The “Spark Spread:” An Equilibrium Model of Cross-Commodity Price Relationships in Electricity*

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Abstract

This paper presents a competitive rational expectations model of spot and forward prices for multiple commodities that can be stored and/or converted. As a result of the conversion option, an equilibrium theory of basis spreads across commodities is derived. This extends the “theory of storage” models to include goods which are not directly storable. In particular, we consider “upstream” (e.g., natural gas and other fuels) and “downstream” commodities (e.g., electricity). We show that many of the most intriguing empirical features of electricity prices follow naturally from the underlying economics of supply and demand. For example, the model produces mean-reverting prices, price-dependent heteroscedasticity, skewness and unstable electricity-fuel correlations. The general model can incorporate other specifications such as a commodity with multiple uses (e.g., oil and its refined products) or a single commodity at multiple locations (e.g., natural gas).
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1 Introduction

The ongoing deregulation of electricity in the US is transforming a once strictly-regulated, low-volatility, monopolistic industry into an active market with annual sales of over $250 billion.\(^1\) In particular, the decoupling of power generation from power distribution has freed end-users, power generators, and power-marketers (intermediaries) to negotiate and contract directly with each other.

A natural byproduct of this greater flexibility is a closer link between market prices and the underlying economics of supply and demand.\(^2\) Midwest electricity prices in June 1998 provide a dramatic example. Peak-load electricity transacted at prices were as high as $7,500 per megawatt hour.\(^3\) The possibility of such extreme price volatility leads naturally to a demand for OTC and exchange-traded derivative securities for risk management and/or speculation. In addition, electricity supply contracts often include, for operational reasons, a wide range of exotic embedded options relating to power delivery.\(^4\)

A critical input into the valuation of energy derivatives is a specification of the underlying spot and forward price processes. In addition, cross-commodity hedging and Value-at-Risk calculations require models of the joint price processes. This paper provides a model in which the links among commodity prices are endogenous. For example, although electricity itself is not directly storable (in meaningful quantities), potential electricity in the form of hydro, coal, nuclear, oil and natural gas is storable. This insight suggests a natural extension of the competitive storage model developed in Kaklor (1939), Working (1948), Working (1949), Telser (1958), Heinkel, Howe, and Hughes (1990), Williams and Wright (1991), Deaton and Laroque (1992), Deaton and Laroque (1996), Chambers and Bailey (1996), Routledge, Seppi,

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\(^1\)See http://www.adtrading.com/adtr32/ehot.htm. Similar moves toward deregulation are also underway in Scandinavia, in the UK and in Australia.

\(^2\)Of course, many observers addressing the current electricity crisis in California have suggested that the absence of some linkages between market prices and the economics of supply and demand has contributed to the crisis.

\(^3\)For comparison, power at Cinergy (an active Midwest hub) typically trades at prices of $25 to $50 per MWh. See P.E.R.C. (1998).

\(^4\)One common exotic structure is a swing option that grants end-users the right to buy additional electricity at a fixed (strike) price on, say, five days (of their choosing) out of the month. Jailet, Rom, and Tompaidis (1997) and Pilipovic and Wengler (1998) show that swing options can be valued as generalized American calls.
and Spatt (2000) and Pirrong (1998), to model the joint equilibrium evolution of spot prices and forward curves for electricity and various fuels. In particular, we specify processes for a sequence of random shocks to the supply and demand of potential fuels and to the net demand for electricity and then derive the equilibrium storage (for any fuels which are storable) and conversion outcomes (i.e., which fuels are burned to generate electricity). These, in turn, determine the equilibrium fuel and electricity price processes. Differentiating the electricity and fuel (e.g., natural gas) price processes gives the process for the *spark spread*.

Our model is a blend of two polar extremes in modeling electricity prices. On the one hand are structural “operations research” style models with detailed specifications of regional transmission capacity and the operational characteristics of individual power plants. These models are used by utilities to project electricity prices under various hypothetical scenarios. On the other hand are reduced-form “finance” models with exogenously specified spot and/or forward price processes. Black (1976), Brennan (1991), Gibson and Schwartz (1990), Schwartz (1997), Amin, Ng, and Pirrong (1995) (based on Heath, Jarrow, and Morton (1992)) and Eydeland and Geman (1998) are examples of this second approach. Our model (like the OR models) captures the physical properties of storage and conversion for electricity and component fuels, but (like the finance models) delivers a tractable framework for derivative security pricing.

Kaminski (1997) lists some key empirical facts with which models of electricity must, he argues, be consistent to be of practical use. These include:

- Skewed price distributions (i.e., “floor reversion” with price “spikes”).
- Heteroscedasticity with higher (lower) volatility when prices are high (low).
- Changing correlations between electricity and potential fuel prices.

All of these follow naturally from a plausible specification of the underlying supply and demand for fuels and electricity. First, skewness and heteroscedasticity are immediate consequences of a supply schedule, called the *supply stack*, which is relatively flat at low levels of output and steep when output is high. In particular, capacity limits in generation (or inter-regional transmission) will make the supply stack vertical at some maximum output. Fluctuation in electricity demand at low output levels will have little impact on price. This defines a “floor” or baseload price level with low price volatility. Fluctuation in demand when output is high, however, will flow through to the price, leading to high price levels and high volatility. In the paper, we illustrate how, even if the shock distributions are symmetric, the equilibrium electricity price distribution can be left skewed while the storage of an input
fuel can produce right skewness. Convertibility concentrates electricity prices to the left and
the prices of an input fuel to the right by altering the relative market-clearing conditions for
these goods. Second, changing (i.e., state dependent) conditional correlations of electricity
and fuel prices are a natural consequence of the option to decide which fuel (or fuels) to use
to generate the marginal unit of electricity. In particular, the prices of fuels “higher up” in
the supply stack (e.g., natural gas) will be less correlated with electricity prices when demand
is low (and gas is not used to generate electricity) and more correlated when demand is high
(and gas is used). However, the relationship need not be monotone. Once the burn of gas
reaches its maximum, the two markets are decoupled as gas prices are determined by gas
market conditions and electricity prices are determined by the next marginal fuel price. In
addition, the shape of the electricity supply stack itself depends on fuel prices and, therefore,
on supply and demand conditions in those markets.

In order to develop an equilibrium model for electricity and various production fuels, we
extend the single commodity storage models of Deaton and Laroque (1992), Deaton and
Laroque (1996), Chambers and Bailey (1996) and Routledge, Seppi, and Spatt (2000) to a
general multi-commodity framework. We show that a unique and well-behaved equilibrium
exists for a model with \( K \) storable and transformable commodities. Although the general
model applies to any production economy, we focus on production technology parameters
where the non-negativity constraints are binding in equilibrium. We derive useful compara-
tive statics of the model. The electricity market, which is the focus of much of this paper,
is a special case of the general model we present. Despite the fact that electricity is not
directly storable, we show how the convertibility option implies that much of the theory of
storage applies to non-storable commodities like electricity. The framework also applies to
an input commodity that can be irreversibly converted into several output commodities. For
example, our model can be applied to the “crack spread” since crude oil can be refined into
gasoline, heating oil, jet fuel, etc. In addition, the model can be used to study commodity
prices at different locations. For example, natural gas at Henry Hub in Louisiana can be
shipped at cost to a variety of markets such as Chicago or Boston. Real options created by
the alternative uses of commodities are central to understanding the level and volatility of
the term-structure of forward prices and the corresponding cross-commodity spreads. This
paper offers the first equilibrium analysis of price spread dynamics.

The paper is organized as follows. In Section 2 we present a general model of convertible
and/or storable commodities. Section 3 analyzes some features of the equilibrium. Section

\footnote{This is similar to the “cheapest to deliver” option in bond futures contracts (see Eydeland and Geman (1998)). However, in our model we determine the endogenous price of the “deliverable.”}
specializes our framework to the electricity market and presents a numerical example. Section 5 concludes and proofs are gathered in the Appendix.

