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On the Fundamental Eigenfunctions of a Clamped Punctured Disk

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INTRODUCTION. This note is concerned with the sign of the fundamental eigenfunctions of the eigenvalue problem

(1.1) $$\Delta^2 u = \lambda u\ , \quad \text{in } \Omega\ ,$$

(1.2) $$u = \frac{\partial u}{\partial n} = 0\ , \quad \text{on } \partial\Omega\ ,$$

on a plane domain Ω; $\mathbf{n}$ denotes the outer normal on $\partial\Omega$. In [5] it was shown that if Ω is the annulus

$$D_\varepsilon = \{(x,y)\colon \varepsilon^2 < x^2 + y^2 < 1\}$$

then the fundamental eigenfunctions (i.e. those corresponding to the lowest eigenvalue) are not of one sign when ε is small. Indeed when ε is sufficiently small ($\varepsilon^{-1} > 762.36$, [3]) <u>the fundamental eigenfunctions possess diametrical nodal lines and span a two–dimensional eigenspace</u>.

The proof of this assertion, [5], [3], relies in an essential way on either appeal to tables or to precise computation for Bessel function values. However the qualitative result, i.e. that the above underlined statement holds for all sufficiently small positive ε, follows by continuity and without explicit computation from the corresponding assertion for the punctured disk. Even for the punctured disk the proof of this assertion, [5],[3], although simpler than in the case of the annulus, still depended on an appeal to the tables. (This appeal is to verify the ordering of the quantities denoted in §4 by α and γ; see the last paragraph of §4). The purpose of this note is to give the details of a proof for the case of the punctured disk that was indicated but not completed in [3] and which requires neither computation nor appeal to the tables.

In [3] we observed that the result for the punctured disk would follow from the inequality

$$\pi x J_0(x) Y_0(x) < 1 \,, \tag{1.3}$$

and the well–known ordering of the zeros of $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$; cf. (3.2). Below we prove the (sharp) inequality (1.3) and, for completeness, also the indicated ordering of zeros. These results follow from the asymptotic formula, [14],

$$\lim_{x \to \infty} e^{-i(x-\pi/4)} \sqrt{\pi x/2}\,(J_0(x) + iY_0(x)) = 1. \tag{1.4}$$

and standard comparison methods for second order ordinary differential equations.

One of the first to question whether the fundamental mode of vibration of the clamped plate was of one sign and the corresponding eigenvalue simple appears to have been A. Weinstein. Interest in the question was increased by the paper [12] of G. Szego where it was shown that were the fundamental mode positive in general then among all plates of a given area the fundamental eigenvalue would be smallest for the circular one. Interestingly, although the hypothesis is false, it is nevertheless true that the ratio of the fundamental eigenvalue of (1.1), (1.2) for a general plain domain Ω to that for the circular region of the same area is never less than .977, Talenti, [13]; the analogous lower bound is also computed for higher dimensions in [13].

The example under consideration here represents the first counter–example to the indicated positivity conjecture. A second counter–example is that of a square, for which the fundamental mode has nodal lines in the corners; numerical evidence for this was provided in [1] and an analytical proof was given in [2]. In fact the result of [2] applies to any plane region with piecewise smooth boundary and a right–angle corner in the boundary; the argument applies for other than just right angles. A result like that in [2] was also proved subsequently in [10].

We remark finally on the connection between the positivity conjecture under consideration here and the so–called "Hadamard conjecture" concerning the positivity of the biharmonic Green's function, (in fact Hadamard, [8], attributes this conjecture to T. Boggio; see in particular pp. 541–542, t. II, of the Oeuvres); for background on this problem see [6]. As is well–known from the theory of positive operators, if the Green's function is positive then the fundamental eigenvalue is simple and the corresponding eigenfunction is of one sign, but the converse is not true. Hadamard noted almost immediately in the (apparently little–known) paper [9] that the annulus with small inner radius provided a counter–example to this conjecture, see also Nakai and Sario [11], for a discussion of this example. Hadamard reaffirmed in [9] his belief in the validity of the conjecture for convex regions. In [7] it was shown that infinite strip provides a counter–example to this revised conjecture. Finally we remark that from the latter counter–example one can exhibit the failure of the conjecture for a long narrow rectangle or an ellipse of large eccentricity; this can be done using a limiting argument in conjunction with methods developed by Hadamard in [8]; cf. [4].

