Dynamic Sparse State Estimation Using $\ell_1-\ell_1$ Minimization: Adaptive-rate Measurement Bounds, Algorithms and Applications

Joao F.C. Mota  
*University College, London*

Nikos Deligiannis  
*University College, London*

Aswin C. Sankaranarayanan  
*Carnegie Mellon University, saswin@ece.cmu.edu*

Volkan Cevher  
*Ecole Polytechnique Federale de Lausanne*

Miguel Rodrigues  
*University College, London*

Follow this and additional works at: [http://repository.cmu.edu/ece](http://repository.cmu.edu/ece)

Published In  
DYNAMIC SPARSE STATE ESTIMATION USING $\ell_1$-$\ell_1$ MINIMIZATION: ADAPTIVE-RATE MEASUREMENT BOUNDS, ALGORITHMS AND APPLICATIONS

João Mota*, Nikos Deligianniss*, Aswin C. Sankaranarayanan†, Volkan Cevher‡, Miguel Rodrigues*

* Electronic and Electrical Engineering Dept., University College London, UK
† Electrical and Computer Engineering Dept., Carnegie Mellon University, USA
‡ Laboratory for Information and Inference Systems (LIONS), EPFL, Switzerland

ABSTRACT

We propose a recursive algorithm for estimating time-varying signals from a few linear measurements. The signals are assumed sparse, with unknown support, and are described by a dynamical model. In each iteration, the algorithm solves an $\ell_1$-$\ell_1$ minimization problem and estimates the number of measurements that it has to take at the next iteration. These estimates are computed based on recent theoretical results for $\ell_1$-$\ell_1$ minimization. We also provide sufficient conditions for perfect signal reconstruction at each time instant as a function of an algorithm parameter. The algorithm exhibits high performance in compressive tracking on a real video sequence, as shown in our experimental results.

Index Terms— State estimation, sparsity, background subtraction, motion estimation, online algorithms

1. INTRODUCTION

We study the reconstruction of sparse, time-varying signals from a limited number of linear measurements. Let $x[k] \in \mathbb{R}^n$ denote the target signal at time $k$ and let $y[k] \in \mathbb{R}^{m_k}$ denote $m_k$ measurements of $x[k]$. We consider the dynamical model

\[ x[k] = f_k(x[k-1]) + \epsilon[k] \]
\[ y[k] = A_k x[k] + \eta[k], \]

where $f_k : \mathbb{R}^n \to \mathbb{R}^n$ describes the evolution of the signals $x[k]$, $k = 1, 2, \ldots$ between consecutive time instants, and $A_k \in \mathbb{R}^{m_k \times n}$ is the matrix of measurements at time $k$. The quantities $\epsilon[k]$ and $\eta[k]$ capture model inaccuracies and measurement noise, respectively.

One of the oldest problems in control theory is to estimate the state sequence $\{x[k]\}_{k \geq 1}$ from the measurements $\{y[k]\}_{k \geq 1}$. The classical solution is the Kalman filter [1], a recursive algorithm, known to be least-squares optimal when the model is linear ($f_k(x) = F_k x$) and $\epsilon[k]$ and $\eta[k]$ are zero-mean Gaussian. Several extensions have been proposed for the case where these two assumptions are not met, e.g., [2–4]. The Kalman filter and these extensions, however, cannot easily integrate additional knowledge of the signal’s structure, e.g., sparsity, and suffer from lack of observability when the number of measurements is limited, i.e., $m_k \ll n$.

