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ON THE PROPAGATION OF SINGULARITIES OF SEMI-CONVEX FUNCTIONS

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Abstract. The paper deals with the propagation of singularities of semi-convex functions. We obtain lower bounds on the degree of the singularities and on the size of the singular set in a neighborhood of a singular point. These results apply to viscosity solutions of Hamilton-Jacobi-Bellman equations. In particular, they provide sufficient conditions for the propagation of singularities, depending only on the geometry of the superdifferential at the singular point.

Key words, convexity, semi-concavity, propagation of singularities, Hamilton-Jacobi equations

INTRODUCTION

In a recent paper [1], *upper* bounds on the dimension of singular sets of semiconvex functions were derived by measure theoretic arguments.

To briefly describe these upper bounds, let $u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a semi-convex function (Definition 1.2 below). Define

$$S^k(u) = \{x \in \mathbb{R}^n : dim(du(x)) = k\},$$

where $k \in [0,n]$ is an integer and du(x) denotes, as usual, the subdifferential of u. Clearly, $\{S^k(u)\}_{=:0}^n$ is a partition of \mathbb{R}^n and $S^{\bullet}(u)$ is the set of all points of differentiability of u. Since we are interested in first order singularities, we call a point x singular for u if $x \in S^k(u)$ for some $k \ge 1$.

In [1] it is proved that $S^k(u)$ is countably $W^{n} e^{fc}$ -rectifiable. In particular,

$$H$$
- $dim(S^k(u)) \leq n$ -fc,

where 7i — dim is the Hausdorff dimension.

The purpose of the present work is to obtain *lower* bounds on the dimension of $S^k(u)$. More precisely, we will describe the structure of $S^k(u)$ in a neighborhood of x, knowing the geometry of $du\{x\}$.

A motivating application of these results concerns the analysis of singularities of solutions to the Hamilton-Jacobi-Bellman equation

$$(1) H(x,u,Vu) = 0.$$

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In fact, if the data are smooth, viscosity solutions of such PDE's (and, in particular, the solutions that are relevant to optimal control) enjoy well known semi-concavity properties (see for instance [12], [13], [15], [16]).

The present work is related to [4] and [5], in which viscosity solutions of (1) are shown not to have any isolated singularity if H is strictly convex with respect to p. In [4], [5], however, no attention is paid to the dimension of du at such singular points, and no attempt is made to estimate the Hausdorff measure of the singular sets.

Different approaches to the analysis of singularities of Hamilton-Jacobi equations are obtained for the one dimensional case in [14] and using characteristics in [21].

Semi-convexity was the only property used in [1] to prove upper bounds on singular sets. On the contrary, to obtain lower bounds we need additional information. This fact is the essential difference between [1] and the present paper. In order to understand the nature of the additional information, let us consider the set of reachable subgradients

$$\mathbf{V}>\mathbf{u}(\mathbf{x}) = \{ \quad \mathbf{Um} \quad Vu(x_h) : x_h \ \mathbf{G} \ S^{\bullet}(u) \setminus \{\mathbf{x}\}, \ x_h \ -\!\!\!-\! \mathbf{x} \}.$$

The above set is a set of generators of du(x) in the sense of convex analysis. Then, we show that the strict inclusion

(2)
$$V_{\mathbf{u}}(\mathbf{z}) du(x)$$
.

is a sufficient condition for the propagation of any singularity $x \in S^k(u)$, $1 \le k < n$ (see Example 2.1 below). The inclusion (2) is satisfied by any viscosity solution of (1) with a strictly convex Hamiltonian, as V+u(x) is contained in the zero level set of H(x,tt(x),-).

Moreover, if x is an isolated singularity, by adapting a variational argument of Tonelli (see the proof of the implicit function theorem in [20]), we show that V+u(x) coincides with \acute{c} ?u(x), see Theorem 2.1 below.

Furthermore, inserting nonsmooth analysis into this procedure, we obtain a more detailed description of the singular sets. In Theorem 2.2 we prove that singularities propagate along directions related to the geometry of $du\{x\}$. These directions are orthogonal to the exposed faces of du(x). In Theorem 2.3 we give a lower bound on the maximum integer $m \le k$ such that x is a cluster point of

$$E'' \gg = Q S'(u),$$

and in (2.7) we estimate from below the Hausdorff (n — fc)-dimensional measure of $S^m(tx)$. Roughly speaking, the computation of m takes into account how many vectors in $V^*u(x)$ are necessary to generate du(x).

We conclude with an outline of the paper. The first section contains preliminary material on Hausdorff measures, semi-convex functions, and the estimates of [1]. In §2 we develop our main results on propagation of singularities of semi-convex functions. The last section is devoted to applications to Hamilton-Jacobi-Bellmann equations and to the discussion of some examples.

1. NOTATION AND PRELIMINARIES

We briefly introduce some notation. We denote by $B_p(x)$ the open ball in \mathbb{R}^n centered in x with radius p, and we abbreviate $B_p = B_P(Q)$.

