coloured so that no colour appears no more than $k$ times, $k = \lfloor n/A \ln n \rfloor$, for some sufficiently large $A$. We show that there is always a Hamiltonian cycle in which each edge is a different colour. The proof technique is probabilistic.

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1 Introduction

Let the edges of the complete graph $K_n$ be coloured so that no edge is coloured more than $k = k(n)$ times. We refer to this as a $k$-bounded colouring. We say that a Hamilton cycle of $K_n$ is polychromatic if each edge is of a different colour. We say that the colouring is good if each edge is of a different colour. Clearly the colouring is good if $k = 1$ and may not be if $k = n - 1$, since then we may colour all edges incident with vertex 1 the same colour. The question we address here then is that of how fast can we allow $k$ to grow and still guarantee that a $k$-bounded colouring is good.

Hahn and Thomassen [2] were the first people to consider this problem and they showed that $k$ could grow as fast as $n^{1/3}$. In unpublished work Rödl and Winkler [4] in 1984 improved this to $n^{1/2}$. In this paper we make further progress and prove

**Theorem 1** There is an absolute constant $A$ such that if $n$ is sufficiently large and $k$ is at most $[n/A \ln n]$ then any $k$-bounded colouring is good.

**Proof** Throughout the proof assume that $A$ is a large constant and $n$ is large.

Let

$$B = 10^{1/3} A^{2/3} \quad \text{and} \quad D = \frac{4B^2}{A} + 20.$$ 

Let $p = \frac{B \ln n}{n}$ and construct a random graph $H$ as follows:

**Step 1:** let $G = G_{n,p} = ([n], E)$.

(Recall that $G_{n,p}$ is the random graph with vertex set $[n] = \{1, 2, \ldots, n\}$ in which each possible edge occurs independently with probability $p$.)
Step 2: let $Y$ denote the set of edges whose colour appears more than once in $E$.

Let $H = ([n], E/Y)$.

Thus no two edges of $H$ are of the same colour. We prove our theorem by showing that

$$\Pr(H \text{ is Hamiltonian }) = 1 - o(1).$$

(Big $O$ and little $o$ notation refer to $n \to \infty$.)

This clearly implies that $K_n$ must have at least one polychromatic Hamilton cycle, provided $n$ is sufficiently large. The proof can be broken into two lemmas.

For $v \in [n]$ let $d_v$ denote the number of edges in $Y$ which are incident with $v$.

**Lemma 1** $\Pr(\exists v \in [n]: d_v \geq D \ln n) = o(1)$

**Lemma 2** If starting with $G = G_{n,p}$ we delete an arbitrary set of edges $Y$ to obtain a graph $H$ and in the process no vertex loses more than $D \ln n$ edges then $H$ is almost surely Hamiltonian.

Our Theorem is clearly an immediate consequence of these two lemmas.

2 Proof of Lemma 1

Let $d = d_1$ and let $S_1, S_2, \ldots, S_m$ be the partition of the edges of $K_n$ incident with vertex 1 into sets of the same colour $i = 1, 2, \ldots, m$. Let $E_i$ be the
set of edges of $K_n$ which have colour $i$. Let $|S_i| = l_i$ and $|E_i| = k_i \leq k$ for $i = 1, 2, \ldots, m$.

An edge $e \in S_i$ is deleted in Step 2 if either

(a) $E \cap S_i = \{e\}$ and $E_i/S_i \neq \emptyset$

or

(b) $e \in E$ and $|E \cap S_i| \geq 2$.

Let

$$D_x = \{ \text{edges incident with vertex 1 which are deleted via case } (x) \},$$

$x=a$ or $b$.

Observe that if $i \neq j$ then the sets $D_a \cap S_i$ and $D_b \cap S_j$ are independent (as random sets.)

The size of $D_a$

Clearly

$$|D_a \cap S_i| = 0 \text{ or } 1, \quad i = 1, 2, \ldots, m.$$ 

Also

$$\Pr(|D_a \cap S_i| = 1) = l_i p(1 - p)^{l_i - 1}(1 - (1 - p)^{k_i - l_i})$$

$$\leq l_i (k_i - l_i) p^2$$

$$\leq (k - 1) l_i p^2.$$ 

Thus

$$\mathbf{E}(|D_a|) \leq (k - 1) p^2 \sum_{i=1}^{m} l_i$$

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Now by Theorem 1 of Hoeffding [1]
\[ \Pr \left( |D_a| \geq \frac{2B^2 \ln n}{A} \right) \leq \exp \left\{ -\frac{B^2 \ln n}{3A} \right\} \leq n^{-2}. \]

The size of \( D_b \)

