11-20-2007

Asset Prices in Business Cycle Analysis

David K. Backus
New York University

Bryan R. Routledge
Carnegie Mellon University, routledge@cmu.edu

Stanley E. Zin
Carnegie Mellon University

Asset prices in business cycle analysis*

David K. Backus,[†] Bryan R. Routledge,[‡] and Stanley E. Zin[§]

November 20, 2007
Preliminary and incomplete

Abstract

Asset prices are well known to lead the business cycle, yet most modern models generate movements in prices and quantities that are roughly contemporaneous. In US data, for example, equity returns and the short-term interest rate lead GDP growth by one or two quarters, while growth rates of consumption, investment, and employment growth move more or less together with GDP. We show how all of these features can be reproduced in variant of the Kydland-Prescott model with recursive preferences and a predictable component in productivity growth. A loglinear approximation is featured throughout; as a result, interest rates have a linear structure similar to popular affine models.

JEL Classification Codes: D81, D91, E1, G12.

Keywords: economic indicators, recursive preferences, interest rates, equity prices, stochastic volatility.

---

*Preliminary and incomplete: no guarantees of accuracy or sense. Comments welcome. We thank Gian Luca Clementi, Chris Edmond, Paul Gomme, and Jim Nason for helpful advice. The latest version of the paper is available at: http://pages.stern.nyu.edu/∼dbackus/GE asset pricing/ms/BRZ rbc latest.pdf.

† Stern School of Business, New York University, and NBER; david.backus@nyu.edu.

‡ Tepper School of Business, Carnegie Mellon University; routledge@cmu.edu.

§ Tepper School of Business, Carnegie Mellon University, and NBER; zin@cmu.edu.

1 Introduction

Dynamic general equilibrium models of business cycles have implications for asset prices, but the reverse is also true: asset prices tell us something about the dynamic structure of the economy. But what?

The most striking feature of asset prices in this context is that they lead the business cycle. While we sometimes summarize business cycles by saying "everything moves up and down together," a closer look tells us this isn't quite true. Aggregate equity prices, for example, are procyclical — they move up and down (on average) with GDP — but they also lead the cycle. The contemporaneous correlation of broad equity indexes with GDP (quarterly growth rates in both cases) is about 0.2, but the correlation with GDP one or two quarters later is 0.4. Interest rates follow a similar pattern. Term spreads — the difference between yields on 10-year and 3-month treasuries, for example — are more highly correlated with future than current GDP growth. Even quantities exhibit modest departures from strict synchronicity. All of these features of aggregate data have been documented by others.

But what do these leads and lags tell us about the nature of business cycles? We think they suggest an information mechanism: that agents (think "investors") have information about the near-term future of the economy that is reflected in asset prices, but not yet in GDP. We illustrate this mechanism in a variant of the Kydland-Prescott (1982) model. In standard versions of the model the information set consists of the current values of the capital stock and productivity. With this information set, output, consumption, investment, employment, and interest rates invariably move up and down together. Indeed, the Barro-King (1984) challenge to business cycle theory is just that: to induce these variables to move together, when shocks to anything but current productivity tend to drive at least some of them in opposite directions.

We study these patterns of leads and lags in a business cycle model with two novel features: recursive preferences and a predictable component in productivity growth. As in Tallarini (2000), recursive preferences have essentially no impact on the behavior of quantities, but they allow us to generate more realistic asset prices. The key ingredient is the predictable component in productivity growth. As we have learned from Bansal and Yaron (2004), even a small predictable component in quantities can have a significant impact on asset prices. In our setting, a predictable component in productivity growth also affects quantities. One might guess, for example, that an increase in expected future productivity growth would raise current consumption and reduce employment and investment. The question is whether this helps us reproduce the correlations we see in the data or destroys the basic business cycle features of the model for reasons outlined by Barro and King. We'll leave you in suspense for now.

2 Leads and lags in US data

US time series data contains a number of well-documented leads and lags. We emphasize asset prices: specifically equity prices and interest rates. The data are quarterly, 1960:1 to

2006:1. In most cases we look at quarterly growth rates, defined as log-differences ($\log x_t - \log x_{t-1}$). The exceptions are interest rates and spreads, which we use as is, and occasional centered year-on-year growth rates ($\log x_{t+2} - \log x_{t-2}$), which we use to smooth the high-frequency variation in many macroeconomic series. Year-on-year growth rates serve, in a sense, as a poor man's Hodrick-Prescott filter.

We describe the lead/lag pattern between two variables x and y with their cross-correlation function,

$$r_{xy}(k) \quad = \quad corr(x_t, y_{t-k}),$$

plotted as a function of k. If x is an indicator and y is (real) GDP growth, negative values of k correspond to correlations of the indicator with future GDP growth. If the correlations are large, we say the indicator leads GDP. Similarly, positive values of k correspond to correlations of x with past GDP growth; large values suggest a lagging indicator.

Our first example is equity prices. We report cross-correlation functions for growth rates of equity prices and GDP in Figure 1. The upper left panel uses the S&P 500 index; we see that the peak in the ccf occurs at $k = -1$ with a correlation of 0.33, implying that equity prices lead GDP by about a quarter. If we use the broader NYSE composite index, the lead increases to 2 quarters with a maximum correlation of 0.33. The S&P 500 growth rate minus the short rate (a crude excess return measure) and the growth rate of the Nasdaq index are similar: they lead GDP by one or two quarters, with a maximum correlation between 0.3 and 0.4. Returns on various equity portfolios (not reported) are similar: they all lead GDP by one or two quarters. Correlations of year-on-year growth rates show a similar lead, but the maximum correlations are somewhat larger (close to 0.5).

Interest rates also lead GDP, as we see in Figure 2. The most common form of this relation involves the slope of yield curve. The cross-correlation function for "10y–3m" term spread (the difference between the yield on 10-year treasuries and the 3-month treasury bill rate) is pictured in the upper left panel. The maximum correlation (0.31) with GDP growth occurs at $k = -2$, so the term spread leads GDP by about two quarters. The correlation is positive, so a steep yield curve (large spread) is associated with rapid future GDP growth, and a downward-sloping yield curve with slow growth. Using the year-on-year GDP growth rate again increases the magnitude of the correlations but not the shape of the cross-correlation function. The lower left panel suggests that most of the correlation comes from the short rate. The final panel shows that using a real rate (we subtract an inflation expectations measure from the 3-month rate) leads to the same pattern. We use this relation in our theoretical work, since it incorporates the lead of interest rates over GDP growth in a particularly simple form.

Quantities exhibit both higher correlations and more modest leads and lags. Consumption (Figure 3) is more highly correlated with GDP than equity prices or interest rates (the contemporaneous correlation is 0.63), but the correlations fall quickly with both leads and lags. Both are different from familiar correlations constructed from Hodrick-Prescott filtered data, where the contemporaneous correlations are larger and the decay rate smaller. See, for example, Christiano and Eichenbaum (1992) and Kydland and Prescott (1982). Evidently growth rates include more high-frequency noise than HP-filtered data. There's also a slight tendency for consumption to lead GDP. In three of the four panels, the maximum correlation is contemporaneous ($k = 0$), but consumption growth leads GDP growth in the sense that the correlation of consumption with future GDP is larger than that of consumption with past GDP. The exception is consumption of services, which accounts for more than half of total personal consumption. Its growth rate leads GDP's by about a quarter. Investment (Figure 4) is roughly contemporaneous, although equipment and software leads GDP a little and non-residential structures lags. Employment (Figure 5) exhibits a slight tendency to lag GDP. Although the maximum correlation is again contemporaneous, employment growth is more strongly correlated with past GDP growth than future GDP growth.

Similar features of US data have been reported in dozens, perhaps hundreds, of other papers. Prominent recent examples include Ang, Piazzesi, and Wei (2006), Beaudry and Portier (2006), King and Watson (1996), Rouwenhorst (1995), and Stock and Watson (1989, 2003). We think they are interesting from a theoretical perspective, since they suggest more complex dynamics than most existing models possess.

3 A theoretical economy

Our theoretical economic environment is a variant of the Kydland-Prescott (1982) model, a growth model with one good, physical capital, endogenous labor input, and shocks to productivity. We make several changes to an otherwise streamlined version of their model: preferences are recursive, production is CES, changes to the capital stock are subject to costs of adjustment rather than time-to-build, and the productivity process has a unit root. We describe each below, characterize the economy's equilibrium with a Bellman equation, and show how a growing economy can be expressed as a stationary one in scaled variables.

We use two notational conventions throughout. (i) We use letters with time subscripts to denote values of variables at specific dates and the same letters without subscripts to denote steady state values. Thus k_t/y_t is the capital-output ratio at date t and k/y is its steady state value. (ii) We denote derivatives by (non-time) subscripts, so that f_k is the derivative of f with respect to k. Thus f_{kt} is the value of the derivative evaluated at date-t values of the variables on which it depends and f_k (with no time subscript) is the derivative evaluated at steady state values.

Our economy has a single agent who represents a continuum of like agents. The agent's preferences are given by the recursive utility function

$$U_t \;\; = \;\; V[u_t, \mu_t(U_{t+1})],$$

where U_t is "utility from date t on;" u_t is date-t ("current") utility, a function of consumption c_t and leisure $(1 - n_t)$; and $\mu_t(U_{t+1})$ is the certainty equivalent of future utility (the risk-adjusted utility of U_{t+1}). For current utility, we follow Kydland and Prescott and use

$$u_t \;\; = \;\; c_t(1 - n_t)^{\lambda}.$$

We'll see shortly that it's essential that u_t is proportional to c_t (or the equivalent). To accomodate growth, we assume that the time aggregator V and certainty equivalent function μ are each homogeneous of degree one (hd1). More specifically, we use the constant elasticity functions

$$V(u_t, \mu_t) = [(1-\beta)u_t^\rho + \beta\mu_t^\rho]^{1/\rho} \tag{1}$$
$$\mu_t(U_{t+1}) = \left[E_t U_{t+1}^\alpha\right]^{1/\alpha}, \tag{2}$$

with $0 < \beta < 1$ and $\rho, \alpha < 1$. When $\alpha = \rho$, equations (1) and (2) are equivalent to additive power utility. We refer to $\sigma = 1/(1-\rho)$ as the intertemporal elasticity of substitution (IES) and $1 - \alpha$ as the coefficient of relative risk aversion (or simply risk aversion). This class of preferences was proposed by Kreps and Porteus (1978); the constant elasticity versions were suggested and applied to asset pricing by Epstein and Zin (1988) and Weil (1989).