2. PRELIMINARIES. One of the preliminaries that must be discussed is the appropriate form of the boundary conditions for the punctured disk. For this purpose we first make the following observation. Let Ω' denote the region that results when a single point (x_0,y_0) is deleted from the region Ω. Then [4], $H_0^{1,2}(\Omega')$ (our notation is standard) is of co–dimension one in $H_0^{1,2}(\Omega)$ and thus whenever $u \in H_0^{1,2}(\Omega)$ and $u(x_0,y_0) = 0$ then $u \in H_0^{1,2}(\Omega')$. (A simpler argument for this than that given in [4] is suggested by [9]. By Weyl's lemma $L^2(\Omega)$ (for any region Ω) is the orthogonal direct sum of the subspace $L_h^2(\Omega)$ of $L^2(\Omega)$ that consists of (equivalence classes of) functions harmonic on Ω and $\Delta H_0^{1,2}(\Omega)$. $L_h^2(\Omega)$ is of co–dimension one in $L_h^2(\Omega')$ and thus since $L^2(\Omega)$ and $L^2(\Omega')$ are indistinguishable and Δ: $H_0^{1,2}(\Omega) \to L^2(\Omega)$ is non–singular, the assertion follows.) Consequently, [3], the proper formulation of the boundary conditions on the punctured disk, as was already found in [5], is

$$u = \frac{\partial u}{\partial n} = 0, \quad x^2 + y^2 = 1, \text{ and } u(0,0) = 0. \tag{2.1}$$

and the positive eigenfunction of (1.1), (2.1) (expressed in terms of polar coordinates (r,θ)) has the form

$$\varphi(re^{i\theta}) = C_1[Y_0(\mu r) + \tilde{K}_0(\mu r)] + C_2[I_0(\mu r) - J_0(\mu r)] \tag{2.2}$$

where

$$\tilde{K}_0(x) = (\frac{2}{\pi})\, K_0(x)\ , \tag{2.3}$$

and μ is the least positive root of

$$[Y_0(\mu) + \tilde{K}_0(\mu)][I_0'(\mu) - J_0'(\mu)] - [Y_0'(\mu) + \tilde{K}_0'(\mu)][I_0(\mu) - J_0(\mu)] = 0\ ; \tag{2.4}$$

(we use standard notation for the Bessel functions, see e.g [14]). Among the eigenfunctions with a single diametric nodal line is one which, when expressed in terms of polar coordinates, is of the form

$$\psi(re^{i\theta}) = [d_1I_1(\nu r) + d_2J_1(\nu r)]\sin\,\theta \tag{2.5}$$

with ν the least positive root of

$$I_0(\nu)J_0'(\nu) + I_0'(\nu)J_0(\nu) = 0. \tag{2.6}$$

(this function is also an eigenfunction of (1.1), (1.2) for $\Omega = D$, the unit disk).

We make one final general observation. The positive eigenfunction for the unit disk D has the form

(2.7) $$c_1 J_0(\tau r) + c_2 I_0(\tau r) ,$$

where τ is the least positive root of

(2.8) $$I_0(\tau)J_0'(\tau) - I_0'(\tau)J_0(\tau) = 0.$$

Since $H_0^{1,2}(D_0) \subseteq H_0^{1,2}(D)$ and in view of the well–known fact that the least eigenvalue for the disk is simple and corresponds to a positive eigenfunction it follows that the least positive roots μ,ν of (2.4) and (2.6) lie to the right of the least positive root of (2.8),

(2.9) $$\tau < \mu,\nu$$

Our main result can now be stated as follows.

Theorem 1. <u>Let</u> μ <u>denote the least positive root of</u> (2.4) <u>and let</u> ν <u>denote the least positive root of</u> (2.6). <u>Then</u>

(2.10) $$\nu < \mu.$$

The eigenvalues that correspond to the eigenfunctions (2.3) and (2.5) respectively are μ^4 and ν^4.

3. MAIN LEMMA. The positive roots of

(3.1) $$J_i(x) = 0, \quad \text{and} \quad Y_i(x) = 0,$$

will be denoted by $j_{i,n}$, $y_{i,n}$, respectively, and indexed (by n) in increasing order. The main result of this section is the following.