Contributions. Our goal is to reconstruct each signal $x[k] \in \mathbb{R}^n$ from a small number of measurements $m_k \ll n$, when $x[k]$ is sparse and has unknown support. Furthermore, we assume sparse model inaccuracies $\epsilon[k]$ and bounded measurement noise $\eta[k]$, i.e., there exists $\sigma_\epsilon \geq 0$ such that $\|\epsilon[k]\|_2 \leq \sigma_\epsilon$. Under these conditions, the Kalman filter has poor performance [5] (especially for non-Gaussian $\epsilon[k]$’s). Assuming that the entries of each $A_k$ are i.i.d. Gaussian, we propose estimating $x[k]$ recursively as follows:

\[ \hat{x}[k] \in \text{Argmin}_{x} \|x\|_1 + \|x - f_k(\hat{x}[k-1])\|_1 \]
\[ \text{s.t. } \|A_k x - y[k]\|_2 \leq \sigma_\epsilon, \]

where $\hat{x}[k-1]$ is the signal estimate at time $k - 1$. In (2), $\|x\|_1 := \sum_k |x_k|$ is the $\ell_1$-norm and $\|x\|_2 := \sqrt{\sum_k x_k^2}$ is the $\ell_2$-norm. Note that, in general, (2) may have more than one solution. Based on the results in [8, 9], we propose a recursive mechanism to compute the number of measurements $m_k$ at each time $k$. This scheme minimizes $m_k$ while guaranteeing perfect reconstruction in the noiseless scenario, $\eta[k] = 0$, or stable reconstruction (i.e., $\|\hat{x}[k] - x[k]\|_2 \leq 2\sigma_\epsilon/\epsilon$, for some $0 < \epsilon < 1$) in the noisy scenario. Furthermore, note that there are no parameters (weights) to tune in (2).

Applications. The model in (1) is actually applicable to non-sparse signals, provided they have sparse representations in a suitable domain. Let $z[k] \in \mathbb{R}^n$ be a non-sparse signal that has a sparse representation $x[k] = \Psi z[k]$, where $\Psi : \mathbb{R}^{n \times m} \to \mathbb{R}^n$ is the sparsifying transform. Suppose $z[k]$ evolves as $z[k] = f_k(z[k-1]) + \epsilon[k]$, and we observe $y[k] = A_k z[k] + \eta[k]$. Then, the sparse coefficients $z[k]$ evolve as in (1) with $f_k(x) = \Psi f_k(\Psi^{-1}x)$ and $A_k = A_k \Psi$. Thus, the class of signals described by our model is quite broad, and the applications are diverse. They include, for example, dynamic MRI [10, 11], radar [12], and background subtraction [13].

2. RELATED WORK

Prior work that incorporates signal structure in state estimation problems includes [14–16]. This work splits the problem of estimating a dynamic sparse signal into the problems of estimating its support, which is addressed with compressive sensing techniques, and estimating its values, which is addressed with a Kalman filter. This method, however, assumes that the support of the signal varies slowly in time. Other approaches assume the coefficients of the support also vary slowly [17, 18], or the signal varies smoothly [18, 19], including with an evolution governed by a linear dynamical system [20]. Instead of assuming smoothness or slow-varying supports, our scheme assumes that the quality of the prediction given by $f_k$ does not vary much between consecutive instants.

*This is common in systems designed to sample signals according to the compressed sensing paradigm [6, 7].
The work in [5] studies three reconstruction schemes, the best of which is a Lagrangian version of (2), i.e., there are no constraints and the objective has the additional term $\beta_2 ||Ax - y|k||_1$. It was experimentally shown in [5] that the Lagrangian version of (2) has an excellent performance and outperforms Kalman filtering, even when the model inaccuracies $\eta[k]$ are Gaussian (and thus not sparse). Note that the solutions of (2) and its Lagrangian version coincide when $\beta_2$ is chosen properly. However, the advantages of solving (2) w.r.t. its Lagrangian version are twofold. First, in practice, it is easier to obtain bounds on the magnitude of $\eta[k]$ than it is to tune the parameter $\beta_2$. Second, the recent results in [8, 9] establish reconstruction guarantees for (2) in the case of static signals; those results also establish an optimal value for the parameter $\beta$ (equal to 1), making (2) parameter-free.

Finally, while prior work studies reconstruction schemes where the number of measurements is the same in all time instants [5, 14, 15, 17–21] (a notable exception is [22], where cross-validation is used to estimate the required number of measurements), our reconstruction scheme adapts the number of measurements recursively.