The single good is produced with capital and labor, as described by the production function

$$y_t = f(k_t, z_t n_t) = [\omega k_t^\nu + (1-\omega)(z_t n_t)^\nu]^{1/\nu},$$

where y_t is output (GDP) at date t, k_t is the stock of physical capital, and z_t is labor productivity (the shock that generates fluctuations in the model). The function f is homogeneous of degree one in k and n. The constant elasticity version on the right defines the elasticity of substitution between capital and labor as $1/(1-\nu)$. The resource constraint ties output to consumption and investment i_t:

$$y_t = f(k_t, z_t n_t) = c_t + i_t. \tag{3}$$

The law of motion for capital is

$$k_{t+1} = g(i_t, k_t) = (1-\delta)k_t + k_t[(i_t/k_t)^\eta (i/k)^{1-\eta} - (1-\eta)(i/k)]/\eta, \tag{4}$$

where g is also hd1. Strict convexity of g is equivalent to adjustment costs for capital, which are necessary if the model is to generate realistic asset prices, particularly equity prices; see Cochrane (1991). The constant elasticity version includes the parameters $\eta \leq 1$, $0 < \delta < 1$ ("depreciation"), and i/k (the steady state ratio of investment to capital). Similar functions have been used by Fernandez de Cordoba and Kehoe (2000) and Jermann (1999). If $\eta = 1$, we have the familiar $g(i_t, k_t) = (1-\delta)k_t + i_t$. In this case, g is linear in both arguments and its second derivatives are zero. But if $\eta < 1$, g is strictly concave, which implies costs of adjusting the capital stock. Taken together, the components of technology lead to the law of motion

$$k_{t+1} = g(y_t - c_t, k_t) = g[f(k_t, z_t n_t) - c_t, k_t], \tag{5}$$

which is inherits linear homogeneity from f and g.

The final component of the economy is the process generating productivity. We start with a vector x of exogenous state variables that have loglinear dynamics:

$$\log x_{t+1} = (I - A)\log x + A \log x_t + B w_{t+1}, \tag{6}$$

where $\{w_t\} \sim \mathrm{NID}(0, I)$, A is stable, and $\log x$ is vector of constants (the unconditional mean of $\log x_t$). Productivity growth is the first element of x:

$$\log z_{t+1} - \log z_t \;\; = \;\; \log x_{1t+1}$$

or

$$z_{t+1} \;\; = \;\; z_t \, x_{1t+1} \; = \; z_t \, e_1^\top x_{t+1}, \tag{7}$$

where $e_1^\top = (1, 0, 0, ...)$. The unit root in $\log z$ is essential to generate realistic asset returns; see Alvarez and Jermann (2005). If x is one-dimensional and $A = 0$, productivity growth is white noise. Multiple dimensions and nonzero As allow us to generate a wide range of behavior for actual and expected productivity growth.

We compute equilibrium quantities by solving a planning problem: choose consumption and labor in each state to maximize utility subject to the laws of motion for capital and productivity and initial conditions. We then compute prices, as needed, from marginal rates of substitution and transformation at equilibrium quantities. The planning problem is summarized by the Bellman equation

$$J(k_t, x_t, z_t) \;\; = \;\; \max_{c_t, n_t} \; V \left\{ c_t (1 - n_t)^\lambda, \mu_t [J(k_{t+1}, x_{t+1}, z_{t+1})] \right\} \tag{8}$$

subject to (5,6,7) and initial conditions.

The next step is to transform a growing economy into a stationary one, which extends similar results in Christiano and Eichenbaum (1992), King, Plosser, and Rebelo (1988), and Tallarini (2000) to a more general setting. In (8), growth in productivity means that the economy is not stationary: other variables inherit the unit root of the productivity process. That's a problem, in practice, because we use local approximations of decision rules. The solution is to transform the problem into one that is stationary in scaled variables. This works because V, μ, f, and g are all hd1. A direct consequence is that the value function J is hd1 in (k, z). To see this, suppose an hd1 J satisfies the Bellman equation. Since the Bellman equation has a unique solution, J must be hd1. To see if this works, suppose the J on the right is hd1 and we divide by a positive number a. Then the Bellman equation becomes

$$\begin{aligned} J(k_t, x_t, z_t) \;\; &= \;\; \max_{c_t/a, n_t} \; aV \left\{ (c_t/a)(1 - n_t)^\lambda, \mu_t [J(k_{t+1}/a, x_{t+1}, z_{t+1}/a)] \right\} \\ \text{subject to:} \qquad & k_{t+1}/a \; = \; g[f(k_t/a, n_t) - c_t/a, k_t/a] \\ & z_{t+1}/a \; = \; (z_t/a) x_{1t+1} \end{aligned}$$

plus (6) and initial conditions. Thus $J(k_t, x_t, z_t) = aJ(k_t/a, x_t, z_t/a)$ and J is hd1. In words: if we divide capital and productivity by two, we get half as much consumption and still satisfy the Bellman equation.

Now consider dividing by z_t. The challenge here is that the scaling differs across periods. If we define the scaled variables $\tilde{k}_t = k_t/z_t$ and $\tilde{c}_t = c_t/z_t$, we can rewrite the Bellman equation as

$$\begin{aligned} J(\tilde{k}_t, x_t, 1) \;\; &= \;\; \max_{\tilde{c}_t} \; V \left\{ \tilde{c}_t (1 - n_t)^\lambda, \mu_t [x_{1t+1} J(\tilde{k}_{t+1}, x_{t+1}, 1)] \right\} \qquad (9) \\ \text{subject to:} \qquad & \tilde{k}_{t+1} \; = \; g[f(\tilde{k}_t, n_t) - \tilde{c}_t), \tilde{k}_t]/x_{1t+1} \end{aligned}$$

plus (6) and initial conditions. Scaling this way introduces the productivity growth rate $x_{1t+1} = z_{t+1}/z_t$ into the law of motion for capital and the certainty equivalent of the Bellman equation, but in other respects the problem is the same as one without growth. We use (9) in what follows, with a slight change in notation: we drop the "1" from J.

4 Loglinear approximation

Models like this can be solved by a variety of methods. We use loglinear approximations to the decision rules, which are relatively easy to derive and interpret. Similar approximations are the norm in business cycle research, but recursive preferences raise enough new issues that it's worth describing the approximation method in some detail.

Our goal is a pair of approximate decision rules of the form

$$\hat{c}_t = h_{ck}\hat{k}_t + h_{cx}^{\top}\hat{x}_t \tag{10}$$
$$\hat{n}_t = h_{nk}\hat{k}_t + h_{nx}^{\top}\hat{x}_t, \tag{11}$$

where $(h_{ck}, h_{cx}, h_{nk}, h_{nx})$ are coefficients to be derived. By convention, the "hat" notation refers to deviations of logarithms of variables from constant values: $\hat{c}_t = \log \tilde{c}_t - \log \tilde{c}$, $\hat{n}_t = \log n_t - \log n$, $\hat{k}_t = \log \tilde{k}_t - \log \tilde{k}$, $\hat{x}_t = \log x_t - \log x$, and $(\tilde{c}, n, \tilde{k}, x)$ (without time subscripts) are constant values (the means of the logarithms). We compute an approximation "by hand" following the route of Campbell (1994), Hansen and Sargent (1980), and Lettau (2003). The key insight comes from Hansen and Sargent: in linear-quadratic control problems with a single controllable state variable, the only complicated dynamics are those of that state variable, and they do not depend on the behavior of the forcing variables. That means, in our case, that h_{ck} can be derived independently of the process for productivity growth, so that we can incorporate relatively complex dynamics into the productivity process without comparable increase in the complexity of our calculations. We deal with additional issues raised by recursive preferences using methods similar to Hansen, Heaton, and Li (2005) and Uhlig (2006). Our theoretical economy is then (approximately) a vector autoregression in $(\hat{k}, \hat{x})$ that we can study with linear time series methods.

The method

We're looking for a solution to the scaled planning problem (9) with the functional forms (1,2) for time and risk preference. To keep the notation manageable, let $J_t = J(\tilde{k}_t, x_t)$, $J_{kt} = \partial J(\tilde{k}_t, x_t)/\partial \tilde{k}_t$, and so on. The first-order conditions for $\tilde{c}_t$ and n_t imply

$$(1-\beta)\tilde{c}_t^{\rho-1}(1-n_t)^{\rho\lambda} = \beta\mu_t(x_{1t+1}J_{t+1})^{\rho-\alpha}E_t[(x_{1t+1}J_{t+1})^{\alpha-1}J_{kt+1}]g_{it} \tag{12}$$
$$\lambda(1-\beta)\tilde{c}_t^{\rho}(1-n_t)^{\rho\lambda-1} = \beta\mu_t(x_{1t+1}J_{t+1})^{\rho-\alpha}E_t[(x_{1t+1}J_{t+1})^{\alpha-1}J_{kt+1}]g_{it}f_{nt} \tag{13}$$

and the envelope condition for $\tilde{k}_t$ is

$$J_{kt} = J_t^{1-\rho}\beta\mu_t(x_{1t+1}J_{t+1})^{\rho-\alpha}E_t[(x_{1t+1}J_{t+1})^{\alpha-1}J_{kt+1}](g_{it}f_{kt} + g_{kt}). \tag{14}$$

Note that these equations involve not only derivatives of the value function, but the value function itself. In this respect, the problem with recursive preferences differs from one with traditional additive preferences.