Let \( X_i = |E \cap S_i| \) and \( \delta_i = 1_{X_i \geq 2} \). Thus
\[ |D_b| = \sum_{i=1}^{m} X_i \delta_i. \]

Now fix \( i \in [m] \). Unfortunately \( X_i \) and \( \delta_i \) are correlated (positively). So let \( Y_i(= BIN(l_i, p)) \) be distributed as \( X_i \) but be independent of it. Then we claim that
\[ X_i \delta_i \text{ is majorised by } (2 + Y_i) \delta_i \]
i.e. for all \( u \geq 0 \)
\[ \Pr(X_i \delta_i \geq u) \leq \Pr((2 + Y_i) \delta_i \geq u). \] (1)

To see this we take 2 independent sequences \( A_1, A_2, \ldots, A_l, B_1, B_2, \ldots, B_l, l = l_i \) of Bernouilli random variables where each is 1 with probability \( p \) and zero with probability \( 1 - p \).

Let
\[ \rho = \begin{cases} \min\{ r : A_1 + A_2 + \ldots + A_r = 2 \} & \text{if } A_1 + A_2 + \ldots + A_l \geq 2 \\ \infty & \text{if } A_1 + A_2 + \ldots + A_l \leq 1 \end{cases} \]
Let
\[ Z_1 = \begin{cases} 
2 + B_{p+1} + \ldots + B_l & \text{if } \rho < \infty \\
0 & \text{if } \rho = \infty.
\end{cases} \]

Z$_1$ has the same distribution as \( X_i \delta_i \).

Let
\[ Z_2 = \begin{cases} 
2 + B_1 + \ldots + B_l & \text{if } \rho < \infty \\
0 & \text{if } \rho = \infty.
\end{cases} \]

Z$_2$ has the same distribution as \((2 + Y_i)\delta_i\) and (1) follows immediately.

Thus \(|D_b|\) is majorised by \( \sum_{i=1}^{m} (2 + Y_i)\delta_i \).

Now
\[
\Pr(\delta_i = 1) \leq \binom{l_i}{2} p^2
\]
and so
\[
\mathbb{E}\left( \sum_{i=1}^{m} \delta_i \right) \leq p^2 \sum_{i=1}^{m} \binom{l_i}{2}
\]
\[
\leq p^2 \frac{n}{k} \binom{k}{2}
\]
\[
\leq \frac{B^2}{2A} \ln n.
\]

Hence
\[
\Pr \left( \sum_{i=1}^{n} \delta_i \geq \frac{B^2}{A} \ln n \right) \leq \exp \left\{ -\frac{B^2}{6A\ln n} \right\}
\leq n^{-2}.
\]

Consider now the distribution of \( \sum_{i=1}^{m} (2 + Y_i)\delta_i \) conditional on \( \sum_{i=1}^{m} \delta_i \leq m_0 = [(B^2 \ln n)/A] \). This is majorised by
\[
\frac{2B^2}{A} \ln n + \sum_{i=1}^{m_0} Z_i
\]
where \( Z_1, Z_2, \ldots, Z_{m_0} \) are independent binomials \( BIN(k, p) \) and so \( Z = \sum_{i=1}^{m_0} Z_i = BIN(m_0 k, p) \). Thus

\[
E(Z) \leq (1 + o(1)) \frac{B^3}{A^2} \ln n - \frac{n}{A \ln n} \frac{B \ln n}{n}
\]

\[
= (1 + o(1)) \frac{B^3}{A^2} \ln n
\]

\[
\leq 11 \ln n
\]

So

\[
\Pr(Z \geq 20 \ln n) \leq \exp\left\{ -\frac{1}{3} \left( \frac{9}{11} \right)^2 11 \ln n \right\}
\]

\[
= O(n^{-2}).
\]

Hence

\[
\Pr\left( d \geq \frac{2B^2}{A} \ln n + \frac{2B^2}{A} \ln n + 20 \ln n \right) = O(n^{-2}).
\]

Multiplying by a factor \( n \) to account for all vertices gives the lemma. \( \square \)

### 3 Proof of Lemma 2

We modify the proof of Posá [3] to account for the deletion of edges. So assume now that \( G = G_1 \cup G_2 \cup G_3 \) where \( G_1 \) and \( G_2 \) are independent copies of \( G_{n,p/2} \) and where \( G_3 \) is an independent copy of \( G_{n,p'} \), where \( p' \) satisfies the equation \( 1 - p = (1 - p/2)^2 (1 - p') \). \( G_3 \) plays no further role in the analysis.

We first show that \( G_1/Y \) almost surely contains a Hamilton path. If it doesn’t then there exists \( i \in [n] \) such that

there exists a longest path of \( G_1/Y \) which does not go through \( i \)

which implies
no longest path of \( \Gamma_i = (G_i/Y)/\{i\} \) has an end-vertex adjacent to \( i \) in \( G_1 \).