2 Model

In this section, we allow for multiple (possibly) storable commodities and for their conversion. As in the earlier literature, the goal of the model, is to specify the consumer-side of the economy as simply as possible to focus on the effects of storage and conversion. In single-good commodity storage models, the production and consumer side of the economy is specified by a reduced-form net-demand function $d(p, a)$, where demand is downward sloping in price $p$ and may depend on the stochastic factor $a$. This provides a straightforward way to construct a unique equilibrium in storage and market-clearing prices (see Deaton and Laroque (1992)). In a model with more than one storable commodity, making an analogous assumption about the vector of demands, $D(p, a)$ (where $D$ and $p$ are $K$-vectors) is problematic since solving for equilibrium storage and prices involves solving a system of overlapping non-linear equations. To circumvent this difficulty, we explicitly specify the consumer-producer problem. In this way, we can describe the equilibrium with a social planner’s problem. In previous storage models, assumptions about (aggregate) consumer preferences are made implicitly through the specification of the net-demand function.

2.1 Structure

At each date $t = 1, 2, ..., $ there is a competitive market for a set of $K = \{0, ..., K\}$ commodities. The vector of spot prices for the commodities is $p$ with elements $p^k$. The price of the numeraire good, good zero, is normalized everywhere to one.

The states of the economy $\omega \in \Omega$ and information arrival are generated by a $m$-state irreducible Markov process with transition probabilities $\pi(a|a')$ for $a, a' \in A$. Let $A_t(\omega) = (a_0(\omega), a_1(\omega), ..., a_t(\omega))$ represent one of the $m^t$ possible paths of the economy from date 0 to $t$. We assume agents know $A_t$ when making decisions at date $t$ (i.e., the path to date $t$ is an element of the filtration, $A_t \in F_t$). The probability of $A_t(\omega)$ is $\text{Prob}\{A_t(\omega)\} = \pi(A_t) = \prod_{\tau=1}^{t} \pi(a_\tau(\omega)|a_{\tau-1}(\omega))$. Without loss of generality let the initial state $a_0 = 0$ and, hence, $A_0 = 0$. 
We assume financial markets are complete (or dynamically complete).\footnote{There is some potential for confusion about completeness since we include a storage technology in our model. Intuition for the spot-forward price connection is often described as a physical-financial “arbitrage” with a short-sale constraint (i.e., non-negative storage). An option to convert natural gas to electricity (but not vice versa) leads to a similar intuition. To be precise, the assumption that markets are complete means that an Arrow-Debreu security can be replicated using only financial assets. After presenting our equilibrium, it is easy to construct a set of forward and call options for the $K$ commodities that imply markets are complete. Equilibrium construction will ensure that “arbitrage” using the physical technology of storage or conversion does not exist.} The price of an Arrow-Debreu security that pays one unit of the numeraire good (whose spot price is everywhere normalized to one) at date $t$ given path $A_t$ is denoted $s(A_t) > 0$. This price is expressed in units of the consumption good as of date zero. Therefore, $s(0) = 1$.

The Markov structure of the shocks is consistent with the mean reversion that is common in many commodity prices (Schwartz (1997) or Deaton and Larque (1992)). It is general enough to incorporate features like seasonality or special cases such as a binomial tree.\footnote{However, in Routledge, Seppi, and Spatt (2000), the single-commodity storage model was not able to match data from the oil market since the Markovian structure does not produce enough volatility for long-horizon forward prices. That paper shows how a permanent shock can be incorporated in a manner that preserved the tractable Markovian features of the model, while matching the properties of long-horizon contracts.}

Agents have identical\footnote{Given the assumption that financial markets are complete, this assumption can be relaxed.} and time-separable preferences, $E \left[ \sum_{t=0}^{\infty} \beta^t u(c(A_t), a_t) \right]$, where $0 < \beta < 1$ is the rate of time preference and $c(A_t) = \{c^k(A_t)\}_{k \in K}$ is the vector of consumption at date $t$ given path $A_t$. Preferences are strictly increasing, differentiable, quasi-concave, and, satisfy the Inada conditions in each commodity. Preferences may depend explicitly on the current shock realization, $a_t$. This captures shocks to commodity demand such as those from weather. The agent knows the current realization $a_t$ when making decisions at date $t$.

Agents have access to a costly storage technology and a costly conversion technology. Storage is costly due to a constant depreciation or wastage factor, $\delta^k \in (0, 1]$. The storage of $q^k$ units of good $k$ at $t - 1$ yields $(1 - \delta^k)q^k$ of the commodity at $t$. Storage must be non-negative. Non-storable goods are a special case where $\delta^k = 1$. Without loss of generality the initial storage for all goods is assumed to be zero. The $K$ commodities also can be transformed or refined. This captures, for example, the ability to produce electricity from natural gas. The technology $g$ transforms the matrix of inputs $b = [b^k_j]_{k, j \in K}$ into a vector of outputs, $g(b) = [g^k(b)]_{k \in K}$. Both inputs and outputs are non-negative. We will denote $b^k$ as the total amount of good $k$ used for conversion. Terminal goods that cannot be further converted, like electricity, are just a special case of the technology. The conversion technology has the standard properties of smoothness, concavity, and $g^k(0) = 0$. In addition,
we allow for a constraint on the maximum amount of refinement in that $\theta_j^k \leq \theta_j^k$. Given the concavity of $g$, the maximum throughput constraint, $\theta_j^k$, can be set high enough so that it does not bind in equilibrium. However, capacity constraints can play an important role in understanding markets like natural gas and electricity. The physical capacity of natural-gas-fired turbines used to produce electricity is reached during extreme peak loads. We will make more specific assumptions about the conversion process later in the paper. As with consumption, the storage $q(A_t)$ and conversion $b(A_t)$ decisions are made after observing the realization $a_t$.

Finally, there is an exogenous endowment of each of the commodities, $\epsilon^k(a_t)$, that is finite and positive ($0 < \epsilon^k(a_t) < \bar{\epsilon}$ for all $k \in K$ and $a_t \in A$). This endowment captures the base-level or primary supply of each good. For agricultural commodities, $\epsilon^k(a_t)$ represents the harvest. In the electricity setting it may capture hydro-electric base-load supply. The endowment may depend on the current shock realization, $a_t$, though not the prior portion of the path. For example, bad weather in the Gulf of Mexico can affect the availability of natural gas. For convenience, we use the same notation to capture shocks to user-demand in $u(c, a_t)$ and primitive supply $\epsilon(a_t)$. Of course, this is without loss of generality given the flexibility in defining the states $A$ of the Markov process.

### 2.2 The Agent’s Consumption, Storage, and Refining Problem

Since we have assumed that financial markets are complete, an agent’s problem can be expressed as a static optimization (as in Debreu (1959), Karatzas, Lehoczky, and Shreve (1987) or Cox and Huang (1989)). Competitive (price-taking) agents solve the following problem.

\[
\max_{c(A_t), q(A_t), b(A_t)} E \left[ \sum_{t=0}^{\infty} \beta^t u(c(A_t), a_t) \right]
\]

subject to:

\[
\sum_{t \in A_t(\omega)} s(A_t) \left( \sum_k p^k(A_t) \left( \epsilon^k(A_t) + (1 - \delta^k)q^k(A_{t-1}) - q^k(A_t) + g_k(b(A_t)) - \theta^k(A_t) \right) \right) - \sum_{t \in A_t(\omega)} s(A_t) \left( \sum_k p^k(A_t) \epsilon^k(A_t) \right) \geq 0
\]

\[
q^k(A_t), \theta^k_j(A_t) \geq 0
\]

The budget constraint states that the time-zero-value of the stream of endowment, net-storage, and net-conversion must be at least as large as the value of the consumption stream.
The assumptions on the preferences and technology are standard, so the problem has a unique solution with the optimal policies for \( c(A_t), q(A_t), \) and \( b(A_t) \) characterized by the first-order conditions (see equations (A1), (A2), and (A3) in the Appendix).

### 2.3 Equilibrium

The equilibrium is defined by optimality for each agent and market clearing. Spot goods markets must clear at all times \( t \) for all paths \( A_t(\omega) \)

\[
e^k(A_t) = e^k(a_t) + (1 - \delta^k)q^k(A_{t-1}) - q^k(A_t) + g_k(b(A_t)) - b^k(A_t) \tag{4}
\]

As is standard in a complete market, we can characterize prices from a social planner’s problem. The social planning problem is

\[
\max_{c(A_t), q(A_t), b(A_t)} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t u(c(A_t), a_t) \right] \tag{5}
\]

subject to (4) and the same non-negativity constraint as (3). Again, the assumptions on the preferences and technology are standard, so the problem has a unique solution characterized by the first-order conditions (see Appendix). Not surprisingly, the equilibrium is determined by these conditions. (Note \( u^*_k(A_t) = \frac{\partial u(c(A_t), a_t)}{\partial a_t} \)).