Our approach combines guess-and-verify with a ruthless determination to compute loglinear approximations to equations that are not naturally loglinear. We guess a value function of the form

$$\log J(\tilde{k}_t, x_t) \;=\; \log p_0 + p_k \log \tilde{k}_t + p_x^\top \log x_t. \tag{15}$$

With this guess, we derive an approximate solution in much the same way we solve linear-quadratic problems:

1. Derive loglinear approximations of the laws of motion (5,6). This is where the ruthless determination comes in: the sums that show up in the technology and the time aggregator are not naturally loglinear, but with enough determination we can approximate them by loglinear functions nonetheless.

2. Derive loglinear approximations of the first-order conditions (12,13). This uses the value function guess (15) and the loglinear approximations of the laws of motion we derived earlier. From them, we compute the coefficients of the decision rules (10,11) as functions of the value function parameters.

3. Derive a loglinear approximation of the envelope condition (14) using the same inputs. If we substitute the decision rules, we get Riccati-like equations that determine the value function parameters (p_k, p_x).

4. Solve the Riccati equations for the value function parameters. This is where the Hansen-Sargent result comes in: p_k is the solution to a quadratic equation that does not depend on the properties of productivity growth. Given p_k, p_x is linear.

5. Use value function parameters to compute the coefficients of the decision rules.

If the procedure is straightforward, the calculations are not. See Appendix B for the gruesome details.

An example: the growth model

We illustrate the approximation method in what we term the growth model: a special case in which the analytical burden is significantly lighter. We fix labor supply (by setting $\lambda = 0$) and eliminate adjustment costs (by setting $\eta = 1$, which results in constant first derivatives $g_{it} = 1$ and $g_{kt} = 1 - \delta$). With these simplifications, the first-order and envelope conditions become

$$(1-\beta)\tilde{c}_t^{\rho-1} \;=\; \beta\mu_t(x_{1t+1}J_{t+1})^{\rho-\alpha}E_t[(x_{1t+1}J_{t+1})^{\alpha-1}J_{kt+1}] \tag{16}$$
$$J_{kt} \;=\; J_t^{1-\rho}\beta\mu_t(x_{1t+1}J_{t+1})^{\rho-\alpha}E_t[(x_{1t+1}J_{t+1})^{\alpha-1}J_{kt+1}](f_{kt}+g_{kt}). \tag{17}$$

Our objective is to use these two equations, the laws of motion, and the guess of the value function to derive an approximate loglinear decision rule.

Despite the simplifications, the mathematical structure is similar to the full-blown model. The first-order and envelope conditions, for example, contain both a certainty equivalent and a conditional expectation — in fact, the same ones. We need loglinear approximations of each. To see how this works, consider an arbitrary lognormal random variable x whose logarithm is normal with mean κ_1 and variance κ_2. You should be able to convince yourself that

$$\begin{aligned} E(x) &= \exp(\kappa_1 + \kappa_2/2) \\ E(x^\alpha) &= \exp(\alpha\kappa_1 + \alpha^2\kappa_2/2) \\ \mu(x) &= [E(x^\alpha)]^{1/\alpha} \;=\; \exp(\kappa_1 + \alpha\kappa_2/2). \end{aligned}$$

Note that the impact of risk (the κ_2 terms) is multiplicative. Since risk is constant (this changes when we introduce stochastic volatility), we can group these terms with the discount factor in the first-order and envelope conditions. Apparently any change in risk or risk aversion can be offset by changing the discount factor.

We use this insight ("ignore the variance") to derive a loglinear approximation of

$$M_t \;\equiv\; \beta\mu_t(x_{1t+1}J_{t+1})^{\rho-\alpha}E_t[(x_{1t+1}J_{t+1})^{\alpha-1}J_{kt+1}]$$

(think "M" for massive expression). Our value function guess (15) implies

$$\begin{aligned} \widehat{J}_{t+1} + \hat{x}_{1t+1} &= p_k\hat{k}_{t+1} + (e_1 + p_x)^\top \hat{x}_{t+1} \\ \widehat{J}_{kt+1} &= (p_k - 1)\hat{k}_{t+1} + p_x^\top \hat{x}_{t+1}. \end{aligned}$$

Using conditional means to evaluate the certainty equivalent and conditional expectation, we find

$$\begin{aligned} \widehat{M}_t &= (\rho p_k - 1)E_t\hat{k}_{t+1} + [(\rho-1)e_1 + p_x]^\top E_t\hat{x}_{t+1} \\ &= -P_kE_t\hat{k}_{t+1} + [(\rho-1)e_1 + \rho p_x]^\top E_t\hat{x}_{t+1}, \end{aligned}$$

where $P_k = 1 - \rho p_k$ is a convenient composite.

With this preliminary work out of the way, we follow the steps in order:

1. Loglinear approximation of the laws of motion. The forcing process is naturally loglinear:

   $$\hat{x}_{t+1} \;=\; A\hat{x}_t + Bw_{t+1}.$$

   The resource constraint (3) is approximated by

   $$\hat{y}_t \;=\; (f_kk/y)\hat{k}_t \;=\; (c/y)\hat{c}_t + (i/y)\hat{i}_t \;\Rightarrow\; \hat{i}_t \;=\; (f_kk/i)\hat{k}_t - (c/i)\hat{c}_t.$$

   Therefore, the law of motion for capital can be written

   $$\begin{aligned} \hat{k}_{t+1} &= (i/kx_1)\hat{i}_t + (g_k/x_1)\hat{k}_t - \hat{x}_{1t+1} \\ &= [(f_k + g_k)/x_1]\hat{k}_t - (c/kx_1)\hat{c}_t - e_1^\top(A\hat{x}_t + Bw_{t+1}). \end{aligned}$$

and

$$\widehat{M}_t \;=\; -P_k[(f_k+g_k)/x_1]\hat{k}_t + P_k(c/kx_1)\hat{c}_t + [(\rho-1+P_k)e_1+\rho p_x]^\top A\hat{x}_t.$$

2. Loglinear approximation of the first-order condition. The loglinearized version of the first-order condition (16) is

$$(\rho-1)\hat{c}_t \;=\; \widehat{M}_t,$$

which implies

$$\begin{aligned} [(\rho-1)-P_k(c/kx_1)]\hat{c}_t &= -P_k[(f_k+g_k)/x_1]\hat{k}_t + [(\rho-1+P_k)e_1+\rho p_x]^\top A\hat{x}_t \\ [1+\sigma P_k(c/kx_1)]\hat{c}_t &= \sigma P_k[(f_k+g_k)/x_1]\hat{k}_t + [(1-\sigma P_k)e_1+(1-\sigma)p_x]^\top A\hat{x}_t. \end{aligned}$$

The second line follows the first from dividing by $\rho-1=-1/\sigma$, where σ is the IES. Solving for $\hat{c}_t$, we see that the decision rule parameters are

$$h_{ck} \;=\; \frac{\sigma P_k(f_k+g_k)/x_1}{1+\sigma P_k(c/kx_1)}, \quad h_{cx}^\top \;=\; \frac{[(1-\sigma P_k)e_1+(1-\sigma)p_x]^\top A}{1+\sigma P_k(c/kx_1)} \tag{18}$$

These relations depend on the value function parameters, but both numerator and denominator are linear in them.

3. Loglinear approximation of the envelope condition. The loglinearized version of the envelope condition (17) is

$$\widehat{J}_{kt}+(\rho-1)\widehat{J}_t \;=\; \widehat{M}_t+\hat{d}_t,$$

where $d_t = f_{kt}+g_{kt}$. The loglinear version of this expression is

$$\hat{d}_t \;=\; (f_k+g_k)^{-1}f_{kk}k\hat{k}_t.$$

If we substitute the intermediate results and collect terms, we have

$$\begin{aligned} -P_k\hat{k}_t+\rho p_x^\top\hat{x}_t \;=\;& -P_k[(f_k+g_k)/x_1]\hat{k}_t + P_k(c/kx_1)\hat{c}_t \\ &+ [(\rho-1+P_k)e_1+\rho p_x]^\top A\hat{x}_t + (f_k+g_k)^{-1}f_{kk}k\hat{k}_t. \end{aligned}$$

If we divide by $\rho-1$ and substitute the decision rule (10) for $\hat{c}_t$, we get

$$\begin{aligned} \sigma P_k\hat{k}_t+(1-\sigma)p_x^\top\hat{x}_t \;=\;& \sigma P_k[(f_k+g_k)/x_1]\hat{k}_t - \sigma P_k(c/kx_1)(h_{ck}\hat{k}_t+h_{cx}^\top\hat{x}_t) \\ &+ [(1-\sigma P_k)e_1+(1-\sigma)p_x]^\top A\hat{x}_t - \sigma(f_k+g_k)^{-1}f_{kk}k\hat{k}_t. \end{aligned}$$

For this equation to hold for all $(\hat{k}_t,\hat{x}_t)$, the coefficients must satisfy

$$\begin{aligned} \hat{k}_t: &\quad \sigma P_k \;=\; \sigma P_k(f_k+g_k)/x_1 - \sigma P_k(c/kx_1)h_{ck} - \sigma(f_k+g_k)^{-1}f_{kk}k \\ \hat{x}_t: &\quad (1-\sigma)p_x^\top \;=\; -\sigma P_k(c/kx_1)h_{cx}^\top + [(1-\sigma P_k)e_1+(1-\sigma)p_x]^\top A. \end{aligned}$$