For any set $A \subset \mathbb{R}^n$ we denote by co(A) the convex hull of A. Moreover, the following sets of convex combinations of points of A will be often used in the sequel.

$$I_j(A) = \left\{ \sum_{i=1}^{j} \lambda_i p_i : p_i \in A, \ \lambda_i \ge 0, \ \sum_{i=1}^{j} \lambda_i = 1 \right\}$$

for any integer $j \ge 1$. We also define

$$mi(A) = \max_{A} j \ge 0 : 1; (A) \# co(A)$$
.

Clearly $I \setminus (A) = A$, hence m(A) = 0 if and only if A is a convex set. Moreover, by Carathéodory's Theorem (see for example [18, p.155]) we know that Jfc+i (A) = co(A), where k is the dimension of co(A). Therefore $m(A) \leq dim[co(A)]$. However, the integer m(A) does not depend just on the dimension of co(A). For example, if A is a finite set of affinely independent points, then m(A) equals the dimension of co(A). On the other hand, if A is the boundary of a fc-dimensional ball, then m(A) = 1.

For any set $S \subset \mathbb{R}^n$ we define

$$S^{I}' = \{p \in \mathbb{R}^n : q \longrightarrow (\sqrt[3]{p}) \text{ is constant on } 5\},$$

and

$$T(5,x) = \{r\theta: r > 0, 6 = \lim_{h \to +\infty} \frac{x_h - x}{p^h - x}, x_h \in S \setminus \{x\}, x_h + x\}.$$

The set T(5, x) defined above is the so-called *contingent* cone to S at x ([3], [6]).

For any real number $r \in]0,n]$ we denote by $H^r(B)$ the Hausdorff r-dimensional measure of $B \subset \mathbb{R}^n$, defined by

$$\mathcal{H}^{r}(B) = \underset{2}{\overset{\wedge}{\text{supinf}}} \underset{\delta>0}{\overset{\vee}{\sum}} \underbrace{\underbrace{\overset{\infty}{\text{Udiam}}(\mathfrak{L}_{t}))^{r}}_{i=1} I B C \bigcup_{i=1}^{\infty} B_{i}, \operatorname{diam}(B_{i}) < \delta \bigvee_{i=1}^{\infty} B_{i}$$

where o_{ir} is the Lebesgue measure of the unit ball in R^r if r is an integer, any positive constant otherwise. We also denote by $7i^{\bullet}(B)$ the cardinality of B. The Hausdorff dimension of B is defined by

$$H - \dim(B) = \inf\{r > 0 : H^r(B) = 0\}.$$

For an introduction to the properties of Hausdorff measures see for example [10], [17]. We merely recall that H^r is a Borel regular measure in \mathbb{R}^n , and

(1.1)
$$n'(B) < +00 => W^{m}(B) = 0 V m > r.$$

We now recall the definition of semi-convexity and the main properties of semi-convex functions.

DEFINITION 1.2. Let ft C Rⁿ be an open convex set, and u: ft -+ R. We say that u is semi-convex in ft if there is a non decreasing upper semicontinuous function u: $[0, +\infty[-> [0, +\infty[$ such that $v(\theta) = 0$ and

(1.2)
$$tu\{x_t\} + (1 - t)u\{x_2\} - ufo\} \ge -t(1 - t)|_{X_t} - x_2|_{u>(|x_t - x_2|)}$$
$$x_t = txa + (1 - t)x_2, xi, x_2 \in ft, t \in [0,1].$$

We call semi-convexity modulus of u the least function u) satisfying (1.2). If u: ft —» R is semi-convex and $x \in \text{ft}$, we say that $p \in \mathbb{R}^n$ is a subgradient of u at x if

$$\liminf_{y\to x}\frac{u(y)-u(x)-\langle p,y-x\rangle}{|y-x|}\geq 0.$$

Borrowing the notation of convex analysis, we denote by du(x) the set of subgradients of u at x, call it the subdifferential of u at x. It is easy to see that du(x) is a compact, nonempty, convex set. Moreover,

(1.3)
$$pedu(x)$$
 <=> $u(y)-u(x)\sim (p,y-x) \ge -|y-x|w(|y-x|)$, Vj/€ ft.

It can also be shown that du(x) is a singleton if and only if u is differentiable at x. Hence, the set of non differentiability points of u can be classified according to the dimension of the subdifferential at the singular point.

DEFINITION 1.3. Let $x \in \{1, \text{ and let } k \in \{0, \dots, n\}$ be an integer. We define

$$S^k(u) = \{x \in \text{ft} : dim(du\{x)) = \text{fc}\},$$

and

$$\mathfrak{L}^*(\mathbf{u}) = |\mathbf{J} \, \mathbf{S}^* \mathbf{u}) = \{ x \in \mathbf{ft} : \dim(5\mathbf{u}(\mathbf{x})) > \mathbf{ib} \}.$$

In order to find sufficient conditions for the propagation of singularities, it will be useful to consider the set V*u(x) of reachable subgradients.