**Proposition 1 (Equilibrium)**

\( \{q(A_t), b(A_t), c(A_t), s(A_t), p(A_t)\} \) is an equilibrium in which \( q(A_t) \) and \( b(A_t) \) solve (5), \( c(A_t) \) is defined by (4), and for all \( A_t \) and \( k \in K \) prices are given by:

\[
s(A_t) = \beta^t \pi(A_t) \frac{u'_0(c(A_t), a_t)}{u'_0(c(0), a_0)} \tag{6}
\]

\[
p^k(A_t) = \frac{u^*_k(c(A_t), a_t)}{u^*_0(c(A_t), a_t)} \tag{7}
\]

By inspection of (2) and (4), note that the current exogenous shock, \( a_t \), and the previous level of endogenous storage, \( q_{t-1} \), are sufficient to describe the state of the system.\(^9\) Therefore, we can rewrite the policies and prices of Proposition 1 as functions of these state variables. By the “Principle of Optimality” (Stokey and Lucas (1989), Chapter 9) the optimal plan for (5) of \( q(A_t), b(A_t), \) and \( c(A_t) \), can be generated by the functions \( Q(a, q), B(a, q), \) and \( C(a, q). \)

\(^9\)Given that the storage technology is non-stochastic, it does not matter whether the endogenous state variable is defined by the previous level of storage, \( q_{t-1} \) or the incoming depreciated inventory \( (1 - \delta^k)q_{t-1}^k \).
That is for storage, \( q(A_t) = Q(a_t, q(A_{t-1})) \) which we will abbreviate \( q_t = Q(a_t, q_{t-1}) \),\(^{10}\) and analogously for optimal refinement and consumption. Therefore, we can restate the equilibrium prices in equations (6) and (7) as functions of the state variables, i.e., \( S(a, q) \) and \( P(a, q) \).

\[
S(a_t, q_{t-1}) = \beta' \pi(A_t) \frac{u_0'(C(a_t, q_{t-1}), a_t)}{u_0'(C(a_0, 0), a_0)}
\]

\[
P^k(a_t, q_{t-1}) = \frac{u_k'(C(a_t, q_{t-1}), a_t)}{u_0'(C(a_t, q_{t-1}), a_t)}
\]

The results that are presented below are driven by the first-order conditions on storage and refinement. In equilibrium, the first-order conditions for (1) and (5) are as follows. Denote \( u'_k(a_t, q_{t-1}) = \frac{\partial u(C(a_t, q_{t-1}), a_t)}{\partial a_t} \) and \( g'_{jk}(a_t, q_{t-1}) = \frac{\partial g_j(B(a_t, q_{t-1}))}{\partial a_t} \).

\[
g'_{jk}(a_t, q_{t-1}) \leq \frac{P^k(a_t, q_{t-1})}{P^j(a_t, q_{t-1})} = \frac{u_k'(a_t, q_{t-1})}{u_j'(a_t, q_{t-1})} \quad \text{if } B^k_j(a_t, q_{t-1}) = 0
\]

\[
\geq \frac{P^j(a_t, q_{t-1})}{P^j(a_t, q_{t-1})} = \frac{u_j'(a_t, q_{t-1})}{u_j'(a_t, q_{t-1})} \quad \text{if } B^k_j(a_t, q_{t-1}) \in (0, \beta_j^k)
\]

\[
\geq \beta(1 - \delta)E\left[u'_k(a_{t+1}, q_t)\right] a_t \quad \text{if } Q^k(a_t, q_{t-1}) > 0
\]

\[
\geq \beta(1 - \delta)E\left[u'_k(a_{t+1}, q_t)\right] P^k(a_{t+1}, q_t) \quad \text{if } Q^k(a_t, q_{t-1}) = 0
\]

where \( q^k_t = Q^k(a_t, q_{t-1}) \). This last equation can also be written in terms of prices as

\[
P^k(a_t, q_{t-1}) \geq \beta(1 - \delta)E\left[\frac{u'_0(a_{t+1}, q_t)}{u'_0(a_t, q_{t-1})} P^k(a_{t+1}, q_t)\right] a_t \quad \text{if } q^k_t > 0
\]

\[
P^k(a_t, q_{t-1}) \geq \beta(1 - \delta)E\left[\frac{u'_0(a_{t+1}, q_t)}{u'_0(a_t, q_{t-1})} P^k(a_{t+1}, q_t)\right] \quad \text{if } q^k_t = 0
\]

The first-order condition on refinement inputs, equation (10), links the prices of different commodities at a given time. The first-order condition on storage, equation (11) or (12), links the price of a commodity across time. In theory-of-storage models, many of the interesting features of equilibrium prices occur due to the non-negativity constraint on storage. For example, in these models, convenience yield is positive (or the forward curve is downward sloping) only when inventory hits zero (see, e.g., Routledge, Seppi, and Spatt (2000)). In the cross-commodity model we consider here both the non-negativity constraint on inventory and the constraints on refinement inputs are important, since linkages implied by the first-order conditions are “broken” when the constraints bind. Note that the non-negativity constraints play an important role in determining prices through their effect on marginal utilities. However, they do not directly impact the equilibrium construction of prices in

\(^{10}\)Recall the initial level of inventory is zero for all goods, so \( q_0 = Q(a_0, 0) \).
equations (8) and (9).

In the remainder of the paper we will consider more specific preferences and technologies to capture the salient features of commodity markets like natural gas and electricity.

2.4 Specific Assumptions

We are interested in how endogenous storage and conversion decisions transmit shocks across commodities and through time. In order to focus on this aspect, we assume that the consumer utility and conversion technology are separable. This lets us abstract from the utility-based substitution of end-user consumption. For example, consumers may demand both natural gas and electricity, but cannot substitute electricity for natural gas through the use of an electric heater. Similarly, abstracting from joint production means that the use of natural gas to produce electricity does not require the use of additional oil. This approach is similar in spirit to the model of Deaton and Laroque (1992). They assume demand shocks are intertemporally i.i.d. in order to isolate the auto-correlation due to the storage decision.\footnote{Note, however, as in Deaton and Laroque (1996), our demand shocks need not be i.i.d.}

Specifically we assume:

(A) \( g_k(b) = \sum_{j \in k} g_k^j(b_k^j). \)

(B) \( u(c, a) = \sum_{k \in \mathcal{K}} v^k(c_k, a). \)

Assumption (A) states that the amount of good \( k \) obtained from conversion, \( g_k \), is separable in the inputs. The amount of good \( k \) generated from good \( j \) depends only on the amount of good \( j \) used, \( b_k^j \). (The notation for the input \( b_k^j \) should be read as using good \( j \) in the production of good \( k \).) Similarly, assumption (B) implies that the marginal utility of good \( k \) depends only on the amount of consumption of good \( k \), \( c_k \). Note that (B) includes a Cobb-Douglas utility function as a special case.

Our focus is on the equilibrium role of the option-like features of storage and production constraints. Therefore, as in single-good storage models, we assume risk-neutrality and a constant risk-free interest rate.\footnote{Jagannathan (1985) concentrates on the pricing kernel implications of multiple commodities in an endowment economy.} From (6), the necessary and sufficient condition for risk neutrality is \( u_0(C(A_t), a_t) = \bar{u} \) for all \( A_t \) (where \( C(A_t) \) solves (5)). Given the specific utility and technology of assumptions (B) and (A), there are two alternative conditions that are sufficient for risk neutrality. We can assume that the marginal utility of the numeraire good
is constant, i.e., $v^\alpha$ is constant. Alternatively, we can restrict commodity refinement and conversion to the $K$ non-numeraire goods (i.e., $q^0_k = d^0_k = 0$) and assume the numeraire endowment is constant ($\bar{v}^0(a) = \bar{v}^0$ for all $a \in \mathcal{A}$). An immediate implication of this assumption is that the equilibrium forward price for any commodity $k$ agreed to at $t$ for delivery at date $T$ is:

$$F_{t,T}^k(a_t, q_{t-1}) = E \left[ P^k(a_T, q_{T-1}) \big| a_t, q_{t-1} \right]$$

(13)

3 Equilibrium Properties

In this section we examine various properties of the equilibrium model to understand their implications for the dynamics of commodity pricing.