4. Solution of the Riccati equations. Given the decision rule coefficients (18), the previous set of equations determines the parameters of the approximate value function, just as in linear-quadratic problems. Note that the coefficients of $\hat{k}_t$, including those implicit in h_{ck}, depend on P_k, but not on p_x or any other feature of the process for productivity growth. This is the Hansen-Sargent result: we find P_k, hence p_k and h_{ck}, without reference to the shocks, including p_x. After substituting for h_{ck}, the coefficients of $\hat{k}_t$ imply

$$
\begin{aligned}
0 \quad = \quad & \sigma^2 P_k^2 (c/kx_1) + \ \sigma P_k \left[1 - (f_k + g_k)/x_1 + \sigma(c/kx_1)(f_k + g_k)^{-1} f_{kk} k\right] \\
& + \sigma (f_k + g_k)^{-1} f_{kk} k. \qquad (19)
\end{aligned}
$$

This is the Riccati equation, which we solve for P_k. Since the last term is negative, it has one solution of each sign. We take the positive solution, so that the value function is increasing in $\hat{k}$. Given a solution for P_k, the solution for p_x is linear. If we define $q = 1 + \sigma P_k(c/kx_1) > 1$, then equating the $\hat{x}_t$ terms gives us

$$
(1-\sigma) p_x^\top \quad = \quad (1 - \sigma P_k) e_1^\top (q^{-1} A)(I - q^{-1} A)^{-1}.
$$

5. Calculation of decision rule. Now that we have the value function parameters, we compute the decision rule coefficients from (18). The coefficients of the productivity shock are

$$
h_{cx}^\top \quad = \quad (1 - \sigma P_k) e_1^\top (q^{-1} A)(I - q^{-1} A)^{-1}. \qquad (20)
$$

Discussion

Now that we've worked our way through the algebra, we can step back and think about its content. Some remarks:

Remark 1 (risk aversion). Tallarini (2000) shows that risk aversion (α in our notation) has little effect on the behavior of quantities. In our version, the impact on the approximation is exactly nil: α appears nowhere in our calculations. Why the difference? His model has an IES of one and endogenous labor input, but neither of these differences matters. The key difference is that he fixes the discount factor β, then allows the steady state to change as he varies risk aversion. We instead fix the steady state, with the consequence that risk aversion has no impact on the behavior of quantities: any change is implicitly reversed by appropriate adjustment of the discount factor. Similar results are reported by Kocherlakota (1990) and Hansen, Sargent, and Tallarini (1999). This is an approximation result, to be sure, but related work by Campanale, Castro, and Clementi (2007) suggests that the difference between the approximation and the true solution is small, even with more complicated certainty equivalent functions. We'll see shortly that risk aversion does have an impact on asset prices and returns, so we can identify α and β that way.

If the irrelevance of risk aversion for the behavior of quantities seems like a disappointment (all this work for nothing?), we think of it as a strength: we can vary it as much as we like without damaging the business cycle properties. Habits, in contrast, affect quantities as well as asset prices, which raises additional challenges in accounting for the joint behavior of quantities and asset prices. See, for example, Boldrin, Christiano, and Fisher (2001) and Jermann (1998).

Remark 2 (other loglinear approximation methods). Many loglinear aproximation methods, including Tallarini's, start with value functions that are quadratic in the logarithm of the state, as in

$$J(\log k) \;=\; p_0 + p_k(\log k)^2.$$

The derivative is therefore linear in $\log k$. Value functions of this form are the industry standard in business cycle research. With recursive preferences, Tallarini uses a result from risk-sensitive control to evaluate the certainty equivalent of the value function when $\rho = 0$ (the IES is one). We use a value function like

$$J(k) \;=\; p_0 k^{p_k}.$$

Here the logarithm of the derivative is linear in $\log k$, which has similar analytical convenience. With additive preferences, the two methods are identical. With recursive preferences, the loglinear structure allows us to compute certainty equivalents and expectations as we did earlier.

Remark 3 (predictability of productivity growth). Equation (20) shows how predictability of productivity growth affects the consumption decision, but it's less than perfectly transparent. If A is a matrix of zeros, h_{cx} is a vector of zeros: productivity growth is not predictable and the current growth rate has no direct effect on consumption (although it does affect consumption through the scaled capital stock). In general, the impact works like this:

$$\begin{aligned}
h_{cx}^\top \hat{x}_t &= (1-\sigma P_k) e_1^\top \left(q^{-1}A + q^{-2}A^2 + q^{-3}A^3 + \cdots \right) \hat{x}_t \\
&= (1-\sigma P_k) e_1^\top \left(q^{-1}E_t\hat{x}_{t+1} + q^{-2}E_t\hat{x}_{t+2} + q^{-3}E_t\hat{x}_{t+3} + \cdots \right) \\
&= (1-\sigma P_k) \sum_{j=1}^{\infty} q^{-j} E_t \hat{x}_{1t+j}.
\end{aligned}$$

If we had characterized the solution as an expectational difference equation, as Hansen and Sargent (1980) do, q would be the unstable root and the sum is would be the usual expected discounted value of the future state. In our case, the sum expresses the impact of expected future productivity growth on current consumption, which we interpret as an income effect: higher expected future growth leads to more consumption now.

Remark 4 (equilibrium dynamics). The equilibrium dynamics of the state follow the linear process

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{x}_{t+1} \end{bmatrix} = \begin{bmatrix} h_{kk} & h_{kx}^\top \\ 0 & A \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} -e_1^\top B \\ B \end{bmatrix} w_{t+1}$$

or

$$s_{t+1} \;=\; A_s s_t + B_s w_{t+1},$$

where

$$h_{kk} \;=\; [(f_k+g_k)/x_1]-(c/kx_1)h_{ck}, \quad h_{kx}^\top \;=\; -(c/kx_1)h_{cx}^\top - e_1^\top A.$$

The unconditional variance of the state is

$$G(0) \;=\; E\left(s_t s_t^\top\right) \;=\; A_* G(0) A_*^\top + BB^\top.$$

We compute $G(0)$ iteratively using Hansen and Sargent's (2005) Matlab program `doublej.m`. Autocovariances follow from

$$G(k) \;=\; E\left(s_t s_{t-k}^\top\right) \;=\; \begin{cases} A_*^k G(0) & k>0 \\ G(0)(A_*^k)^\top & k<0. \end{cases}$$

Since $G(-k)=G(k)^\top$, positive k is sufficient. When we look at growth rates, we do the same thing with the expanded state vector (s_t, s_{t-1}).

Remark 5 (intertemporal elasticity of substitution). The key parameter in the dynamics of the capital stock is ρ: as $\rho \to -\infty$ [and $\sigma = 1/(1-\rho) \to 0$], $h_{kk} \to 1$. See Campbell (1994). [Later: show this by expressing the Riccati equation in terms of h_{kk}.] Since h_{kk} is the source of endogenous dynamics, it's a critical feature to us. You see a similar effect from the curvature of the production function: as $f_{kk} \to 0$, as it would if $\nu \to 1$ (we have an "Ak" model in the limit), $h_{kk} \to 1$.

Remark 6 (growth rates). We're interested in (among other things) growth rates of various macroeconomic variables: GDP, consumption, investment, and (later on) employment. We compute them in our model from the dynamics of the state. The growth rate of GDP is

$$\begin{aligned} \log y_t - \log y_{t-1} &= \hat{y}_t - \hat{y}_{t-1} + e_1^\top \hat{x}_t \\ &= (f_k k/y)(\hat{k}_t - \hat{k}_{t-1}) + e_1^\top \hat{x}_t. \end{aligned}$$

Similarly, the growth rates of consumption and investment are, respectively,

$$\begin{aligned} \log c_t - \log c_{t-1} &= \hat{c}_t - \hat{c}_{t-1} + e_1^\top \hat{x}_t \\ &= h_{ck}(\hat{k}_t - \hat{k}_{t-1}) + h_{cx}^\top(\hat{x}_t - \hat{x}_{t-1}) + e_1^\top \hat{x}_t \end{aligned}$$

and

$$\begin{aligned} \log i_t - \log i_{t-1} &= \hat{i}_t - \hat{i}_{t-1} + e_1^\top \hat{x}_t \\ &= h_{ik}(\hat{k}_t - \hat{k}_{t-1}) + h_{ix}^\top(\hat{x}_t - \hat{x}_{t-1}) + e_1^\top \hat{x}_t. \end{aligned}$$

This last follows from the coefficients

$$h_{ik} \;=\; (f_k k/i) - (c/i)h_{ck}, \quad h_{ix}^\top \;=\; -(c/i)h_{cx}^\top.$$

Each expresses a growth rate as a linear function of s_t and s_{t-1}.

Remark 7 (interest rate). With these preferences and constant leisure, the pricing kernel is

$$m_{t+1} \;=\; \beta\,(c_{t+1}/c_t)^{\rho-1}\,[U_{t+1}/\mu_t(U_{t+1})]^{\alpha-\rho}\,.$$

The (continuously-compounded one-period) real interest rate is

$$r_t = -\log E_t m_{t+1}.$$

With constant volatility, this is just

$$\begin{aligned} r_t &= \text{constant} + \sigma^{-1} E_t(\hat{c}_{t+1} - \hat{c}_t + \hat{x}_{1t+1}) \\ &= \text{constant} + \sigma^{-1} h_{ck}(h_{kk} - 1)\hat{k}_t + \sigma^{-1}[h_{ck} h_{kx}^\top + h_{cx}^\top (A - I) + e_1^\top A]\hat{x}_t. \end{aligned}$$

If $A = 0$, productivity growth is white noise and the second term is zero. The interest rate is an AR(1), with persistence generated by the dynamics of the capital stock. Kaltenbrunner and Lochstoer (2007) make a similar point. More generally, the interest rate reflects the dynamics of both the capital stock and productivity growth.