### 3. BACKGROUND: STATIC SIGNAL RECONSTRUCTION

Our scheme is motivated by the recent results in [8]. This reference studies problem (2) in a static scenario, i.e., when only one iteration of (2) is performed. We summarize those results next.

Let $x^* \in \mathbb{R}^n$ be a sparse signal of which we have $m$ linear noisy measurements $y = Ax^* + \eta$, where $A \in \mathbb{R}^{m \times n}$ and $||\eta||_2 \leq \sigma$, for a known $\sigma \geq 0$. We assume access to a signal $w \in \mathbb{R}^n$ similar to $x^*$, in the sense that $||Ax - w||_1$ is expected to be small. Suppose we attempt to reconstruct $x^*$ by solving the $\ell_1$-$\ell_2$ minimization problem:

$$\begin{align*}
\text{minimize} & \quad ||x||_1 + \beta ||x - w||_1 \\
\text{subject to} & \quad ||Ax - y||_2 \leq \sigma,
\end{align*}$$

(3)

where $\beta > 0$. The following result from [8] establishes reconstruction guarantees for (3). To state it, we need the following quantities:

$$\begin{align*}
\bar{H} & := \big| \{i : x_i^* > 0, x_i^* > w_i \} \cup \{i : x_i^* < 0, x_i^* < w_i \} \big| \\
\xi & := \big| \{i : w_i \neq x_i^* = 0 \} \big| - \big| \{i : w_i = x_i^* \neq 0 \} \big|,
\end{align*}$$

(4a)

(4b)

where $\cdot$ denotes the cardinality of a set. Note that $0 \leq \bar{H} \leq s$, where $s$ is the sparsity of $x^*$.

**Theorem 1** ([8]). Let $x^*, w \in \mathbb{R}^n$ and suppose we take $m$ linear measurements $y = Ax^* + \eta$, where $||\eta||_2 \leq \sigma$, for $\sigma \geq 0$. Assume $\bar{H} > 0$ and that there exists at least one index $i$ for which $x_i^* = w_i = 0$. Let the entries of $A \in \mathbb{R}^{m \times n}$ be i.i.d. Gaussian with zero mean and variance $1/m$,

1. If $\sigma = 0$ or, equivalently, $y = Ax^*$, and

$$m \geq 2\bar{H} \log \left( \frac{n}{\bar{H} + \xi/2} \right) + \frac{7}{5} \left( s + \frac{\xi}{2} \right) + 1,$$

(5)

then, with probability at least $1 - 3\exp\left(-\frac{3\sigma^2}{m}\right)$, $x^*$ is the unique solution of (3) with $\beta = 1$.

2. If $\sigma > 0$, define $0 < \epsilon < 1$ and let

$$m \geq \frac{1}{(1 - \epsilon)^2} \left[ 2\bar{H} \log \left( \frac{n}{\bar{H} + \xi/2} \right) + \frac{7}{5} \left( s + \frac{\xi}{2} \right) + \frac{3}{2} \right] + 1.$$  

(6)

Then, any solution $\hat{x}$ of (3) with $\beta = 1$ satisfies $||\hat{x} - x^*||_2 \leq 2\sigma/\epsilon$ with probability at least $1 - 3\exp\left(-\frac{3\sigma^2}{m\epsilon^2}\right)$.

**Theorem 1** establishes lower bounds on the number of measurements that guarantees that (3) with $\beta = 1$ recovers $x^*$ perfectly (resp. stably) in a noiseless (resp. noisy) measurement scenario, with high probability. The bounds in (5) and (6) are a function of the signal dimension $n$, the signal sparsity $s$, and the quantities $\bar{H}$ and $\xi$. Note that $\bar{H}$ and $\xi$ depend only on the signs of each entry of the vectors $x^*$ and $w - x^*$, but not on their values. As these quantities are not known in practice (they depend on the unknown signal $x^*$), we propose in section 4 an adaptive scheme to estimate them using previous signals. Note that $\bar{H}$ and $\xi$ measure the quality of the approximation of $x^*$ by $w$. When this approximation is reasonable, problem (3) requires much less measurements than standard $\ell_1$ minimization, i.e., (3) with $\beta = 0$. For example, in a noiseless acquisition scenario, standard $\ell_1$ minimization requires $2s \log(n/s) + (7/5)s + 1$ measurements for perfect reconstruction with the same probability as in Theorem 1 [23]. When the dominant terms are the log’s, (5) can be much smaller than this bound, since $\bar{H} \leq s$.