3.1 Characterizing Prices for a given storage

To understand the joint effects of storage and conversion, it is helpful to consider the effect of conversion by itself on equilibrium prices. In this subsection, we look at conversion for exogenously determined net storage. Since $q^j_k$ in (10) is monotonic, we can write the equilibrium conversion policy as a function of the ratio $p^k / p^j$. That is: $B_k^j(a_t, q_{t-1}) = b_k^j(p^k / p^j)$ where the function $b(\cdot)$ is monotonically decreasing. This is helpful in characterizing prices since, when coupled with the assumption on the separability of utility in (B), excess demands satisfy gross substitutability. Therefore, the price of good $k$ is strictly increasing in net-storage of good $k$ and weakly increasing in the storage of all other goods.

Proposition 2 (Market-Clearing-Prices)

For demand state $a \in \Omega$ and for a specified exogenous net-storage levels $\Delta q_t$

(a) The market-clearing price function $p^k = \rho(a, \Delta q)$ exists and is continuous in net storage $\Delta q$.

(b) The price of good $k$ is strictly increasing and the price of all other goods is weakly increasing in the net storage of good $k$, $\Delta q^k$.

(c) The net conversion of good $k$, $b_k^j$ is weakly decreasing in $\Delta q^k$, while the net conversion of all other goods, $b^j$, is weakly increasing in $\Delta q^k$.

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Footnote: The gross-substitutability property states that excess demand of good $k$, denoted $X^k(p, a, \Delta q)$ (for a given state and net inventory demand) have cross-derivatives that are positive and own derivatives that are negative. That is: $dX^k / dp^j \geq 0$ for all $k \neq j$ (Mas-Colell, Whinston, and Green (1995)).
(d) The net conversion of good $j$ is strictly increasing in the net storage of good $k$ if there exists a chain of commodities, $k = i_1, i_2, \ldots, i_m = j$ such that $0 < B_{i_{m+1}}^k < B_{i_{m+1}}^k$ or $0 < B_{i_{m+1}}^k < B_{i_{m+1}}^k$.

The comparative static results are intuitive. An (exogenous) increase in net storage for good $k$ shifts the price of good $k$ up. The equilibrium response is to decrease (increase) the use $b_j^k$ (production $g_k^j(b_j^k)$) of good $k$ and increase (decrease) the use (production) of good $j$, which increases the price of good $j$. Since optimal conversion, $b_j^k = B_j^k(p^k/p)$, is monotonic, the market-clearing price function specifies the equilibrium conversion. The final result highlights that the linkage in the prices of the commodities is endogenous. In the context of electricity, an increase in the price of oil may affect the natural gas market if both fuels are being used to produce electricity. Note that the commodities need not be directly related. For example, a shock to the jet-fuel market may affect the crude oil market, which affects the electricity market, which impacts natural gas. However, for this chain reaction to occur, it is necessary that there is some discretion in each of the conversion decisions; that is the production cannot be constrained at either zero or its maximum. Note that the link between commodities can be through a chain of input-outputs, through inputs having a common output, through an input having multiple outputs or some combination of these.

### 3.2 The Role of Inventory

Costly storage implies that inventory is bounded. In particular, an increase to inventory at period 0 has a limited and diminishing effect on inventory in period $T$. Optimality also requires a positive probability of stockout. This along with the Markovian structure of the exogenous shocks will limit the effect that inventory has on equilibrium prices.

**Proposition 3 (Inventory is Temporary)**

Consider a sequence of demand shock realizations, $\{a_t\}_{t=1}^\infty$, and equilibrium inventories $\{q_t\}_{t=1}^\infty$ with $q_t = Q(a_t, q_{t-1})$ and an arbitrary vector of initial inventories, $q_0$. A positive perturbation to initial inventories $q_0 + \epsilon$ with $\epsilon > 0$ implies that:

(a) the effect of the perturbation decreases monotonically in time

(b) the effect is temporary with probability one.

Similar to single-good storage models, the positive storage costs imply that inventory cannot always be strictly positive and reaches zero, $q_t = 0$ (a “stockout”) with probability one.\(^{14}\)

\(^{14}\)Note that the statement is that $q_t^k = 0$ for all $k$ with probability one. Of course, this implies the weaker $q_t^k = 0$ for some $k$. 

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This is important since stockouts (or their likelihood) are central to many of the features of our model such as non-constant cross-price correlations. Stockouts also play a useful computational role. The asymmetric behavior induced by the non-negativity constraint in inventory and conversion induces a path dependence in the state \((a_t, q_{t-1})\). However, a stockout breaks the dependence on the past and the process renews. This implies that the long-run distribution of inventory and prices is independent of the current demand shock, \(a_t\), and level of storage, \(q_{t-1}\).

For many commodities, the storage and conversion decisions are not observable for econometric analysis. It is important, therefore, to understand the implications of storage and conversion for prices. In fact, since even spot prices may not be observable (as in Schwartz (1997)), the implications for forward and derivative prices are important. The following proposition derives some of these implications.

**Proposition 4 (Inventory and Prices)**

(a) Spot and derivative prices are mean reverting.

(b) The limiting distribution (as \(h \to \infty\)) is independent of the current demand state and previous inventory.

(c) If equilibrium inventory is non-decreasing in previous storage, \(dQ^k/dq^j \geq 0\), then the probability distributions for the vector of current and future spot prices, 
\[ P(a_{t+h}, q_{t+h-1}) \text{ for } h = 0, 1, \ldots \text{ are weakly ranked by first-order stochastic dominance as } q_{t-1} \text{ is decreased. Therefore, if } dQ^k/dq^j \geq 0, \text{ then forward and call prices are weakly decreasing in previous storage.} \]

The first two points follow trivially from the renewal property of inventory. Since inventory remains in a compact interval, prices are also bounded and hence mean reverting. As a result, the long-run distribution is independent of the current state. Under the restriction in the final portion of the proposition, not only do current prices (weakly) fall with previous storage, but all future prices are also (weakly) lower. Since spot prices in future periods are lower for each sequence of potential realizations of the demand state, this implies that the price of a derivative contract, whose price is monotonically increasing in future spot prices, like a forward or call, must (weakly) decrease.

The general intuition underlying the conclusion that prices across commodities move weakly together after a shock to a particular commodity’s inventory is natural given the social planner’s formulation of the equilibrium. The central planner desires to smooth by

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allocating additional inventory supply across different products and time dates, thereby reducing all prices (weakly), if he is able to smooth in this manner. This logic raises the question of why a somewhat restrictive condition is imposed (and required) in this portion of the proposition. The reason is that the maximum constraints on the conversion option may interfere with the ability of the social planner to spread supply across products. In such situations the resulting storage adjustments on other products that are efficient can undue the conclusion that additional supply reduces all prices. Of course, in a situation in which there is a single storable good (and for example, many nonstorable goods) such difficulties cannot arise and the restriction stated in the proposition is always satisfied.

The final implication of Proposition 4 is to illustrate the link between the inventory of one commodity and the price of another. For example, electricity is not storable. However, input fuels like coal and natural gas are storable. The inventory of input fuels will affect the price of non-storable commodities. Under the restriction assumed in the final portion of Proposition 4, all prices are weakly decreasing in the previous storage. In the context of electricity and its potential input fuels, consider the comparative static exercise of increasing the incoming storage of oil. In equilibrium, the increase in oil inventory is allocated to increased (at least not decreased) inventory of oil for the future and to more oil being burned into electricity. The net storage-refining demand for oil decreases, resulting in more oil available for immediate consumption, and hence, a lower price for oil. If the conversion of oil had been strictly positive and hence is strictly decreased by the change in previous storage, then the price of electricity falls. Through the refinement linkage, the storage perturbation is passed through to the electricity market. While the change to previous inventory and the optimal response in storage and refining of oil does not directly influence the conversion of gas into electricity (recall that assumption (A) stated that the technologies for converting oil to electricity and gas to electricity are separable), in equilibrium, the conversion and storage of gas may be affected. If gas is currently being converted to electricity, the lower price of electricity results in strictly less gas being converted to electricity. This leaves more natural gas for immediate consumption and gas storage, and hence a lower price of gas. The perturbation to oil inventory affects the price of oil and is potentially propagated to the electricity market and then to the gas market through the technologies for converting (burning) oil and gas into electricity. However, in order for the transmission to occur, there must be some discretion in the amount of fuels being used. This occurs when fuels are marginal in that equation (10) holds with equality and the amount of fuel being converted is neither zero nor its maximum. Unless an unconstrained amount of oil is used to produce electricity, there is no transmission of its inventory perturbation to the electricity market.
Unless both oil and gas are unconstrained in producing electricity, there is no transmission of the inventory perturbation in oil to the natural gas market.