5 Properties of the growth model

We show how the growth model works, starting with the growth model, the example from the previous section. We consider the complete model in the next section. We start with the benchmark case of white noise productivity growth, then go on to consider examples in which productivity growth is predictable.

Parameter values

There's some controversy about informal "calibration" exercises like this one. Perhaps it's best to confess up front that this is less than a serious statistical exercise. The model itself differs from the US economy in many obvious respects: no government, no international borrowing or lending, no variation in the relative price of consumption to investment goods, no demographics, and so on. Nevertheless, our hope is that some of the properties of the model will give us some insight into similar properties of the US and other aggregate economies. In any case, most of the parameter values are common ones, taken in our case from Kydland and Prescott (1982). We do our best later on to show that some of the parameter values are central to the properties we report, but long experience suggests that many are not.

Here's a quick summary of the logic behind the parameter choices for the growth model:

- Preferences. We use Kydland and Prescott's $\rho = -0.5$, which corresponds to an IES of 2/3. There's a range of opinion on this, so we'll explore alternatives later. Labor is fixed ($\lambda = 0$).
- Technology: production. Like most others, Kydland and Prescott use a Cobb-Douglas production function. We imitate by setting $\nu = 0$, but plan to explore this further at some future date. The value of ω follows from data on capital's share of gross domestic income, which we label s_k, and the capital-output ratio k/y. Kydland and Prescott

use $s_k = 0.36$, which we round to $1/3$, and $k/y = 10$, but there are conceptual and measurement issues with each. See, for example, the discussion in Gomme and Rupert (2007). In the general case,

$$s_k \;=\; f_k(k/y) \;=\; \omega(k/y)^{\nu-1}(k/y) \;=\; \omega(k/y)^{\nu}.$$

With $s_k = 1/3$ and $\nu = 0$, we have $\omega = 1/3$.

- Productivity growth. Our starting point is a random walk process for productivity:

$$\log z_t - \log z_{t-1} \;=\; \log x_{1t} \;=\; x_1 + Bw_{t+1}.$$

  This is more persistence than Kydland and Prescott (in their version, productivity is persistent but stationary), but the random walk has been used by Christiano and Eichenbaum (1992), King, Plosser, and Rebelo (1988), Tallarini (2000), and probably many others. We use the Christiano-Eichenbaum number $x_1 = 1.004$, the mean growth rate of GDP per hour worked. As in these other papers, there are no dynamics in productivity growth ($A = 0$). The conditional standard deviation is $B = 0.015$. This comes from Kydland and Prescott (they have two components, whose combined conditional standard deviation is about 0.01), scaled up by 3/2 because their process is for total factor productivity, and ours is for labor productivity.

- Technology: law of motion. We use Kydland and Prescott's $\delta = 0.025$ and set $\eta = 1$, which turns off the adjustment cost function.

These choices are summarized in Table 1.

With these inputs, we can compute numerical values for the various expressions used to compute the solution. The derivatives of the technology are $g_k = 1 - \delta$ and $f_k = s_k(y/k)$. The second derivative term is

$$f_{kk}k \;=\; -(1-\nu)\omega(y/k)^{-\nu}\left[1-\omega(y/k)^{-\nu}\right] \;=\; -(1-\nu)(1-s_k)s_k(y/k) \;=\; -0.0222.$$

The steady state investment-to-capital ratio follows from equation (5):

$$x_1 \;=\; 1 - \delta + (i/k),$$

or $i/k = x_1 - 1 + \delta = 0.0290$. Then $(c/k) = (y/k) - (i/k) = 0.0710$ and $c/kx_1 = 0.0707$.

Properties of the benchmark version

The benchmark version of this model, with white noise productivity growth ($A = 0$), cannot account for the leads and lags we see in US data. We focus on the cross-correlation between the short rate and GDP growth, comparing the features of the model with those documented in Section 2.

Growth rates in this model are largely a reflection of the shock. Growth rates of GDP, consumption, and investment are each a combination of the white noise shock and a persistent component due to capital dynamics, but with our parameters the former accounts for most of the variance. As a result, contemporaneous growth rates of consumption and investment are highly correlated with GDP growth; see Figure 6. These contemporaneous correlations overstate what we see in US data (compare Figures 3 and 4), an apparent result of the one-shock process. In contrast, correlations at non-zero leads and lags are modest. They're not exactly zero, but they're close.

The interest rate is much different from the growth rates: as expected consumption growth, it inherits the persistent dynamics of capital accumulation. Its contemporaneous correlation is positive (an increase in productivity raises the expected growth rate of consumption and marginal product of capital). Correlations at leads and lags decline slowly, reflecting the same dynamics.

Bottom line: this model, like most others in the literature, has the "everything moves up and down together" property. All of this is pretty much built in; even relatively large changes in parameter values have little impact on it.

Predictable productivity growth

We consider two processes with dynamics in productivity growth. The first is a two-period lead, in which productivity growth is known two periods before it occurs. The second is a combination of white noise and a predictable component designed to capture the useful features of each.

The two-period information lead works like this: productivity growth is a random walk, but it's announced two periods in advance. The representative agent knows productivity growth over the next two periods before making consumption and investment decisions. As a result, consumption goes up immediately. Since output doesn't change initially, investment falls. The opposing movements in consumption and investment are a clear example of the Barro-King result, and a sign that this isn't the solution to our problem.

The interest rate doesn't change much initially either, but expected future consumption growth generates an increase in the expected future interest rate. In this model, we would see an increase in the slope of the yield curve. The model thus reproduces the positive correlation between the slope of the yield curve and future GDP growth that we saw in US data.

Our last example is an attempt to combine the two: a productivity process that is mostly a random walk, but has an additional predictable component that generates the cross-correlation between the interest rate and GDP growth we see in the data. There are lots of ways to do this; one that comes close is an ARMA(1,1) model that introduces a persistent component in the spirit of Bansal and Yaron (2004):

$$x_{1t} \;=\; (1-\varphi)x_1 + \varphi x_{1t-1} + w_t - \theta w_{t-1}.$$

A unit impulse generates the sequence: $1, \varphi+\theta, \varphi(\varphi+\theta), \varphi^2(\varphi+\theta)$. Thus φ governs the persistence of this component and $\varphi+\theta$ the magnitude. To get the negative correlation between the interest rate and GDP growth, we need $\varphi+\theta < 0$, which means that productivity overshoots: part of the initial increase is reversed in subsequent periods.

We haven't experimented extensively, but Figure 8 shows how this might work. We set $\varphi = 0.5$ and $\theta = 0.8$. As a result, we have something resembling the S-shaped cross-correlation function we see in the data. Since the persistent component is smaller than the initial shock, the correlations of consumption and investment with output remain strongly positive. In this sense we have reconciled a predictable component of productivity growth with the Barro-King challenge to business cycle theory.

6 Employment, adjustment costs, and stochastic volatility

[later.]

Employment dynamics

[Later. Back to the full-blown model. The interesting aspect of this version is that predictable increases in future productivity growth generate declines in current labor input through the same income effect we saw with consumption. Too much of this will make employment countercyclical a la Barro and King, but can a little bit generate dynamics like Gali's (1999)?]

Adjustment costs

[Later. Important mainly for equity prices, since it allows q to be different from 1.]

Multivariate productivity process

[Later. The idea is to allow different sources of information, which is apparent in yield curve dynamics.]

Stochastic volatility

[Later. Breaks the observational equivalence result: recursive preferences play a central role here. Example with white noise productivity growth but persistent volatility generates interesting movements in interest rates and output: high volatility lowers the interest rate, other things equal, without changing output. Mention Bloom (2007).]

7 Related work

There's lots of related work, we'll discuss it some time. A quick list in the meantime:

- News about future productivity: Beaudry and Portier (2007), Jaimovich and Rebelo (2006), Opazo (2006).
- Exotic preferences in business cycle models: Campanale, Castro, and Clementi (2007), Croce (2006).

8 Final remarks

We've shown how modest changes to the Kydland-Prescott model can reproduce some of the leads and lags evident in US macroeconomic data, particularly the tendency for interest rates to lead GDP growth. Most of the work here is done by the productivity process, which includes a small but important predictable component. Recursive preferences also play a role when the conditional variance of productivity growth is stochastic. We use a real model, but future work may explore the impact of monetary policy on this feature of the data. Indeed, we would guess that one of the roles of monetary policy is to respond to signals of future events, so in that sense this line of work would serve as a useful input.

All of these properties have been derived using a convenient loglinear approximation to the equilibrium. This approach makes the calculations more transparent than usual and helps us to understand the nature of the equilibrium dynamics. When we look at asset prices, the loglinear approximation leads to the same kind of structure that has been used so successfully in affine pricing models. As a result, it supplies an interpretation of their parameters in terms of preferences, technology, and shocks.

A Data definitions and sources

[Later.]

B Loglinear approximation

The objective is to approximate the solution of various business cycle models with loglinear laws of motion and decision rules. The method follows the steps mentioned in Section 4, but the algebra is more complicated than in the growth model.

[Work in progress.]

Endogenous labor input

Here we approximate the solution of the scaled planning problem (9) with decision rules of the form (10,11).

Preliminaries. Most of the work is to compute loglinear approximations of various expressions: first-order Taylor series in logs of variables. We start with the resource constraint and laws of motion. The resource constraint (3) becomes (approximately)

$$\begin{aligned} \hat{y}_t &= (f_k k/y)\hat{k}_t + (f_n n/y)\hat{n}_t \\ &= (c/y)\hat{c}_t + (i/y)\hat{i}_t \\ \Rightarrow \quad \hat{i}_t &= (f_k k/i)\hat{k}_t + (f_n n/i)\hat{n}_t - (c/i)\hat{c}_t \end{aligned}$$

Laws of motion. The law of motion (6) for the shocks is loglinear to start with:

$$\hat{x}_{t+1} = A\hat{x}_t + Bw_{t+1}.$$

The law of motion (5) for capital implies

$$\begin{aligned} \hat{k}_{t+1} &= (i/kx_1)\hat{i}_t + (g_k/x_1)\hat{k}_t - \hat{x}_{1t+1} \\ &= [(f_k + g_k)/x_1]\hat{k}_t - (c/kx_1)\hat{c}_t + (f_n n/kx_1)\hat{n}_t - e_1^\top \hat{x}_{t+1}. \end{aligned}$$

The first line uses $g_i = 1$, a common convention that is satisfied by our functional form.

