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On Distribution Functions of the Derivatives of Weakly Differentiable Mappings

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A recent investigation into the behavior under homogenization of second order materials with negative capillarity [CMM] led to the formulation of certain questions concerning distribution functions of absolutely continuous mappings on a real interval,none of which appears to have been examined previously. The present article describes and resolves one of these questions: *For a function on the unit interval possessing an absolutely continuous first derivative, does the joint distribution of function and first derivative already determine the distribution of the second derivative?*

Given any measurable mapping [for brevity, we will also apply this term to L_N equivalence classes] $\mathbf{w}: \Omega \to \mathbb{R}^m$, with Ω a bounded measurable subset of $\mathbb{R}^N$, the (mass) distribution $\mu = \mu^{\mathbf{w}}$ denotes that Borel measure on $\mathbb{R}^m$ defined by

(0) $\quad \mu(B) = L_N \{x \in \Omega: \mathbf{w}(x) \in B\}, \qquad B \in \mathbb{B}(\mathbb{R}^m),$

where L_N denotes N-dimensional Lebesgue measure and $\mathbb{B}(\mathbb{R}^m)$ denotes the Borel σ-algebra on $\mathbb{R}^m$.

Given a function $u \in W^{2,1}(0,1)$, where $W^{2,1}(0,1)$ denotes the Sobolev space of real functions on (0,1) possessing two summable generalized derivatives, there are several associated (mass) distributions to be considered: one for each of the functions u, u', u" as well as for the mappings $\mathbf{f} = (u, u')$, $\mathbf{g} = (u, u', u'')$, etc. We are here interested in the linkage between certain of these distributions, in particular between the distribution $\mu := \mu^{u''}$ and the distribution $\pi := \mu^{\mathbf{f}}$; π is hereafter called a *joint* distribution for emphasis. Our main result shows that μ is characterized by π; in particular, if u, v are such that

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$\mu^{(u,u')} = \mu^{(v,v')}$ then it necessarily follows that $\mu^{u''} = \mu^{v''}$.

Our arguments will utilize the following result [F, Theorem 2.10.10].

<u>Proposition</u> Suppose X and Y are separable metric spaces, ν is a metric outer measure on Y, f maps X into Y, and f(A) is ν-measurable whenever A is a Borel subset of X. If

$$\zeta(S) = \nu[f(S)] \quad \text{for} \quad S \subset X$$

and ψ is the metric outer measure over X resulting from ζ by Caratheodory's construction on the family of all Borel subsets of X, then

$$\psi(A) = \int N(f|A, y)\, d\nu(y) \quad \text{for every Borel set } A \subset X,$$

where for any mapping $g: Z \to Y$

$$N(g, y) = \#\{x \in Z \mid g(x) = y\} \in \{0, 1, \ldots, n, \ldots, \infty\}.$$

<u>Theorem A</u> For each u in $W^{2,1}(0, 1)$, the joint distribution $\pi = \mu^{(u,u')} =: \mu^{\mathbf{f}}$ determines the distribution $\mu = \mu^{u''}$. More precisely, for every A in $\mathbb{B}(\mathbb{R})$ one has the line integral formula [with $y = (y_0, y_1) \in \mathbb{R}^2$] :

$$(*) \quad \mu(A) = \int_{E_A} [N(\mathbf{f}, y)/y_1]\, dy_0 + L_1(Z)\, \delta_{\{0\}}(A),$$

where

$E_A = \{y \in \mathbf{f}([0, 1]) : y_1 dy_1/dy_0 \in A\}$, $Z = \{x \in [0,1] : u'(x) = 0\}$

and $\delta_{\{0\}}$ is the unit mass at $\{0\}$. Here π determines the integrand in (*) through the relation

$$(1) \pi(B) = \int_{F_B} [N(\mathbf{f}, y)/y_1]\, dy_0 + \pi^s(B \cap (\mathbb{R} \times \{0\})), \quad \text{all } B \in \mathbb{B}(\mathbb{R}^2),$$

where $F_B = \mathbf{f}([0, 1]) \cap B$, and π^s is supported on an L^1 null set of $\mathbb{R} \times \{0\}$ with $\pi^s(\mathbb{R} \times \{0\}) = L_1(Z)$.

<u>Remarks:</u> The line integrals in (*) and (1) are to be interpreted as integrals with respect to arclength, i.e. Hausdorff 1-measure. The symbol dy_1/dy_0 appearing in the definition of E_A refers to the slope of the (bilateral) tangent

line to the set $\mathbf{f}([0, 1])$ at the point $y \in \mathbf{f}([0, 1])$ (so that formally, $y_1 dy_1/dy_0 = u''$), provided that y is a point where the tangent ["contingent"] cone $T_S(y)$ to $S = \mathbf{f}([0, 1])$ consists of a single line. Here we are utilizing terminology stemming from the following definition of Bouligand [S,p263][F,p233] [MM1,p304], see also [AE].