**Proposition 5 (Propagation)**

An increase in the previous storage of good \( k, q_{t-1}^k \), will strictly decrease the price of good \( j, p_t^j \), if there exists a chain of commodities from \( k \) to \( j \) as in Proposition 2(d) there exists a commodity \( i \) with \( dQ_i^j/dq_i^k \neq 0 \) and a chain of commodities from \( i \) to \( j \), \( i = i_1, i_2, \ldots, i_m = j \) such that \( 0 < \bar{b}_{i_{n+1}}^l < \bar{b}_{i_n}^l \) or \( 0 < \bar{b}_{i_{n+1}}^b < \bar{b}_{i_n}^b \).

The proposition is slightly more complicated than the preceding example since there are multiple possible ways the storage of good \( k \) can affect the price of good \( j \). For example, the change in \( \Delta q^k \) may affect the price of good \( j \) through conversion as in Proposition 2(d). Alternatively, the storage of good \( k \) may alter the equilibrium storage of good \( i \) and good \( i \) may transmit the perturbation to the price of good \( j \).

### 3.3 The Effect of Storage and Conversion on Price Distributions

As mentioned in the introduction, the distribution of electricity prices has some puzzling features. They are heavily skewed and “floor reverting.” They are heteroskedastic and the correlations with input commodities are not constant. Natural gas, on the other hand, displays heteroskedasticity, but is far less skewed. These features are derived from the correlation structure induced by the equilibrium conversion and storage decisions.

#### 3.3.1 Skewness

An important feature of electricity is that compared to related commodities like natural gas, the distribution is much more skewed. In this section we consider how the unconditional skewness can arise from conversion and storage. In order to explore this, we begin from an economy, absent of storage and conversion, where spot prices are distributed symmetrically. In this sub-section, we assume (C) and (D).

(C) The Markov process and the end-user demands are such that the prices that solve \( D(a^k, p^k) = 0 \) have a symmetrical distribution.

Again, the example considered in Section 4 satisfies this assumption. Note that since the end-user demands are not necessarily linear, the assumption cannot be stated in terms of the Markov process alone.
The discussion following Proposition 4 indicates when a shock to one market can propagate to another. In order to explore the nature of skewness and cross-price correlations, consider the special case where the Markov process demand shocks can partitioned into $K$ independent processes. In addition, we consider the case where the end-user demand for good $k$ depends only on the $k^{th}$ shock. In this special case of independent shocks, prices (or price changes) would be independent across commodities in the absence of storage and convertibility.$^{16}$ Formally, this assumption is stated as:

$\Omega$ is partitioned into $\Omega^k$ where $a_i^k$ and $a_j^l$ are independent and $D(a, p^k) = D(a^k, p^k)$

The option to convert a good (e.g., gas) increases its value by increasing the good’s uses. In contrast, the option reduces the value of output goods (like electricity) by increasing their supply. The contrasting price effects for input and output commodities produce skewed price distributions. Consider three economies. First, in an economy where good $e$, electricity, and good $n$, natural gas, are neither storable nor convertible, the price distributions are symmetric (by assumption). Second, consider what happens if $n$ can be converted into $e$ and good $e$ remains a terminal good. In this case, conversion will be positive only when the demand state of good $n$ is low and good $e$ is high. The equilibrium price of good $n$ will be higher and the price of $e$ is lower. In the opposite situation, the non-negativity constraint on $b_e^n$ binds and the prices are the same as when the goods are not convertible. Since the two situations are equally likely (the assumption of symmetry), the conversion skews the price of good $n$ to the right (fewer observations of the low price) and skews the price of good $e$ to the left (fewer observations of the high price). Thirdly, adding storage makes things more complicated. If good $n$ is storable, it will be stored in the low demand states (pushing up the price) and used in the high-demand states (otherwise reducing the price). The net effect of storage is to concentrate the distribution. However, since stockouts happen (see Proposition 3) not all of the high-demand states can be perfectly buffered. This tends to leave the distribution with a mass on the high price (right skew). This point is discussed further for the numerical example in Section 4 (Figure 5).

3.3.2 Heteroskedasticity

The conditional variance of spot prices, like the conditional correlations, depends on the endogenous level of storage. Unfortunately, the effect of inventory on conditional variance cannot be signed. Intuitively, one would expect inventory to buffer shocks and hence reduce

$^{16}$The numerical example considered in Section 4 has this feature.
3.3.3 Cross-commodity Price Correlation and Hedging

Allowing for conversion, the correlations across prices will no longer be zero when the shocks are independent (assumption (D)). Furthermore, allowing for storage means that the correlations conditioning on the current demand state and the incoming storage, will change with inventory.

**Proposition 6 (Price Correlations)**

Given an economy where (D) holds,

(a) \( \text{cor}_t(p_{i, t+1}^k, p_{j, t+1}^l) \) is non-zero if and only if in at least one possible state, \( a_{t+1} \), there is a chain of commodities, \( j = i_1, i_2, \ldots, i_m = k \) such that \( 0 < B_{k, t+1}^{k_n} < \widetilde{B}_{k, t+1}^{k_n}, \) or \( 0 < B_{k, t+1}^{k_n} < \widetilde{B}_{k, t+1}^{k_n}, \) or \( dQ_{k, t+1}^{k_n}/dq_{k, t+1} \neq 0 \)

(b) \( \text{cor}_t(p_{i, t+1}^k, p_{j, t+1}^l) \) is constant only if there is no demand-state and previous level of storage, \( (a_t, q_{t-1}) \) such that (a) occurs. In this case, the constant correlation is zero.

Building on the example in Section 3.3.1, consider the situation where storable natural gas and oil may be used to produce electricity. First, in the case of no storage or conversion, equilibrium prices are, by assumption (D), uncorrelated. Second, even if gas and oil are assumed to be non-storable, the option to convert will induce correlation in the prices since the possibility of conversion of oil into electricity next period will simultaneously affect next period’s demand for oil and supply of electricity. In addition, since commodities can be linked through discretionary conversion as in Proposition 2(d), the conversion can induce correlation between oil and gas. Since the amount of the oil and electricity conversions will depend on the demand state, the conditional correlation will be constant (in an economy without storage) only if the demand shocks are intertemporally i.i.d.

Storage makes the conditional price correlations more complicated in two ways. First, the dependence of prices and conversion on the state variable, previous storage, \( q_{t-1} \) makes the conditional correlations non-constant even if demand shocks are intertemporally i.i.d. Note that since the inventory follows a renewal process, the conditional correlations will

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\(^{17}\) Whether or not the \( \text{var}(P_{i, t+1}^k | a_t, q_{t-1}) \) is increasing or decreasing in \( q_{t-1} \) depends on whether \( E[(P_{i, t+1}^k)^2 | a_t, q_{t-1}] \) is increasing faster than \( E[(P_{i, t+1}^k)^2 | a_t, q_{t-1}]^2 \).
inherit this feature. Second, the price of commodities at period \( t + 1 \) can be related even if there is no possibility of a linkage through immediate conversion at date \( t + 1 \). Suppose the demand state, \( a_t \) and inventory levels, \( q_{t-1} \) are such that no oil or gas is currently being used to produce electricity and furthermore, there is no state, \( a_{t+1} \), where oil or gas would be converted at \( t + 1 \). It can be the case that \( t + 1 \) prices are correlated through the possibility of conversion at some future date \( \tau > t + 1 \). This option of future conversion means that oil and gas inventory are substitutes since either may be used for electricity. This linkage in the inventory demand induces a non-zero conditional correlation as of date \( t \).

### 3.4 The Supply Stack

An important component of the electricity market is the marginal cost curve for electricity called the supply stack. As mentioned previously, electricity is a special case of our model where one terminal and non-storable good, electricity (denoted \( e \)), can be made with multiple fuels (denoted \( k \)) like oil, natural gas, and coal.