Finally, we mention that [8] also provides bounds for the case $\beta \neq 1$, but they are significantly more complex than (5) and (6). Interestingly, those bounds are minimized for $\beta = 1$, a value that leads to a practical performance close to optimal. For this reason, we set $\beta = 1$ henceforth.

**Algorithm 1** Adaptive-Rate Sparse State Estimation

**Initialization:** choose $0 \leq \alpha < 1$, $\beta > 0$, and estimate $s_1$ and $s_2$, the sparsity of $x[1]$ and $x[2]$, respectively.

1. for the first two time instants $k = 1, 2$ do

   2. Set $s_k = 2s_k \log(n/s_k) + (7/5)s_k + 1$
   3. Generate Gaussian matrices $A_k \in \mathbb{R}^{m_k \times n}$
   4. Acquire $m_k$ measurements of $x[k]; y[k] = A_k x[k]$
   5. Find $\hat{x}[k]$ such that

$$\begin{align*}
\hat{x}[k] & \in \text{Argmin} \quad ||x||_1 \\
& \quad \text{s.t.} \quad A_k x = y[k],
\end{align*}$$

6. end for

7. Set $w[2] = f_2(\hat{x}[1])$ and compute $\bar{H}_2$ and $\xi_2$ as in (4) with $\hat{x}[2]$ and $w[2]$ in place of $x[2]$ and $w$, respectively.
8. Set $m_2 = 2\bar{H}_2 \log(n/(2s_2 + \xi_2/2)) + (7/5)(s_2 + \xi_2/2) + 1$
9. Set $\phi_3 = m_2$
10. for each time instant $k = 3, 4, 5, \ldots$ do

11. Choose $m_k = (1 + \delta)\phi_k$
12. Generate Gaussian matrix $A_k \in \mathbb{R}^{m_k \times n}$
13. Acquire $m_k$ measurements of $x[k]; y[k] = A_k x[k]$
14. Set $w[k] = f_k(\hat{x}(k) - 1)$ and find $\hat{x}[k]$ such that

$$\begin{align*}
\hat{x}[k] & \in \text{Argmin} \quad ||x||_1 + ||x - w[k]||_1 \\
& \quad \text{s.t.} \quad A_k x = y[k],
\end{align*}$$

15. Compute $\bar{H}_k$ and $\xi_k$ as in (4) with $\hat{x}[k]$ and $w[k]$
16. Set $s_k = \big| \{i : \hat{x}[k] \neq 0 \} \big|$
17. Set $m_k = 2\bar{H}_k \log(n/(s_k + \xi_k/2)) + (7/5)(s_k + \xi_k/2) + 1$
18. Update $\phi_{k+1} = (1 - \alpha)\phi_k + \alpha m_k$
19. end for