<u>Definition.</u> An element $\mathbf{v} \in \mathbb{R}^m$ is a **bilateral unit tangent** to the closed set S of $\mathbb{R}^m$ at the point $y \in S$ provided that there exist sequences $\{y_i\},\{y_i'\}$ in S converging to y and satisfying

$$(y_i - y)/|y_i - y| \to \mathbf{v}, \qquad (y_i' - y)/|y_i' - y| \to -\mathbf{v}$$

The collection of all such $\mathbf{v}$ is denoted by $U_S(y)$, and the cone consisting of all lines $\mathbb{R}\mathbf{v}$ with $\mathbf{v} \in U_S(y)$ is denoted by $T_S(y)$ and called the <u>contingent</u> cone to S at y.

A key feature utilized in our development is the fact that since $(u, u') = \mathbf{f}: [0, 1] \to \mathbb{R}^2$ is an absolutely continuous curve in $\mathbb{R}^2$, the following result applies [MM1,§3] [in what follows H_1 denotes one-dimensional Hausdorff measure on $\mathbb{R}^m$]:

<u>Lemma</u> If $S \subset \mathbb{R}^m$ is the track of an absolutely continuous curve $\mathbf{f}$ then for H_1-a.e. $y \in S$

(a) $U_S(y)$ consists of a single pair of opposing unit vectors

(b) $\mathbf{f}^{-1}(y)$ is a finite set

(c) $\mathbf{f}'(t) \neq \mathbf{0}$ for all $t \in \mathbf{f}^{-1}(y)$

(d) $U_S(y) = \{\pm\mathbf{f}'(t)/|\mathbf{f}'(t)|: t \in \mathbf{f}^{-1}(y)\}$.

Furthermore, $\mathbf{f}$ has the following property

(N) $L_1(A) = 0 \Rightarrow H_1(\mathbf{f}(A)) = 0, \qquad A \subset \mathrm{Dom}(\mathbf{f})$.

We proceed to present the proof of Theorem A.

Proof: Note that whenever Y, Z are disjoint measurable subsets of [0, 1] it follows from (0) that the distribution function $\mu_3 = \mu^{\mathbf{f}|Y\cup Z}$ is simply the sum of $\mu_1 = \mu^{\mathbf{f}|Y}$ and $\mu_2 = \mu^{\mathbf{f}|Z}$. In particular this holds for $Y = [0, 1]\setminus Z$, with Z as in the statement of the theorem. Now set for each $\varepsilon > 0$, $Y_\varepsilon = \{x\in[0,1]: |u'(x)| > \varepsilon\}$, $\pi_\varepsilon := \mu^{\mathbf{f}|Y_\varepsilon}$, and $\mu_\varepsilon := \mu^{u''|Y_\varepsilon}$. Then it follows from (0) and the Vitali-Hahn-Saks theorem that

$$\lim_{\varepsilon\to 0}\pi_\varepsilon = \mu^{\mathbf{f}|Y} =: \pi_0 ,$$

in the sense that var $(\pi_0 - \pi_\varepsilon)\to 0$, and similarly,

$$\lim_{\varepsilon\to 0}\mu_\varepsilon = \mu^{u''|Y} =: \mu_0.$$

Now the behavior of μ restricted to $\mathbb{B}(\mathbb{R}\setminus[-\varepsilon,\varepsilon])$ for all positive ε determines μ since $\mu(\mathbb{R}) = 1$. Thus it will suffice for our purposes to prove that for each $\varepsilon > 0$ μ_ε is given by the line integral formula

$$(2)\quad \mu_\varepsilon(A) = \int_{E_{\varepsilon,A}} [N(\mathbf{f},y)/y_1]\, dy_0, \quad \text{all } A\in \mathbb{B}(\mathbb{R}),$$

with $E_{\varepsilon,A} = \{y\in \mathbf{f}(Y_\varepsilon): y_1 dy_1/dy_0 \in A\}$, while π_ε determines the integrand in (2) via

$$(3)\quad \pi_\varepsilon(B) = \int_{F_{\varepsilon,B}} [N(\mathbf{f},y)/y_1]\, dy_0 , \quad \text{all } B\in \mathbb{B}(\mathbb{R}^2),$$

with $F_{\varepsilon,B} = \mathbf{f}(Y_\varepsilon)\cap B$.