First-order and envelope conditions. We start with the components and work up to the conditions themselves. For example,

$$\begin{aligned} \hat{g}_{it} &= g_{ii} i\hat{i}_t + g_{ik} k\hat{k}_t \\ &= (g_{ik} + g_{ii} f_k) k\hat{k}_t - (g_{ii} c)\hat{c}_t + (g_{ii} f_n n)\hat{n}_t \\ \hat{f}_{nt} &= (f_{nk} k/f_n)\hat{k}_t + (f_{nn} n/f_n)\hat{n}_t. \end{aligned}$$

For $d_t = g_{it} f_{kt} + g_{kt}$, we have

$$\begin{aligned}(f_k + g_k)\hat{d}_t &= (f_{kk} + f_k g_{ik} + g_{kk}) k\hat{k}_t + f_{kn} n\hat{n}_t + (f_k g_{ii} + g_{ki}) i\hat{i}_t \\ &= (f_{kk} + 2 f_k g_{ik} + g_{kk} + f_k^2 g_{ii}) k\hat{k}_t - (f_k g_{ii} + g_{ki}) c\hat{c}_t + [f_{kn} + f_n (f_k g_{ii} + g_{ki})] n\hat{n}_t.\end{aligned}$$

Now to the (approximate) value function (15). The loglinear approximation of next period's value function and its derivative are

$$\begin{aligned}\widehat{J}_{t+1} &= p_k \hat{k}_{t+1} + p_x^\top \hat{x}_{t+1} \\ \widehat{J}_{kt+1} &= (p_k - 1)\hat{k}_{t+1} + p_x^\top \hat{x}_{t+1}\end{aligned}$$

With these inputs and the laws of motion, the loglinear approximation to

$$M_t \equiv \beta \mu_t (x_{1t+1} J_{t+1})^{\rho-\alpha} E_t[(x_{1t+1} J_{t+1})^{\alpha-1} J_{kt+1}]$$

is

$$\widehat{M}_t = (\rho p_k - 1)\left\{[(f_k + g_k)/x_1]\hat{k}_t - (c/kx_1)\hat{c}_t + (f_n n/kx_1)\hat{n}_t\right\} + \rho[p_x + (1-p_k)e_1]^\top A\hat{x}_t.$$

We've essential replaced inputs with their conditional means and ignored the impact of risk on the level, which is absorbed, in any case, in the discount factor β. We'll return to this when we consider asset pricing.

With these components, the first-order conditions have the loglinear approximations

$$\begin{aligned}(\rho - 1)\hat{c}_t - \lambda\rho[n/(1-n)]\hat{n}_t &= \widehat{M}_t + \hat{g}_{it} \\ \rho\hat{c}_t + (1-\lambda\rho)[n/(1-n)]\hat{n}_t &= \widehat{M}_t + \hat{g}_{it} + \hat{f}_{nt}\end{aligned}$$

Collecting terms and substituting $P_k = 1 - \rho p_k$, the first-order conditions become

$$\begin{bmatrix} (\rho-1) - P_k(c/kx_1) + g_{ii}c & -\lambda\rho[n/(1-n)] + P_k(f_n n/kx_1) - g_{ii} f_n n \\ \rho - P_k(c/kx_1) + g_{ii}c & (1-\lambda\rho)[n/(1-n)] + P_k(f_n n/kx_1) - g_{ii} f_n n - f_{nn} n/f_n \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ \hat{n}_t \end{bmatrix}$$

$$= \begin{bmatrix} -P_k(f_k + g_k)/x_1 + (g_{ik} + g_{ii} f_k)k & \rho[(1-p_k)e_1 + p_x]^\top A \\ -P_k(f_k + g_k)/x_1 + (g_{ik} + g_{ii} f_k + f_{nk}/f_n)k & \rho[(1-p_k)e_1 + p_x]^\top A \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{x}_t \end{bmatrix}.$$

Similarly, the envelope condition becomes

$$(\rho - 1)\widehat{J}_t + \widehat{J}_{kt} = \widehat{M}_t + \hat{d}_t$$

or

$$\begin{aligned}-P_k\hat{k}_t + \rho p_x^\top \hat{x}_t = &\left[-P_k(f_k + g_k)/x_1 + (f_k + g_k)^{-1}(f_{kk} + 2 f_k g_{ik} + g_{kk} + f_k^2 g_{ii})k\right]\hat{k}_t \\ &+ \rho\left[(1-p_k)e_1 + p_x\right]^\top A\hat{x}_t \\ &+ \left[P_k(c/kx_1) - (f_k + g_k)^{-1}(f_k g_{ii} + g_{ki})c\right]\hat{c}_t \\ &+ \left[-P_k(f_n n/kx_1) + (f_k + g_k)^{-1}[f_{kn} + f_n(f_k g_{ii} + g_{ki})]n\right]\hat{n}_t\end{aligned}$$

or

$$
\begin{aligned}
0 \quad = \quad & \Big[-P_k[(f_k+g_k)/x_1-1]+(f_k+g_k)^{-1}(f_{kk}+2f_kg_{ik}+g_{kk}+f_k^2g_{ii})k\Big]\,\hat{k}_t \\
& +\rho\Big[(1-p_k)e_1^\top - p_x^\top(I-A)\Big]\,\hat{x}_t \\
& +\ \Big[P_k(c/kx_1)-(f_k+g_k)^{-1}(f_kg_{ii}+g_{ki})c\Big]\,\hat{c}_t \\
& +\ \Big[-P_k(f_nn/kx_1)+(f_k+g_k)^{-1}[f_{kn}+f_n(f_kg_{ii}+g_{ki})]n\Big]\,\hat{n}_t
\end{aligned}
$$

This is a complete mess; its essential feature for our purposes is that all of these coefficients are linear functions of the value function parameters (p_k, p_x).

Riccati equation. [Later.]

Decision rules. [Later.]

Parameter values

[Later.]

Stochastic volatility

Here's a simple stochastic volatility model, in which productivity growth has a constant conditional mean but time-varying conditional variance. Let

$$v_{t+1} \quad = \quad (1-\varphi)v + \varphi v_t + \tau w_{0t+1}$$

and replace w_{t+1} with $v_t^{1/2} w_{t+1}$ in (6).

Quick and dirty version of the growth model. Work through the necessary conditions, starting with M_t. Guess a value function of the form

$$\log J(\tilde{k}_t, x_t) \quad = \quad \log p_0 + p_k \log \tilde{k}_t + p_v v_t + p_x^\top \log x_t.$$

Then

$$
\begin{aligned}
\hat{x}_{1t+1} + \widehat{J}_{t+1} \quad &= \quad p_k\hat{k}_{t+1} + p_v v_{t+1} + (e_1+p_x)^\top \hat{x}_{t+1} \\
&= \quad p_k[\hat{k}_t - (c/kx_1)\hat{c}_t] + p_v[(1-\varphi)v + \varphi v_t + \tau w_{0t+1}] \\
& \qquad + [(1-p_k)e_1 + p_x]^\top (A\hat{x}_t + Bv_t^{1/2} w_{t+1}).
\end{aligned}
$$

All we care about are the v_t terms. The certainty equivalent includes

$$p_v\varphi v_t + (\rho-\alpha)\alpha[(1-p_k)e_1+p_x]^\top BB^\top[(1-p_k)e_1+p_x]v_t/2$$

Now the expectation term:

$$
\begin{aligned}
(\alpha-1)(\hat{x}_{1t+1}+\widehat{J}_{t+1})+\widehat{J}_{kt+1} &= (\alpha p_k-1)\hat{k}_{t+1}+\alpha p_v v_{t+1}+[(\alpha-1)e_1+\alpha p_x]^\top \hat{x}_{t+1} \\
&= (\alpha p_k-1)[\hat{k}_t-(c/kx_1)\hat{c}_t]+\alpha p_v[(1-\varphi)v+\varphi v_t+\tau w_{0t+1}] \\
&\quad +\alpha[(1-p_k)e_1+p_x]^\top(A\hat{x}_t+Bv_t^{1/2}w_{t+1}),
\end{aligned}
$$

which generates the v_t terms

$$
\alpha p_v\varphi v_t+\alpha^2[(1-p_k)e_1+p_x]^\top BB^\top[(1-p_k)e_1+p_x]v_t/2.
$$

Adding them together gives us

$$
(1+\alpha)\varphi p_v v_t+\alpha\rho[(1-p_k)e_1+p_x]^\top BB^\top[(1-p_k)e_1+p_x]v_t/2.
$$

References

Alvarez, Fernando, and Urban Jermann, 2005, "Using asset prices to measure the persistence of the marginal utility of wealth," *Econometrica* 73, 1977-2016.

Ang, Andrew, Monika Piazzesi, and Min Wei, 2006, "What does the yield curve tell us about GDP growth?," *Journal of Econometrics* 131, 359-403.

Bansal, Ravi, and Amir Yaron, 2004, "Risks for the long run: A potential resolution of asset pricing puzzles," *Journal of Finance*, forthcoming in the August issue.

Barro, Robert J., and Robert G. King 1984, "Time-separable preferences and intertemporal-substitution models of business cycles, *Quarterly Journal of Economics* 99, 817-839.