### 4. DYNAMIC SIGNAL RECONSTRUCTION

Algorithm 1 describes the scheme we propose for recursive estimation of $x[k]$. For simplicity, we consider only the noiseless measurement scenario, but its adaptation to the noisy one is immediate.
The algorithm is meant to be run on a real-time system, since the measurements taken at each iteration are determined on-the-fly. In steps 1-6, the first two signals, $x[1]$ and $x[2]$, are reconstructed using standard $\ell_1$ minimization. The number of measurements $m_1$ and $m_2$ are computed as in [23], and require an estimate of the signals’ sparsity. Steps 7-9 initialize our “estimator” $\phi_1$ of the true bound on the number of measurements. That is, during the recursive part of the algorithm, i.e., steps 10-19, $\phi$ should approximate the right-hand side of (5) for $s = s_k$, $\mathbf{h} = \mathbf{h}_k$, and $\xi = \xi_k$, where the subscript $k$ indicates that these are parameters associated with $x[k]$. Since $\phi_k$ is only an approximation, we take more measurements than the ones it prescribes, as in step 11, where $\delta$ is a safeguard parameter. Steps 12-14 describe the measurement process and the reconstruction of $x[k]$ using $\ell_1-\ell_1$ minimization. Next, steps 15-16 compute the quantities $\mathbf{h}_k$, $\xi_k$, and $s_k$, and step 17 computes $\mathbf{m}_k$, which, if the reconstruction was perfect, equals the right-hand side of (5) applied to $x[k]$. Note, however, that $\mathbf{m}_k$ is computed only after the measurements of $x[k]$ have been taken and the reconstruction of $x[k]$ has occurred. The value $\mathbf{m}_k$ is then used in step 18 to update $\phi_k$ as an exponential moving average filter with parameter $\alpha$.

To explain the rationale for the filtering step and the safeguard parameter $\delta$, suppose there is no filtering, i.e., $\alpha = 1$. In that case, the estimator $\phi_k$ of $\delta$ applied to $x[k]$ is simply $\mathbf{m}_k - 1$, which, if $x[k-1] > x[k-1]$ (perfect reconstruction at $k-1$), equals (5) applied to $x[k-1]$. Since (5) applied to $x[k-1]$ might differ from (5) applied to $x[k]$, we take more measurements for $x[k]$ than the ones specified by $\mathbf{m}_k - 1$, that is, $m_k = (1 + \delta)\mathbf{m}_k - 1$ (step 11). So, even when there is perfect reconstruction at time $k - 1$, $\delta$ should be large enough to account for variations of (5) from $x[k-1]$ to $x[k]$; see Lemma 1 below for a lower bound. If reconstruction fails at time $k - 1$, i.e., $x[k-1] \neq x[k-1]$, $\mathbf{m}_k - 1$ may be very different from (5) applied to $x[k-1]$ and to $x[k]$. The reason for filtering $\mathbf{m}_k$ in step 18 is to mitigate the effect of these failed reconstructions. We therefore recommend setting $\alpha < 1$.

**Reconstruction guarantees.** The following lemma considers $\alpha = 1$ and derives a lower bound on the probability of reconstruction success at each time, provided $\delta$ is large enough.

**Lemma 1.** Let $\alpha = 1$, $k > 2$, and $m = \min_{i = 1, \ldots, k} m_i$. Let also

$$
\delta \geq \max_{3 \leq i \leq k} \frac{2(\mathbf{h}_i \log(m_i) - \mathbf{h}_i \log(m_i))}{1 + 2\mathbf{h}_i \log(m_i)} + \frac{\xi_i}{\mathbf{u}_i - \mathbf{u}_{i-1}},
$$

(7)

where $\mathbf{u}_i = s_i + \xi_i/2$. Assume $s_i \geq \{i : x_i[|q| \neq 0]\}$, for $q = 1, 2$, i.e., that the initial sparsity estimates $s_1$ and $s_2$ are not smaller than the true sparsity of $x[1]$ and $x[2]$. Then, the probability that Algorithm 1 reconstructs $x[i]$ perfectly in all time instants $1 \leq i \leq k$ is not smaller than

$$
1 - \exp\left(-\frac{1}{2}(m_i - \sqrt{m_i})^2\right),
$$

(8)

**Proof.** Since $\alpha = 1$, step 11 becomes $m_i = (1 + \delta)\mathbf{m}_k - 1$, for all $3 \leq i \leq k$. According to Theorem 1, if $1 + \delta \mathbf{m}_k - 1$ is not smaller than the right-hand side of (5) applied to $x[k]$, that is,

$$
(1 + \delta)\mathbf{m}_k - 1 \geq 2\mathbf{h}_i \log(m_i) + \frac{\xi_i}{\mathbf{u}_i - \mathbf{u}_{i-1}} + 1,
$$