In (2) and (3) we have utilized the fact that, by the definition of Y_ε,

$\mathbf{f}(Y_\varepsilon) = \mathbf{f}([0, 1]) \cap \{y\in \mathbb{R}^2: |y_1|>\varepsilon\}$, so that

$$(4)\quad N(\mathbf{f}|Y_\varepsilon,y) = N(\mathbf{f}, y), \qquad \text{for each } y\in \mathbf{f}(Y_\varepsilon).$$

We have also denoted by $y_1 dy_1/dy_0$ the product of y_1 with the slope of the tangent line to the set $\mathbf{f}([0, 1])$ at the point $y = (y_0, y_1)$ [by the Lemma there is a unique tangent line at H_1-a.e. $y \in \mathbf{f}([0,1])$]. The existence of the integrals in (2) and (3) as integrals with respect to H_1 [taking $dy_0 = \cos \theta(y)\, dH_1$ where $\theta(y)$ denotes the angle with the horizontal made by the tangent line to $\mathbf{f}([0,1])$ at y] follows by the Proposition with $X = [0,1]$ and $\nu = H_1$ on $Y = \mathbb{R}^2$. Namely, by the Lemma the absolute continuity of $\mathbf{f}$ ensures that this mapping has the (N) property of Lusin [S] from which it easily follows that $\mathbf{f}(S)$ is H_1-measurable for each L_1-measurable subset S of [0 ,1]. Thus on defining the set function Π by

$$\Pi(A) = H_1(\mathbf{f}(A)) \text{ on subsets } A \subset [0, 1],$$

and letting ß denote the metric (outer) measure generated by $\Pi | \mathbb{B}([0,1])$, it follows that

$$(5) \quad \beta(A) = \int_{\mathbf{f}(A)} N(\mathbf{f}|A,y)\, dH_1(y) \qquad \text{for all } A \in \mathbb{B}([0,1]).$$

It follows from (5) that the line integral in (3) is also well defined, for each $\varepsilon > 0$. The validity of (3) is now an easy consequence of the formal relation $dx = du/u'$, since the equality is evident for $B = \mathbf{f}(J)$ on each component J of the open sets

$C_+(\varepsilon) = \{x | \, u'(x) > \varepsilon\}, \qquad C_-(\varepsilon) = \{x | \, u'(x) < -\varepsilon\}.$

The validity of formula (2) utilizes the fact that the function u" can be factored as $h \circ \mathbf{f}$ where $h: \mathbf{f}[0, 1] \to \mathbb{R}$ is the measurable mapping defined H_1 -a.e. by $h(y) = y_1 dy_1/dy_0$. More precisely, if C,D,E denote the subsets of [0, 1] consisting, respectively, of points x where u"(x) is not defined as a real number, of points x in Z at which Z doesn't have unit density, and of points x such that $\mathbf{f}([0,1])$ fails to have a unique (nonvertical) tangent line,then

$$L_1(C) = L_1(D) = L_1(E) = 0$$

[the validity of the last relation follows from the proof of the Lemma, cf.e.g. [S,Ch IX Lemma 3.1]]. Hence by setting $X = [0, 1]\setminus(C \cup D \cup E)$ one finds that $\mu = \mu^{u''|X}$. Moreover $u'': X \to \mathbb{R}$ can be factored as $u'' = h \circ \mathbf{f}$, where $h: \mathbf{f}(X) \to \mathbb{R}$ is given by $h(y) = y_1 dy_1/dy_0$. It follows from this that (2) is valid for each set $A \in \mathbb{B}(\mathbb{R})$ such that $A \subset \mathbb{R}\setminus\{0\}$. The validity of (2) for all A follows immediately.

<u>Remarks:</u> We observe that the arguments used in the proof of Theorem A demonstrate that π also characterizes $\mu^{\mathbf{g}}$, where $\mathbf{g} = (u,u',u'')$. So π characterizes $\mu^{(u',u'')}$, and hence if $u \in W^{3,1}(0, 1)$ it also characterizes $\mu^{u'''}$, and so on.

Although Theorem A leads fairly straightforwardly (using [MM2]) to the conclusion that for $\Omega \subset \mathbb{R}^N$ and $u \in W^{2,p}(\Omega)$, $p > N$, the joint distribution $\pi = \mu^{\mathbf{f}}$ where $\mathbf{f} = (u, \text{grad } u)$, characterizes $\mu_i = \mu^{u,x^i x^i}$, $i = 1, \ldots, N$, as well as $\mu^* = \mu^{\triangle u}$, one has to utilize the additional fact that π characterizes the distributions $\mu_{\mathbf{a}} = \mu^{(\partial_{\mathbf{a}})^2 u}$ for all second directional derivatives $(\partial_{\mathbf{a}})^2$ to obtain a multidimensional analogue of Theorem A involving nonsymmetric differential operators. Of course the use of rectifiable currents would be more suitable for this multidimensional context ([F,ch 4] or [M, ch 4]). In any event Theorem A supplies a key result for the relaxation analysis of the second order model in [CMM], as will be shown in a subsequent publication.

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