Lemma. The roots of equations (3.1) satisfy

$$0 < y_{0,1} < y_{1,1} < j_{0,1} < j_{1,1} < y_{0,2} < y_{1,2} < j_{0,2}. \tag{3.2}$$

For all $x > 0$ there holds the inequality

$$\pi x J_0(x) Y_0(x) < 1. \tag{3.3}$$

In view of (3.2) one sees readily from the series expansions at zero that the signs of J_0 and Y_0 and their derivatives are as indicated in the following table. (Note that the $j_{1,n}$ are the zeros of J_0' and the $y_{1,n}$ are the zeros of Y_0'.)

Table 1

	$J_0(x)$	$J_0'(x)$	$Y_0(x)$	$Y_0'(x)$
$(0, y_{0,1})$	+	−	−	+
$(y_{0,1}, y_{1,1})$	+	−	+	+
$(y_{1,1}, j_{0,1})$	+	−	+	−
$(j_{0,1}, j_{1,1})$	−	−	+	−
$(j_{1,1}, y_{0,2})$	−	+	+	−
$(y_{0,2}, y_{1,2})$	−	+	−	−
$(y_{1,2}, j_{0,2})$	−	+	−	+

We shall defer the proof of this lemma till last and show first how Theorem 1 follows from it.

4. PROOF OF THEOREM 1. We first write down some properties of the modified Bessel functions $I_0(x)$ and $K_0(x)$; these satisfy

$$(4.1) \qquad 0 < I_0'(x) < I_0(x),$$

and

$$(4.2) \qquad 0 < K_0(x) < -K_0'(x).$$

It is clear from (4.1) that the least positive root τ of (2.8) lies in the interval $(j_{0,1},j_{1,1})$. As noted in section 2, μ and ν both lie to the right of τ, hence to the right of $j_{0,1}$.

A consultation of Table I shows that $J_0'(x)/J_0(x)$ is positive in $(j_{0,1},j_{1,1})$and decreases from 0 to $-\infty$ as x increases from $j_{1,1}$ to $j_{0,2}$. Thus if α is the root of

$$J_0'(\alpha) = -J_0(\alpha)$$

in $(j_{1,1},j_{0,2})$, then it follows from (4.1) that ν lies in the interval $(j_{1,1},\alpha)$, i.e.

$$(4.3) \qquad \nu < \alpha.$$

We next attempt to locate the least positive root μ of (2.4); as we have seen $\mu > j_{0,1}$. First we note that, as readily follows from inspection of the series, $I_0(x) - J_0(x)$ and its derivative are positive for all positive x while from Table I we have

$$(4.4) \qquad \frac{I_0'(s) - J_0'(s)}{I_0(s) - J_0(s)} < \frac{I_0'(s)}{I_0(S)} < 1 \qquad \text{on } (y_{0,2},j_{0,2}).$$

On the other hand $Y_0'(x) + \tilde{K}_0'(x)$ is negative in $(j_{0,1},y_{1,2})$ so if (2.4) has a root μ in that interval then $Y_0(x) + \tilde{K}_0(x)$ is negative at $x = \mu$ so since that function is positive at $x = y_{0,2}$,

$$Y_0(\beta) + \tilde{K}_0(\beta) = 0$$

must have a root β with

$$y_{0,2} < \beta < \mu < y_{1,2}.$$

However

$$\frac{Y_0'(s) + \tilde{K}_0'(s)}{Y_0(s) + \tilde{K}_0(s)} > \frac{Y_0'(s)}{Y_0(s)}$$

on $(\beta, y_{1,2})$. Thus, if $\mu < y_{1,2}$, then μ must lie to the right of the root γ of

$$Y_0'(\gamma) = Y_0(\gamma)$$

that belongs to $(y_{0,2}, y_{1,2})$. In any case therefore

(4.5) $$\gamma < \mu.$$

In order to prove (2.10) it suffices, in view of (4.3) and (4.5), to prove that

$$\gamma > \alpha.$$

To this end we use the Wronskian relation

(4.6) $$\frac{\pi x}{2} [J_0(x)Y_0'(x) - J_0'(x)Y_0(x)] = 1,$$

Table I and (3.3) to obtain

$$J_0'(\gamma) > -J_0(\gamma),$$

which, since $\gamma \in (y_{0,2}, y_{1,2}) \subseteq (j_{1,1}, j_{0,2})$, implies that $\gamma > \alpha$.

In fact a consultation of the tables shows that γ and α differ by not more than 0.02.