(9)

then the probability of perfect reconstruction at time $i$ is not smaller than $1 - \exp\left(-\frac{1}{2}(m_i - \sqrt{m_i})^2\right)$. In other words,

$$
P(S_i | E_i) \geq 1 - \exp\left(-\frac{1}{2}(m_i - \sqrt{m_i})^2\right),
$$

(10)

where $S_i$ is the event “perfect reconstruction at time $i$” and $E_i$ is the event in (9). Simple algebraic manipulation shows that if we replace the expression for $\mathbf{m}_i$ (in step 17) in (9), we obtain

$$
\delta \geq \frac{2(\mathbf{h}_i \log(m_i) - \mathbf{h}_{i-1} \log(m_i)) + \frac{\xi_i}{\mathbf{u}_i - \mathbf{u}_{i-1}}}{1 + 2\mathbf{h}_{i-1} \log(m_i) + \frac{\xi_{i-1}}{\mathbf{u}_{i-1} - \mathbf{u}_{i-2}}}.
$$

(11)

That is, (11) is event $E_i$. Therefore, condition (7) corresponds to the event $E := E_3 \land E_4 \land \cdots \land E_k$. And we have

$$
P(S_1 \land S_2 \land \cdots \land S_k | E) \geq \prod_{i = 3}^{k} P(S_i | S_1 \land \cdots \land S_{i-1} \land E) \geq P(S_1 | S_2 | E) \prod_{i = 3}^{k} P(S_i | E_i).
$$

(12)

From (12) to (13), we used the fact that $S_1$ and $S_2$ are independent. From (13) to (14), we used the fact that $S_i | E = S_i | E_i$, for $3 \leq i \leq k$, and that the events $S_i$ conditioned on $E_i$, i.e., (9), are independent, for $3 \leq i \leq k$. Now note that, since $\mathbf{m}_i \geq m$ and $1 - \exp\left(-\frac{1}{2}(m - \sqrt{m})^2\right)$ is an increasing function, (10) implies

$$
P(S_i | E_i) \geq 1 - \exp\left(-\frac{1}{2}(m - \sqrt{m})^2\right),
$$

(15)

The right-hand side of (15) also lower bounds $P(S_1)$ and $P(S_2)$ [23]. From (14) and (15), we obtain

$$
P(S_1 \land S_2 \land \cdots \land S_k | E) \geq \left(1 - \frac{1}{2}(m - \sqrt{m})^2\right)^k
$$

and the lemma is proved.

When the conditions of the lemma hold, the probability of successful reconstruction decreases with time, albeit with a very slow rate: for example, if $m = 8$, which is very small for applications, the right-hand side of (8) gives 0.9998 for $k = 10^2$, and 0.9845 for $k = 10^4$. Larger $m$ give even smaller rates.

To get some insight about (7), let $\gamma$ be an index for which the maximum is achieved in the right-hand side of (7). Also, let $n$ be much larger than $s_i$, and $\xi_i$. Then, (7) becomes

$$
\delta \geq \frac{2(\mathbf{h}_\gamma - \mathbf{h}_{\gamma-1}) \log n + o(\log n)}{2\mathbf{h}_{\gamma-1} \log n + o(\log n)},
$$

and, for a large $n$,

$$
\delta \geq \frac{\mathbf{h}_\gamma - \mathbf{h}_{\gamma-1}}{\mathbf{h}_{\gamma-1}}.
$$

(16)

Equation (16) tells us that, for $\alpha = 1$ and for very sparse signals, the oversampling factor $\delta$ in Algorithm 1 should be greater than the largest relative increase between two consecutive $\mathbf{h}_k$'s. Writing $\mathbf{h}_k = \{|i : x_i[k] > 0, \xi_i[k] > 0\} \cup \{|i : x_i[k] > 0, \xi_i[k] > 0\}$ (see (4a)), we conclude that $\mathbf{h}_k$ increases if and only if there is a new index $i$ for which $x_i[k] \neq 0$, and $\xi_i[k] \neq 0$ have the same sign.