Let this final event be denoted by \( \mathcal{E}_i \). Then

\[
\Pr(G_1/Y \text{ has no Hamilton path }) \leq n \Pr(\mathcal{E}_n).
\]

(2)

Let now \( P \) be a longest path of \( \Gamma_n \) and let \( x_0 \) be one of its end-vertices. Let \( END \) be the set of end-vertices of longest paths of \( \Gamma_n \) which can be obtained from \( P \) by a sequence of rotations keeping \( x_0 \) as a fixed end-vertex. (Given a longest path \( Q \) with end-vertices \( x_0, y \) and an edge \( yv \) where \( v \) is an internal vertex of \( Q \), we obtain a new longest path \( Q' = x_0..vy..w \) where \( w \) is the neighbour of \( v \) on \( P \) between \( v \) and \( y \). We say that \( Q' \) is obtained from \( Q \) by a rotation.)

It follows from Posá [3] that

\[
|N(\Gamma_n, END)| \leq 2|END|, \tag{3}
\]

where for a graph \( \Gamma \) and a set \( S \subseteq V(\Gamma) \)

\[
N(\Gamma, S) = \{w \notin S : \exists v \in S \text{ such that } vw \in E(\Gamma)\}.
\]

**CLAIM:** with probability \( 1 - o(n^{-1}) \)

\[
S \subseteq [n-1], |S| \leq \frac{n}{4D \ln n} \text{ implies } |N(G_1/\{n\}, S)| \geq 3D(\ln n)|S|.
\]

(The proof of this claim is deferred to the end of the proof of the lemma.)

Hence in \( \Gamma_n \) we have with probability \( 1 - o(n^{-1}) \)

\[
S \subseteq [n-1], |S| \leq \frac{n}{4D \ln n} \text{ implies } |N(\Gamma_n, S)| \geq D(\ln n)|S|.
\]

It follows from (3) that with probability \( 1 - o(n^{-1}) \)

\[
|END| \geq \frac{n}{12}.
\]
Now consider the edges of $G_1$ from vertex $n$ to $END$. They are independent of $END$ and so are distributed as $B(|END|, p/2)$. Thus their expected number is at least $(B \ln n)/24$. Thus if $A$ and hence $B$ is large there will be at least $(B \ln n)/48$ such edges with probability $1-o(n^{-1})$. But for large $A, D < B/48$ and so not all of these edges can be included in $Y$. Thus $Pr(E_n) = o(n^{-1})$ and (2) implies that $G_1/Y$ almost surely has a Hamilton path.

To finish the proof take a Hamilton path $P$ of $G_1$ and fix one of its end-vertices, $x_0$ say, and using rotations create a set of end-vertices $END$ of Hamilton paths with one end-vertex $x_0$. The above analysis shows that $|END| \geq \frac{n}{12}$ almost surely. Now add the edges of $G_2$, which are independent of $x_0$ and $END$. Again we can argue that there are almost surely too many $x_0-END$ edges in $G_2$ for them all to be included in $Y$ and the lemma follows since the existence of any one not in $Y$ means that $H$ is Hamiltonian.

**Proof of CLAIM**

If the condition in the claim does not hold then there exist disjoint sets $S,T \subseteq [n-1], s = |S| \leq n/(4D \ln n), t = |T| \leq 3D(\ln n)s \leq 3n/4$ such that each vertex of $T$ is adjacent to at least one vertex in $S$ and no vertex in $[n-1]/(S \cup T)$ is adjacent to any vertex of $S$.

Fix $s, t$ and let $t_0 = 3sD(\ln n)$ Then the probability of the above event is bounded by

\[
\binom{n-1}{s} \binom{n-1}{t} \left( \frac{sp}{2} \right)^t \left( 1 - \frac{p}{2} \right)^{s(n-1-s-t)} \leq \left( \frac{ne}{s} \right)^s \left( \frac{ne}{t} \right)^t \left( \frac{sp}{2} \right)^t e^{-sp/10} \\
= \left( \frac{ne}{s} \right)^s \left( \frac{e}{t} \right)^t \left( \frac{B \ln n}{2} \right)^t n^{-sB/10}
\]
for large \( A \). Now multiply this upper bound by \( n^2 \), which bounds the number of possible \( s,t \), in order to prove the claim.

Finally, we remark that we believe the following

**Conjecture:** there exists an absolute constant \( \epsilon > 0 \) such that if \( k < \epsilon n \) then any \( k \)-bounded colouring of \( K_n \) is good.

Hahn and Thomassen made a somewhat stronger conjecture.

**References**