We can define the supply stack for electricity by considering the cost-minimizing way of producing \( b_{e,t} \) units of electricity. That is

\[
C(b_{e,t}) = \min \left\{ \sum_k \eta^{k}_{e,t} b_{e,t}^k \left| \sum_k g^{k}_{e} (t^{k}_{e,t}) \geq b_{e,t} \right\} \right.
\]

(14)

given the information about the demand state, \( a_t \) and storage, \( q_{t-1} \). The first-order conditions are a special case of (10)

\[
\frac{\eta^{k}_{e,t}}{g^{k}_{e}} \begin{cases} 
\geq \lambda_t & \text{if } t^{k}_{e,t} = 0 \\
= \lambda_t & \text{if } t^{k}_{e,t} \in (0, t^{k}_{max}) \\
\leq \lambda_t & \text{if } t^{k}_{e,t} = t^{k}_{max}
\end{cases}
\]

(15)

where \( \lambda_t \) is the Lagrange multiplier on the production constraint and (by the envelope theorem) is the marginal cost of producing another unit of electricity. It is obvious that \( \lambda_t \) must be increasing in \( b_{e,t} \). It is also possible due to the capacity constraints that \( \lambda_t \) is not continuous. For example, the capacity constraint for oil may be reached before the threshold for natural gas production is reached. The equilibrium price of electricity is where \( \lambda_+ \leq p_e^0 \leq \lambda_- \) where \( \lambda_+ \) (\( \lambda_- \)) is the marginal cost of electricity production from the left (right).

Figure 1 illustrates an electricity example with both natural gas and oil as potential
fuels. One marginal cost curve is plotted with three possible inverse net-demand curves for electricity, representing different realizations of the demand shock. Note that $b_e$ (horizontal axis) represents only the electricity produced from oil and gas. The fact that the net-demand curve for electricity can fall below the point where $b_e = 0$ reflects some “baseload” electricity production (e.g., hydro).

4 Numerical Example

In this section we implement a simple form of our model with one non-storable “downstream” commodity, electricity, and one storable “upstream” commodity, natural gas. Natural gas can be burned (converted) as a fuel to generate additional electricity (e.g., beyond the baseload), consumed in some alternate use (e.g., heating), or stored. This implementation includes endogenous decisions about gas storage and how much gas to burn for electricity. It abstracts from the option of choosing among multiple alternative fuels to generate electricity.

We model the Markov stochastic structure as the combination of two independent two-state Markov processes as in (D). In this way, the numerical example highlights the role of endogenous storage and conversion in the connectedness of natural gas and electricity prices. The specific functional forms are as follows.

Assumption (B) implies demand functions that are separable in price. For convenience, we characterize our example directly in terms of the implied demand function net of baseload endowment. In particular, we assume demand for good $k$ of $D^k(a^k, p^k) = a^k - p^k$ with $a^k \in \{a^k_H, a^k_L \}$ for both natural gas ($k = n$) and electricity ($k = e$). The conversion from natural gas to electricity is linear with a maximum fixed generation capacity. We assume $g(b^e_n) = \min(b^e_n, 0.5)$, where $b^e_n \geq 0$. Prices are normalized so that $p^e_n$ is the price of the gas needed to produce one “unit” of electricity. The numerical values are:

$$a^k_H = 2 \quad a^k_L = 1$$

$$\begin{pmatrix}
\pi(a^k_H | a^k_H) & \pi(a^k_H | a^k_L) \\
\pi(a^k_L | a^k_H) & \pi(a^k_L | a^k_L)
\end{pmatrix} = \begin{pmatrix}
0.7 & 0.3 \\
0.3 & 0.7
\end{pmatrix}$$

for $k = n, e$. This produces a symmetric Markov process with 4 states, $(a^n_H, a^n_H), (a^n_L, a^n_H), (a^n_H, a^n_L)$ and $(a^n_L, a^n_L)$, which tend to persist in the high (low) state. The storage cost of natural gas is $\delta = 0.03$ and the risk-free rate is set to zero.

The numerical algorithm used to solve for equilibrium burn (i.e., conversion) and storage
is similar to Routledge, Seppi, and Spatt (2000). The extra complexity of solving for the equilibrium conversion is simplified by constructing the $\rho(a, \Delta q)$ as in Propositions 2.

### 4.1 Numerical Results

*Equilibrium Storage and Conversion:* Figure 2a plots the equilibrium storage, $Q$, as a function of previous inventory, $q_{t-1}$ for different demand states, ($a^n_t$ and $a^e_t$). Natural gas is added to inventory at every level of $q_{t-1}$ in the low gas and low electricity state (i.e., $a^n_t = a^n_L$ and $a^e_t = a^e_L$). In the low gas and high electric state (i.e., $a^n_t$ and $a^e_H$) gas inventory is increased when $q_{t-1}$ is small, but reduced when $q_{t-1}$ is large. In high gas demand states (i.e., $a^e_H$) inventory is reduced with the largest reduction coming when the electricity market is also in the high state (i.e., $a^e_H$). Figure 2b plots the equilibrium conversion against the previous inventory and demand states. For this parameterization gas is never burned to generate electricity when the electricity market is in the low state (i.e., $a^e_L$). The most gas is burned when gas is in the low state and electricity is in the high state (i.e., $a^n_L$ and $a^e_H$) with the gas burn increasing to its maximum when $q_{t-1} \geq 0.71$. Naturally, less gas is burned when both electricity and gas are in their high states (i.e., $a^n_H$ and $a^e_H$) since natural gas is then more valuable.

Figures 3a and 3b plot the unconditional distributions of natural gas inventory, $q_t$, and the amount of gas burned. For these parameters values, the gas market is stocked out (i.e., $q_t = 0$) 41% of the time. With a 38% probability, gas is used to make electricity and the capacity constraint is binding 9% of the time.

We present the equilibrium of this example in terms of the underlying supply and demand curves as illustrated in Figure 4. There is a baseload source of electricity (e.g., hydro) at a price of 1 per MWh for up to 0.4 MWhs. Without gas, the supply schedule is vertical thereafter. The downward sloping end-user demand curve varies according to a demand shock $a_L$ or $a_H$. The demand curve crosses the “baseload” supply schedule at 1 in the low demand case, $a^e_L$, and at 2 in the high demand case, $a^e_H$. In addition to baseload electricity, there is also an option of generating electricity from natural gas. Of course, the cost of electricity generated from gas depends on the price of natural gas. Three (possible) illustrative supply curves are presented. In the low electricity demand state, gas is not an economical source of electricity. However, in the high electricity demand states gas is burned. Note that it is possible that gas is so economical (when previous inventory, $q_{t-1}$ is high) that the generation capacity is reached and the supply stack is again vertical.
**Skewness of spot prices and spreads:** Our economy has both a storage and a conversion option for natural gas. To illustrate the impact of these technological options, Figures 5a, 5b and 5c contrast the unconditional distributions of natural gas and electricity spot prices and the gas/electricity spark spread, $P_t^g - P_t^e$, in three related economies. When gas is neither storable nor convertible (hollow bar in Figure 5) the resulting price and spark spread distributions are symmetric. However, if natural gas is convertible, but still not storable (the gray bar) the price distributions are now skewed. For three combinations of shocks, $(a_H^g, a_H^e)$, $(a_H^e, a_L^e)$, $(a_L^g, a_L^e)$ it is not economical (in this example) to use natural gas to generate electricity. Thus, the spot prices are identical to those when gas is non-convertible. However, gas is used to generate additional electricity in the fourth combination, $(a_L^g, a_H^e)$ which raises the gas price $P_t^g > a_L^g$, and lowers the electricity price, $P_t^e < a_H^e$. The opposing skews in the gas and electricity prices are a consequence of the “one way” convertibility of gas into electricity. When gas is cheap, it is used to generate electricity, but electricity cannot be converted into gas.