**Variations of Algorithm 1.** For example, rather than generating a matrix $A_i$ at each iteration, one can generate a single (Gaussian) matrix $A \in \mathbb{R}^{m \times K}$ at the beginning and, at each iteration, select $m_i$ rows of $A$ randomly. Another variation, motivated by Lemma 1, sets $\alpha = 1$ and recursively updates $\delta$ applying, e.g., an exponential moving average filter to the expression in the right-hand side of (7).
Fig. 1. (a)-(d) background image and first three frames; (e) predicted image using (reconstructed) frames 1 and 2; and (f) reconstruction of frame 3 by \(\ell_1-\ell_1\) minimization.

5. EXPERIMENTAL RESULTS

We assessed the performance of Algorithm 1 by applying it to compressive background subtraction [13], which we explain next.

Compressive background subtraction. Let \(\{z[k]\}\) be a sequence of (vectorized) images with the same background \(b \in \mathbb{R}^n\), assumed known. We have access only to a set of \(m_k\) linear measurements \(u[k] = A_k z[k]\) from each image \(z[k]\), where \(A_k \in \mathbb{R}^{m_k \times n}\) is a measurement matrix. Each \(z[k]\) can then be decomposed as \(z[k] = x[k] + b\), where \(x[k]\) is the image foreground. As noticed in [13], foregrounds are typically sparse and thus can be reconstructed using standard \(\ell_1\) minimization. To do it, we need access to foreground measurements, which can be obtained as follows [13]: take measurements of the known background \(b\) using the same measurement matrix, \(u^0 := A_k b\), and subtract them from \(u[k]\), i.e., \(y[k] := u[k] - u^0 = A_k (z[k] - b) = A_k x[k]\).

Our approach. In our experiments, we modified the model in (1) by assuming that each \(x[k]\) is generated by the two previous signals, i.e., \(x[k] = f_k(x[k-1], x[k-2])\). This modification has no implications on our algorithm or on the associated reconstruction guarantees. However, it allows us to model the action of a motion-compensated extrapolation algorithm [24–26]: given two (consecutive) images, \(z[k-1]\) and \(z[k-2]\), predict the next one, \(z[k]\), assuming linear motion. We perform extrapolation on the image domain rather than on the foreground domain, because the texture of the former is richer and improves the estimation performance. The side information fed to \(\ell_1-\ell_1\) minimization is, of course, in the foreground domain: \(w[k] = e[k] - b\), where \(e[k]\) is the image extrapolated by the motion-compensated algorithm.

Experimental setup. We used the Hall video sequence (http://trace.eas.asu.edu/yuv/), from which we removed the first 18 frames, as they had no foreground. Each image was downsampled to a resolution of \(128 \times 128\). For the motion-compensated extrapolation, we used sub-pel motion estimation with a block size of \(8 \times 8\) pixels and a search range of 6 pixels. The parameters \(\alpha\) and \(\delta\) were 0.5 and 0.1, respectively. The parameters \(s_1\) and \(s_2\) were initialized with the true sparsity of the first two foregrounds. To solve each \(\ell_1-\ell_1\) minimization problem in step 14 of Algorithm 1, we used ADMM [27, 28], where one term of the objective function is \(\|x\|_1 + \|x - w\|_1\), and the other term is the indicator function of the linear system \(Ax = y\). It can be shown that both terms have closed-form proximal operators. The augmented Lagrangian parameter was updated as suggested in [29].

Results. Fig. 2 shows the number of measurements \(m_k\), estimate \(\phi_k\), and right-hand side (5) for \(x[k]\) per frame. It is also shown the bound for standard \(\ell_1\) minimization [23].

6. CONCLUSIONS

We proposed a recursive \(\ell_1-\ell_1\) minimization algorithm for reconstructing time varying sparse signals from a limited number of linear measurements. Based on recent theoretical results on \(\ell_1-\ell_1\) minimization, the algorithm estimates, on-the-fly, the number of measurements required to reconstruct the signal in the next time instant. Experimental results on compressive background subtraction using real test video data demonstrate the validity of our estimation scheme.
and the high reconstruction performance of $\ell_1$-$\ell_1$ minimization.

7. REFERENCES