Beaudry, Paul, and Franck Portier, 2006, "Stock prices, news, and economic fluctuations," *American Economic Review* 96, 1293-1307.

Beaudry, Paul, and Franck Portier, 2007, "When can changes in expectations cause business cycle fluctuations in neo-classical settings?," *Journal of Economic Theory* 135, 458-477.

Bloom, Nick, 2007, "The impact of uncertainty shocks," manuscript, August.

Boldrin, Michele, Lawrence J. Christiano, and Jonas D.M. Fisher, 2001, "Habit persistence, asset returns, and the business cycle," *American Economic Review* 91, 149-166.

Campanale, Claudio, Rui Castro, and Gian Luca Clementi, 2006, "Asset pricing in a general equilibrium production economy with Chew-Dekel risk preferences," manuscript, New York University, December.

Campbell, John Y., 1994, "Inspecting the mechanism: An analytical approach to the stochastic growth model," *Journal of Monetary Economics* 33, 463-506.

Christiano, Lawrence J., and Martin Eichenbaum, 1992, *American Economic Review* 82, 430-450.

Cochrane, John H., 1991, "Production-based asset pricing and the link between stock returns and economic fluctuations," *Journal of Finance* 46, 209-237.

Cogley, Timothy, and James M. Nason, 1995, "Output dynamics in real-business-cycle models," *American Economic Review* 85, 492-511.

Croce, Massimiliano, 2005 "Welfare Costs and Long-Run Consumption Risk in a Production Economy," manuscript, August.

Epstein, Larry G., and Stanley E. Zin, 1989, "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework," *Econometrica* 57, 937-969.

Fernandez de Cordoba, Gonzalo, and Timothy J. Kehoe, 2000, "Capital flows and real exchange rate fluctuations following Spain's entry into the European Union," *Journal of International Economics* 51, 49-78.

Galí, Jordi, 1999, "Technology, employment, and the business cycle: Do technology shocks explain aggregate fluctuations?," *American Economic Review* 89, 249-271.

Gomme, Paul, and Peter Rupert, 2007, "Theory, measurement and calibration of macroeconomic models," *Journal of Monetary Economics* 54, 460-497.

Hansen, Lars Peter, John C. Heaton, and Nan Li, 2005, "Consumption strikes back? Measuring long-run risk," manuscript, University of Chicago, October.

Hansen, Lars Peter, and Thomas J. Sargent, 1980, "Forumulating and estimating dynamic linear rational expectations models," *Journal of Economic Dynamics and Control* 2, 7-46.

Hansen, Lars Peter, and Thomas J. Sargent, 2005, *Recursive Models of Dynamic Linear Economies* manuscript, September.

Hansen, Lars Peter, Thomas J. Sargent, and Thomas Tallarini, 1999, "Robust permanent income and pricing," *Review of Economic Studies* 66, 873-907.

Jaimovich, Nir, and Sergio Rebelo, 2006, "Can news about the future drive the business cycle?," manuscript, February.

Jermann, Urban, 1998, "Asset pricing in production economies," *Journal of Monetary Economics* 41, 257-275.

Kaltenbrunner, Georg, and Lars Lochstoer, 2007, "Long-run risk through consumption smoothing," manuscript, London Business School, August.

King, Robert G., Charles I. Plosser, and Sergio T. Rebelo, 1988, "Production, growth, and business cycles: II. New directions," *Journal of Monetary Economics* 21, 309-341.

King, Robert G., and Mark W. Watson, 1996, "Money, prices, interest rates and the business cycle," *Review of Economics and Statistics* 78, 35-53.

Kocherlakota, Narayana R., 1990, "Disentangling the coefficient of relative risk aversion from the elasticity of intertemporal substitution," *Journal of Finance* 45, 175-190.

Kreps, David M., and Evan L. Porteus, 1978, "Temporal resolution of uncertainty and dynamic choice theory," *Econometrica* 46, 185-200.

Kydland, Finn E., and Edward C. Prescott, 1982, "Time to build and aggregate fluctuations," *Econometrica* 50, 1345-1370.

Lettau, Martin, 2003, "Inspecting the mechanism: closed-form solutions for asset prices in real business cycle models," *Economic Journal* 113, 550-575.

Opazo, Luis, 2006, "The Backus-Smith puzzle: the role of expectations," manuscript, Georgetown University, February.

Primiceri, Giorgio E., Ernst Schaumburg, and Andrea Tambalotti, 2006, "Intertemporal disturbances," NBER Working Paper 12243, May.

Rouwenhorst, K. Geert, 1995, "Asset pricing implications of equilibrium business cycle models," in T.F. Cooley, editor, *Frontiers of Business Cycle Research*, Princeton, NJ: Princeton University Press.

Stock, James H., and Mark W. Watson, 1989, "New indexes of coincident and leading economic indicators," in *NBER Macroeconomics Annual*, O.J. Blanchard and S. Fischer, eds., 352-394.

Stock, James H., and Mark W. Watson, 2003, "Forecasting output and inflation: the role of asset prices," *Journal of Economic Literature* 41, 788-829.

Tallarini, Thomas D., 2000, "Risk-sensitive real business cycles," *Journal of Monetary Economics* 45, 507-532.

Uhlig, Harald, 2006, "Asset pricing with Epstein-Zin preferences," manuscript, March.

Weil, Philippe, 1989, "The equity premium puzzle and the risk-free rate puzzle," *Journal of Monetary Economics* 24, 401-421.

Table 1
Benchmark parameter values for the growth model

Parameter	Value	Source/comment
Preferences		
λ	0	fixes labor supply
ρ	-0.5	Kydland and Prescott (1982)
α	—	does not affect quantities
β	—	identified by endogenous
Technology: production		
ν	0	Kydland and Prescott (1982)
ω	1/3	Kydland and Prescott (1982), rounded
Technology: capital accumulation		
δ	0.025	Kydland and Prescott (1982)
η	1	Kydland and Prescott (1982)
Productivity growth		
x_1	1.004	Christiano and Eichenbaum (1992)
A	0	no predictable component
$B = Var_t(\log x_{1t+1})^{1/2}$	0.015	Kydland and Prescott (1982), adjusted