Lastly, when gas is both storable and convertible (the black bar in Figure 5), the resulting electricity spot price distribution is still skewed to the right due to the “one way” convertibility of gas into electricity. However, the gas price distribution is more symmetric (and less volatile) than in the previous two economies. This is because when the net demand for gas is low, it is also stored (thereby raising $P_t^g$) and then later burned (or used by consumers) when the net demand for gas is high (then lowering natural gas prices). As a result, the corresponding spark spread distribution is now asymmetric (Figure 5c). The spark spread represents the realized value of the option to convert gas into electricity.\(^\text{18}\)

From this point on, we confine our discussion to the economy with both storable and convertible natural gas. The discussion of Figure 5 focused on how skewness arises in our model. In the following we discuss term-structure dynamics, mean reversion, conditional volatility and hedge ratios to demonstrate how our model captures the important empirical features of energy prices identified in Kaminski (1997).

**Term structure dynamics:** Figures 6a-6d illustrate the dynamics of the evolution of the gas and electricity forward price term structures. For each of the four possible combinations of exogenous demand states, we plot the evolution of the gas and electricity term structures from date $t$ to date $t+1$ given either low ($q_{t-1} = 0$) or high ($q_{t-1} = 1.8$) outgoing inventory

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\(^{18}\)An interesting result of the dramatic volatility in electricity prices in the summer of 1998 (F.E.R.C. (1998)) is that there was a large increase in the market price of natural gas fueled turbine generators. The turbines are the simplest way to convert natural gas to electricity. Although outside the scope of our model, the price increase of the capital equipment reflects the increase in the value of the conversion option.
at $t - 1$. These figures illustrate the range of forward curve relations that our model can produce, even in this very simple implementation. For example, in Figure 6a ($a^g_t = a^g_L$, $a^e_t = a^e_L$ and $q_{t-1} = 0$) both the gas and electricity forward curves are in contango at date $t$ (i.e., rising with time to delivery), but can change to lumped-shaped gas and backwarded electricity at $t + 1$ ($a^g_{t+1} = a^g_H$ and $a^e_{t+1} = a^e_H$). In Figure 6b ($a^g_t = a^g_H$ and $a^e_t = a^e_H$), the date $t$ gas curve is in contango, but the electricity curve is backwarded with $P^e_t = P^n_t$ (when $q_{t-1}$ is low) or $P^e_t > P^n_t$ (when $q_{t-1}$ is high and the date $t$ burn, $b^e_t$, is capped at its maximum of 0.5).

In Figure 6c ($a^g_t = a^g_L$ and $a^e_t = a^e_H$) the gas curve is either backwarded (if $q_{t-1}$ is low), lump-shaped (if $q_{t-1}$ is moderate; not shown) or in contango (if $q_{t-1}$ is high). The electricity spot price is unaffected by $q_{t-1}$ and electricity forward prices are in contango. Lastly, in Figure 6d ($a^g_t = a^g_H$ and $a^e_t = a^e_H$), the electricity forward curve is backwarded while the gas forward curve can again be backwarded, lump-shaped (not shown) or in contango.

**Conditional volatility:** Heteroscedasticity is another of the important empirical facts about energy prices in Kaminski (1997). Figure 7a plots the standard deviation of the change in the spot price of natural gas, $SD(P^n_{t+1} - P^n_t)$ against the previous storage, $q^n_{t-1}$, for each of the four possible states. Figure 7b is a similar plot for electricity. Conditional on the shocks, $a^n_t$ and $a^e_t$, Proposition 4 implies that the current spot prices must be inversely related to previous inventory $q_{t-1}$ when there is a single storable good as in this example. Both figures show that for each $(a^n_t, a^e_t)$ pair, volatility is decreasing in inventory, $q_{t-1}$, and thus are (weakly) increasing in the spot prices of both natural gas and electricity (for a given demand state). In each case, conditional on the shock state, $a^n_t$ and $a^e_t$, volatility is decreasing in $q_{t-1}$. However, as noted in Section 3.3.2, this is not a general result.

**Minimum variance hedge ratios:** The covariance between natural gas and electricity prices is important for two reasons. First, futures contracts on marginal fuels, like natural gas, are one possible hedge for forward positions in electricity. Second, Value-at-Risk calculations for (firm-wide) energy derivative portfolios require covariances (as well as variances) of the underlying prices. While the second is clearly important, we focus here on the implication of our model for cross-commodity minimum variance hedge ratios (MVHRs), $\text{cov}_t(F^n_{t_{t+h}}, F^{e_{t_{t+h}}})/\text{var}_t(F^n_{t_{t+h}})$. In particular, Figure 8 plots the minimum-variance hedge ratios when using a one-period natural gas futures contract to hedge a one-period electricity futures position conditional on different inventories and states. While somewhat irregular, the MVHRs for each state $(a^n_t, a^e_t)$ is roughly increasing in $q_{t-1}$ which is (roughly) the likelihood that gas will be burned to make electricity. Similar pictures could be drawn for MVHRs

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for longer horizons.

5 Conclusions

In the paper, we have used a simple competitive rational expectations model to study the effect of physical storage and conversion options on the dynamics of spot and derivative prices of “upstream” and “downstream” commodities. Our baseline example is an economy with non-storable electricity and storable potential fuels such as natural gas and oil. The result is an equilibrium model of energy prices and the “spark spread.” Specific results from our theoretical and/or numerical example are:

- A storage-conversion equilibrium exists.
- The impact of commodity convertibility is highlighted.
- Spot and forward prices of all related commodities are typically weakly decreasing in the storage levels.
- Price distributions are skewed.
- Price distributions are heteroscedastic with higher (lower) volatility when prices are high (low).
- Correlations between electricity and potential fuel prices (e.g., oil and natural gas) are not constant, but depend on the exogenous demand state and endogenous inventory level.

This framework also can also be applied to other cross-commodity situations. For example, geographical price differences (e.g., basis in natural gas at Henry Hub versus Chicago) and other cross-commodity spreads (e.g., the “crack” spread for petroleum, the "crush" spread for soybeans, etc.) are special cases of the general framework we present. In each case physical constraints, such as non-negative storage, non-storability of some related commodities, irreversible convertibility, lead to “option-like” features in the resulting spot and futures prices.
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Appendix

Proof of Proposition 1 Equilibrium
The existence of a unique solution, $Q(A_t)$ and $B(A_t)$, to (5) follows from the assumptions on preferences and technology. Similarly, for the consumer problem, (1), a sufficient condition for a solution to exist is that prices are positive and finite. In the conjectured equilibrium in equations (6) and (7) it is the case that $0 < P^k(A_t) < \bar{p}$ and $0 < S(A_t) < \bar{s}$ for all $k \in \mathcal{K}$ and $A_t$.

For the consumer, the solution to (1) is characterized as follows. The multiplier on the budget constraint is denoted $\lambda$ and let $u'_k(A_t) = \frac{\partial u(c(A_t), x)}{\partial x}$. For all commodities $k$ and paths $A_t$

$$
\beta^t \pi(A_t) u'_k(A_t) - \lambda s(A_t) p^k(A_t) = 0
$$

(A1)

Since the budget constraint is expressed in units of the consumption good as of date zero, $\lambda = u'_0(0) > 0$. Let $g'_{jk}(A_k) = \frac{\partial g_j(b(A_t))}{\partial x}$. For all pairs of commodities, $j, k \in \mathcal{K}$, and paths $A_t$

$$
p^j(A_t) g'_{jk}(A_t) - p^k(A_t) \begin{cases} 
0 & \text{if } b^j_k = 0 \\
= 0 & \text{if } b^j_k \in (0, \bar{b}^j_k) \\
\geq 0 & \text{if } b^j_k = \bar{b}^j_k
\end{cases}
$$

(A2)

Finally, for all $k \in \mathcal{K}$ and $A_t$

$$
-s(A_t) p^k(A_t) + (1 - \delta^k) \sum_{a_{t+1} \in \mathcal{A}} s(A_t; a_{t+1}) p^k(A_{t+1} a_{t+1}) \begin{cases} 
0 & \text{if } q^k_t > 0 \\
\leq 0 & \text{if } q^k_t = 0
\end{cases}
$$

(A3)

Since inventory is intertemporal, the right-hand part of (A3) includes paths, $(A_t(\omega); a_{t+1}(\omega))$ that continue on from $A_t(\omega)$ with shock $a \in \mathcal{A}$. Similarly, note that $A_t(\omega)$ and $A_{t-1}(\omega)$ have, by definition, identical shocks up to date $t - 1$.