The values found in [3] for μ and ν were

$$\mu = 4.768309396, \qquad \nu = 4.61089980.$$

5. PROOF OF THE MAIN LEMMA. From [6,§§17.5,17.6] we have the asymptotic formula,

$$\lim_{x \to \infty} e^{-i(x-\pi/4)}\sqrt{\pi x/2}\,(J_0(x) + iY_0(x)) = 1. \tag{5.1}$$

If we put

$$v(x) = e^{-i(x-\pi/4)}\sqrt{\pi x/2}\,(J_0(x) + iY_0(x)), \tag{5.2}$$

then v satisfies the differential equation,

$$v'' + 2iv' + (2x)^{-2}v = 0. \tag{5.3}$$

It readily follows that v is the unique solution to the integral equation

$$v(x) = 1 + \frac{i}{2}\int_x^\infty (e^{2i(t-x)} - 1)(2t)^{-1}v(t)dt, \tag{5.4}$$

and that

$$v'(x) = \int_x^\infty e^{2i(t-x)}(2t)^{-2}v(t)dt. \tag{5.5}$$

From (5.4) it follows that

(5.6) $$v(x) = 1 + O(x^{-1}), \qquad \text{as } x \to \infty,$$

and when the latter is substituted in (5.5) we obtain

(5.7) $$v'(x) = O(x^{-2}), \qquad \text{as } x \to \infty.$$

From the differential equation (5.3) we find

$$(4x^2|v'|^2 + |v|^2)' = 8x|v'|^2.$$

From this together with (5.6) and (5.7) we conclude that

(5.8) $$|v(x)|^2 < 1, \quad 0 < x < \infty.$$

It follows from (5.2) that

$$\mathrm{Im}(e^{i(x-\pi/4)}v(x))^2 = \pi x J_0(x)Y_0(x),$$

and thus (5.8) implies

$$\pi x J_0(x)Y_0(x) < 1, \ \ 0 < x < \infty,$$

as was to be proved.

Next we establish the ordering (3.2). First we note that $J_0(x)$ is positive and $Y_0(x)$ negative for small positive x as follows from inspection of their series developments. It follows from the Wronskian relation (4.6) that $Y_0(j_{0,1}) > 0$ and thus $0 < y_{0,1} < j_{0,1}$ and hence, by Sturm's theorem, $y_{0,k} < j_{0,k}$ for all k. From the differential equation it follows that xJ_0' is strictly monotone on any interval where J_0 does not change sign and similarly xY_0' is monotone on any interval where Y_0 does not change sign. From this it follows that $j_{0,1} < j_{1,1}$ and $y_{0,1} < y_{1,1}$. Using Rolle's theorem and the above monotonicity assertions we conclude that $j_{0,k} < j_{1,k}$ and $y_{0,k} < y_{1,k}$ for all k. It remains only to show that for all k, $y_{1,k} < j_{0,k}$ and $j_{1,k} < y_{0,k+1}$

A simple computation yields

$$(5.9) \qquad (J_0'(x) + iY_0'(x))/(J_0(x) + iY_0(x)) = (i + v'(x)/v(x) - (2x)^{-1}).$$

Since, as follows from (5.6) and (5.7), $|v'/v| < (2x)^{-1}$ when x is large, (5.9) means that the trajectory in the complex plane described by $J_0'(x) + iY_0'(x)$ leads that described by $J_0(x) + iY_0(x)$ by an angle of slightly more than $\pi/2$ when x is large. The latter trajectory crosses the imaginary axis when $x = j_{0,k}$ and crosses the real axis when $x = y_{0,k}$; the crossings of the imaginary and real axes by the former trajectory correspond respectively to $x = j_{1,k}$ and $x = y_{1,k}$. Thus it readily follows that for large k,

$$(5.10) \qquad y_{0,k} < y_{1,k} < j_{0,k} < j_{1,k} < y_{0\cdot k+1}.$$

Given that (5.10) holds for large k it readily extends to hold for all k. Indeed since J_0' satisfies the Bessel equation of order 1, it follows from Sturm's theorem that J_0' cannot have two zeros in $[y_{0,k},y_{0,k+1}]$ and thus $j_{1,k+1} < y_{0,k}$. Similarly Y_0' cannot have two zeros in $[j_{0,k+1},j_{0,k}]$ and thus $y_{1,k-1} < j_{0,k-1} < j_{1,k-1} < y_{0,k}$.

Thus we have

$$0 < y_{0,k} < y_{1,k} < j_{0,k} < j_{1,k} < y_{0,k+1} < y_{1,k+1} < j_{0,k+1},$$

for all $k > 0$, hence (3.2) holds in particular and the proof of the main lemma is complete.

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