Figure 1
US data: cross-correlation functions for equity prices and GDP

Figure 2
US data: cross-correlation functions for interest rates and GDP growth

Figure 3
US data: cross-correlation functions for consumption and GDP

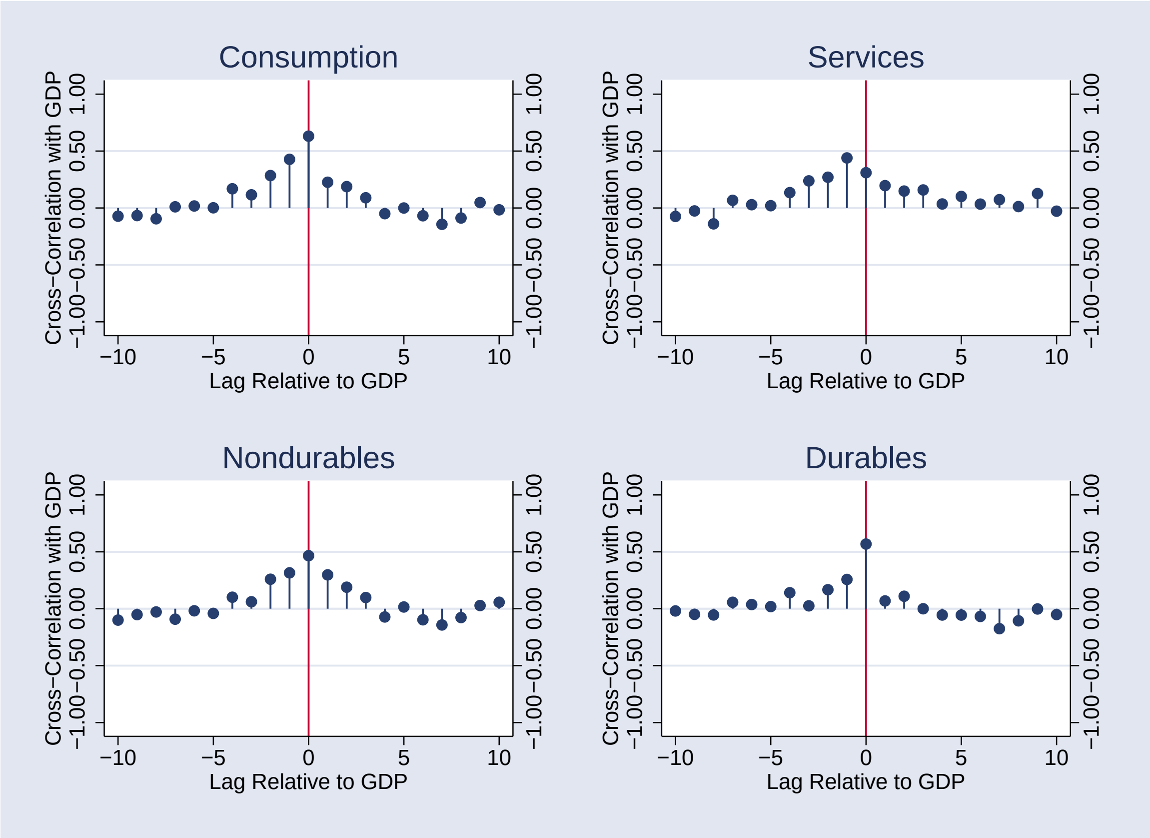

Figure 4
US data: cross-correlation functions for investment and GDP

Figure 5
US data: cross-correlation functions for employment and GDP

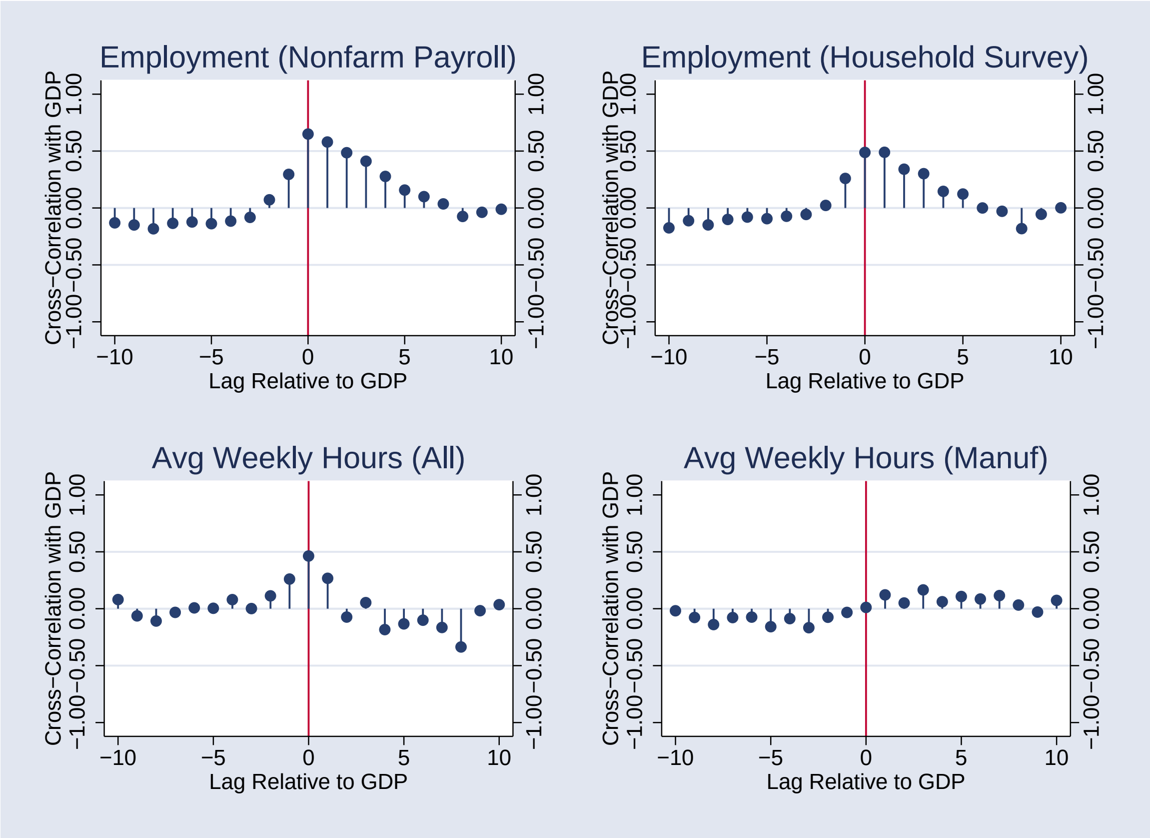

Figure 6
Growth model: cross-correlation functions with random walk productivity

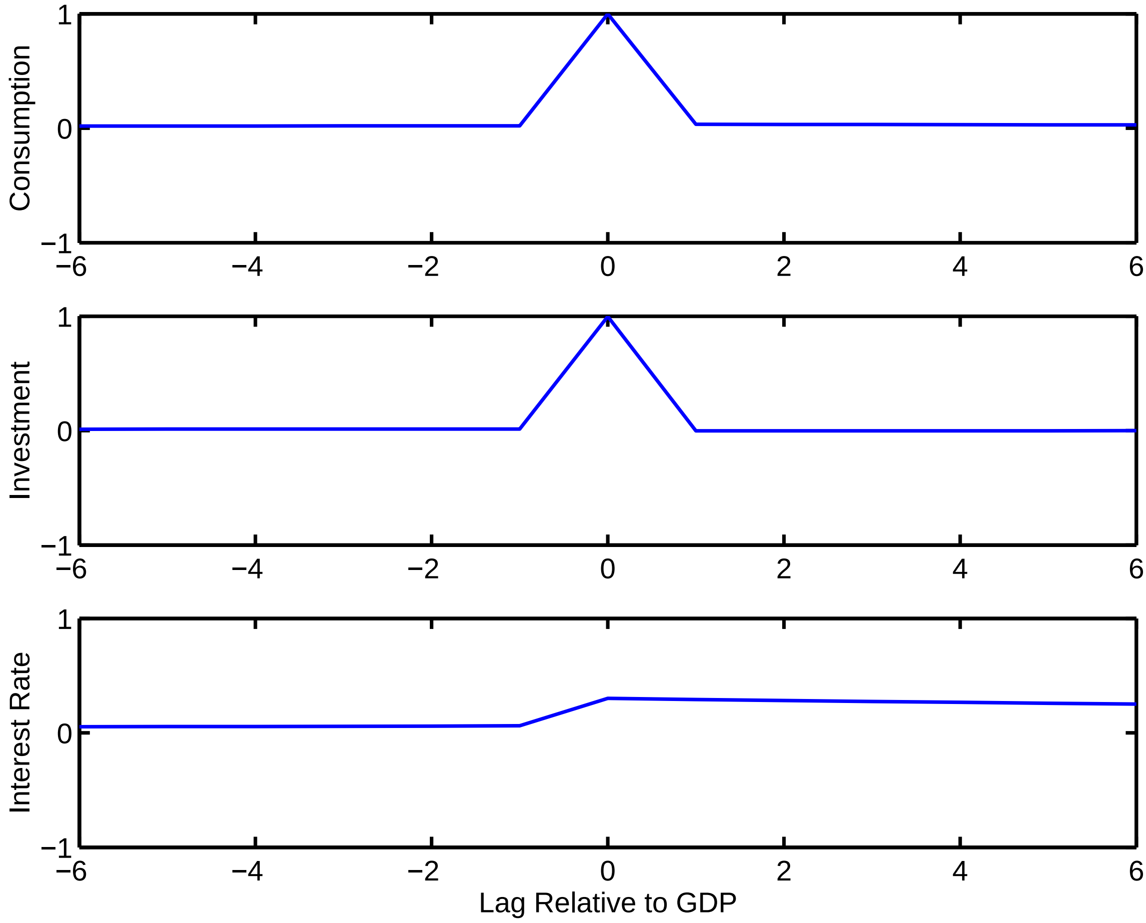

Figure 7
Growth model: cross-correlation functions with two-period information lead

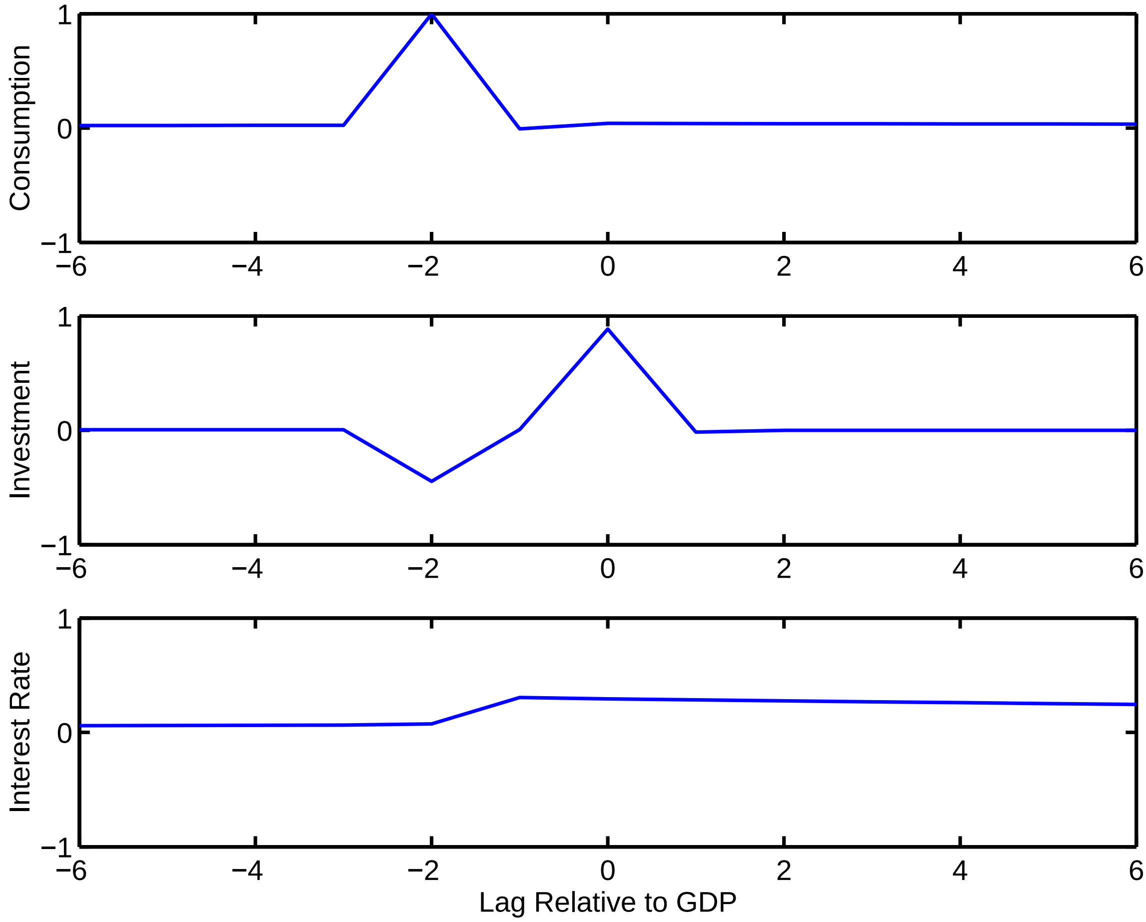

Figure 8
Growth model: cross-correlation functions with persistent component