For the social planner, the optimal policy to for $q(A_t)$ and $b(A_t)$ in (5) are characterized by the first-order conditions. The optimal consumption policy, $c(A_t)$, follows from (4).

$$
\begin{cases} 
\leq 0 & \text{if } b^j_k = 0 \\
= 0 & \text{if } b^j_k \in (0, \bar{b}^j_k) \\
\geq 0 & \text{if } b^j_k = \bar{b}^j_k
\end{cases}
$$

(A4)

$$
-\beta^t \pi(A_t) u'_k(A_t) + \beta^{t+1} (1 - \delta^k) \sum_{a_{t+1} \in \mathcal{A}} \pi(A_t; a_{t+1}) u'_k(A_{t+1}; a_{t+1}) \begin{cases} 
0 & \text{if } q^k_t > 0 \\
\leq 0 & \text{if } q^k_t = 0
\end{cases}
$$

(A5)
Substitute equations (6) and (7) into (A2) to obtain (A4) and into (A3) to obtain (A5). Finally note that substituting (6) and (7) into (A1) produces an equality.

**Proof of Proposition 2 Market-Clearing Prices**

(a) The existence of a price vector \( p^* \) that clears markets, \( X(p^*; a, \Delta q) = 0 \) is a standard fixed-point argument and is omitted here for brevity. To establish that \( p^* \) is unique, consider the own and cross derivative of the excess demands. In the following, note that \( B \) is continuous but is not everywhere differentiable due to the non-negativity and maximum constraints. However, since \( B^k_j \) is weakly decreasing for all \( k \) and \( j \), the left-hand and right-hand derivatives at these points are both non-positive which is all that is needed.

\[
\frac{dX^k(p; a, \Delta q)}{dp^k} = D^k + \sum_{j \neq k} B^k_j \cdot (1/p^j) - \sum_{j \neq k} g^j_k B^j_k \cdot \left(-p^j/(p^k)^2\right) < 0 \quad \text{(A6)}
\]

Since \( g^j_k \geq 0 \) and \( D^k < 0 \), the excess demand is strictly decreasing in own price. Similarly, the cross-derivative is

\[
\frac{dX^k(p; a, \Delta q)}{dp^j} = B^k_j \left(-p^k/(p^j)^2\right) - g^j_k B^j_k \left(1/p^j\right) \geq 0 \quad \text{(A7)}
\]

These conditions, known as gross substitutability, imply that the equilibrium is unique (see Mas-Colell, Whinston, and Green (1995) Theorem 17.F.3). Continuity follows from the Implicit Function Theorem. It is straightforward to verify that the price function is differentiable almost everywhere and continuous everywhere.

(b) The comparative static result is from Mas-Colell, Whinston, and Green (1995) Theorem 17.G.3. Totally differentiating excess demands in (??), \( \nabla_{\Delta q} X(\rho(a, \Delta q), a, \Delta q) = 0 \) implies \( \nabla_{\Delta q} p = -(\nabla_p X)^{-1} \). Since excess demands satisfy gross substitutability, \( \nabla_p X \) is negative on the diagonal and non-negative off diagonal. (As above, this statement refers to both the left and right hand-side derivatives. Hence we need not focus on the non-differentiability.) Choose an \( s > 0 \) such that \( H = 1/s \nabla_p X + I \) has all non-negative entries. Rearranging, \( \nabla_p X = -s(I - H) \). It is sufficient to show that \( (I - H)^{-1} \) exists and is non-negative. To do this, define the sequence, \( H_N = \sum_{n=0}^{N} H^n \). Since \( H \) is non-negative, the sequence is non-negative and increasing. It is also bounded (see below) and hence has a limit, \( H_\infty \), which implies \( H_N \to 0 \). Since, \( H_N(I - H) = I - H^{N+1} \), taking limits as \( N \to \infty \) yields \( (I - H)^{-1} = H_\infty \), which is non-negative by construction. Therefore, from \( \nabla_{\Delta q} p = 1/s H_\infty \) is non-negative. Finally, the diagonal elements of \( \nabla_{\Delta q} p \) are strictly negative since \( dD^k/dp^k < 0 \). (Note that the first term in \( H_\infty \) is \( I \).)
(Note: To establish that $H_N$ is bounded we rely on the fact that the complete system of excess demands including a numeraire good must be homogeneous of degree zero. Therefore, when we fix the price of the numeraire good to one, we get that $\nabla_p X p \leq 0$. This implies that there exists a positive vector, $c$, such that $(I - H) p = c$. Pre-multiplying by $H_N$ gives, $(I - H^{N+1}) p = N c$. Since $H^{N+1} \geq 0$, $H_N c \leq p$. This provides a bound on every element of $H_N$.)

(c) Consider an increase in the net storage of good $k$, $\Delta q^k$. This has no direct impact on the excess demand of good $j \neq k$. However, from part (b), $d p^j \Delta q^k \geq 0$. Since $D^j$ is strictly decreasing in $p^j$, the net use of good $j$, $B^j$, cannot fall. Given the structure on the conversion technology from assumption (A) as seen in equation (??), an increase in $B^j_k$ means a decrease in $-q^j_k(B^j_k)$ (and vice versa). Since the net conversion of all other goods cannot fall, the net conversion of good $k$, $B^k$ cannot rise.

(d) Note in (c) that for $j \neq k$, $d B^j_k \Delta q^k < 0$ if and only if $d p^j / d q^k > 0$. From the construction in (b), $\nabla_{\Delta q^k} = - (\nabla_p X)^{-1} = 1 / s H_\infty$. Therefore, for if $d p^j / d q^k = 0$, it must be the case that element $j k$th of $H_n = 0$ for all $n$. This is equivalent to there being no path from $j$ to $k$ that does not include a zero element of $H$ (exactly as with state-invariant limiting distributions with a finite-state Markov process). By the construction of $H$, this condition pertains to zeros on the off-diagonal elements of $\nabla_p X$. Finally, since $d D^j / d q^k = 0$, the off-diagonal elements are determined entirely by $B^j_k$.

Proof of Proposition 3 Inventory is Temporary
[To be added]

Proof of Proposition 4 Inventory and Prices
[To be added]

Proof of Proposition 5 Propagation
[To be added]

Proof of Proposition 6 Price Correlations
[To be added]
Figure 1 – The “Supply Stack” for Electricity

One marginal cost or supply stack (increasing) and three net-demand functions (decreasing) are plotted. Note the supply stack can be discontinuous.
Figure 2a - Equilibrium Inventory Policy
Each line is for a different demand state

Figure 2b - Equilibrium Burn (Refining) Policy
Each line is for a different demand state. In this example, equilibrium refinement is strictly positive only in the (Low, High) and (High,High) demand states
Figure 2c - Equilibrium Spot Price for Gas
Each line is for a different demand state

Figure 2d - Equilibrium Spot Price for Electricity
Each line is for a different demand state
Figure 3a – Unconditional Distribution of Inventory

Figure 3b – Unconditional Distribution for Burning (Refining) Natural Gas into Electricity
Figure 4 – Illustration of (three possible) Supply and (two possible) Demand and (four possible) Equilibrium Spot Prices for Electricity
Figure 5 a, b and c – Unconditional Distributions for (a) Natural Gas Price, (b) Electricity Price, and (c) Electricity-Gas Price Spread Under Three Scenarios: (1) Storage and Burn of Gas to Electricity, (2) Only Burn, (3) No Storage and no burn
Insert Figure 6 (a), (b), (c), and (d)
(Pages 35-38)
Figure 7 a, b – Standard Deviation of Spot Price of (a) Natural gas and (b) Electricity Conditional on Demand State and Previous storage

\( \text{Previous storage } \left( \var(p^k_t - p^k_{t-1} | a_t, q_{t-1}) \right)^{1/2} \)
Figure 8 – Minimum Variance Hedge Ratio for Hedging Spot Electricity with Spot Gas

\[
\text{Minimum Variance Hedge Ratios} \\
\text{Hedge Spot Electricity with Spot Gas}
\]

Demand States
(No, Electricity)
(Low, Low)
(Low, High)
(High, Low)
(High, High)

Previous Inventory

\[
\frac{\text{cov}(\hat{P}_{i+1}^{n} - P_{i+1}^{n}, P_{i+1}^{e} - P_{i}^{e} \mid a_{i}, q_{i-1})}{\text{var}(\hat{P}_{i+1}^{n} - P_{i}^{n} \mid a_{i}, q_{i-1})}
\]