DEFINITION 1.4. Let $u: \mathrm{ft} \twoheadrightarrow \mathrm{R}$ be a semi-convex function, and let $x \in \mathrm{G}$ ft. We define

$$V+u(x) = \{ lim Vu(x_h) : x_h \in 5^{\circ}(tx), x_h -+ x \}.$$

Then, it is known that du(x) is the convex hull of V«,u(x) (see e.g. [4]).

In the following theorem we list some basic properties of semi-convex functions. We recall (see [3]) that a set-valued map S(x) is said to be upper semicontinuous if the following implication holds:

$$Ph' \in S(x_h), \quad x_h \rightarrow x, \quad PH - P = pe S(x).$$

THEOREM 1.1. Let u : ft = R be a semi-convex function. Then,

(1) u is locally Lipschitz continuous in ft and

$$\frac{du}{\partial a}(\dot{x}) = h\dot{m} - \frac{u(x + t\delta) - u(x)}{-t} = \max\{\langle p, \theta \rangle : p \in \partial u(x)\}$$

for any z G it and any 6 G $R^n \setminus \{0\}$.

- (2) Tie set-valued maps du(x), V*tz(z) are upper semicontinuous in x.
- (3) If z G $S^k(u)$, then !*+! (V.u(x)) = $\delta u(x)$.
- (4) For any $k \in \{0, ..., n\}$ and any p > 0 we have

$$T(Sj(u),z) C [du(x)]^{\pm} Vz G Sj(u),$$

where $S_{\rho}^{k}(u)$ denotes the set of all points $x \in S^{k}(u)$ such that du(x) contains a fc-dimensional ball of radius ρ .

(5) For any integer $k \in \{0,...,n\}$ the set $S^k(u)$ is countably $H^n \sim^k$ -rectifiable, that is it can be covered, up to a $\%^n \sim^k$ -negligible set, with a countable sequence of C^l hypersurfaces $Th \in \mathbb{R}^n$ of dimension (n-k), i.e.

$$n^{n-k}(s^k(u)||Jr_h|| = 0.$$

Moreover,

$$\int_{s^k(u)nn^f} H^k\{du(x)\}d'H^{n-k}(x) < +oo$$

for any open set Q^r CC ft.

Proof. (1) See [1] and [4].

- (2) The upper semicontinuity of the map du(x) easily follows by (1.3), and the upper semicontinuity of V*rx(x) follows directly from its definition.
- (3) Since V*u(x) is closed and its convex hull equals 9u(x), the assertion follows by Carathéodory's Theorem.
 - (4) See [1], Theorem 3.1.
 - (5) See [1], Theorem 4.1. |

REMARK 1.1. Note that (5) provides an upper bound on the Hausdorff dimension of $S^k(u)$, which is not greater than (n-k). It is easy to see that this bound is optimal. Indeed, let

$$u(x_u, x_n) = |x_1| + ... + |x_k|.$$

Then, $S^k(u)$ is the (n - fc)-plane of all $x \in \mathbb{R}^n$ such that z, = 0 for $1 \le i \le k$.

2. EXPOSED FACES AND REACHABLE SUBGRADIENTS

We want to study the structure of the singular set $E^{1}(u)$ in the neighborhood of a singular point z.

DEFINITION 2.1. We define the singularity degree of $z \in G^{-k}(u)$ as the unique integer k such that $z \in G^k(u)$. We say that z is an isolated singularity of degree k if $T(E^{fc}(tx),z) = 0$. We say that a singularity propagates if

$$\mathbf{T}\big(\Sigma^1(u),x\big)\neq\emptyset.$$

Moreover, all vectors $6 e T(E^1(u),x)$ n $dB \setminus are$ called directions of propagation of the singularity at x.

Clearly, a convex function may well have an isolated singularity of degree n. Indeed, if $x \in S\pounds(u)$ for some p > 0, then du(x) contains an n-dimensional ball. Hence, by Theorem 1.1, x is not a cluster point of $S^{\wedge}(u)$. In other words, $S^{\otimes}(u)$ is a discrete set for any p > 0. Moreover, there are convex functions with isolated singularities of degree < n.

EXAMPLE 2.1. Let

$$u(x_1,\ldots,x_n) = \sqrt{(x_1^2+\ldots+x_k^2)+(x_{k+1}^4+\ldots+x_n^4)}.$$

Then, u is a convex function in \mathbb{R}^n and $u \in \mathbb{C}^2(\mathbb{R}^n \setminus \{0\})$. On the other hand, $du(0) = [-1,1]^* \times \{0\}^n \sim^*$, so that 0 is the only point in $S^k(u)$.

Note that, in the above example du(0) = V*u(0). More generally, we will show that a sufficient condition for the propagation of a singularity of degree k < n at x is the strict inclusion V+u(x) du(x). In particular, this condition is satisfied for solutions of some Hamilton-Jacobi equations, see §3.

In the remainder of this paper we always assume that $Q \subset \mathbb{R}^n$ is a convex open set, $u:Q\longrightarrow \mathbb{R}$ is a semi-convex function, and u(t) is the semi-convexity modulus of u. Since our statements are local, we assume that u is Lipschitz continuous in Q and we denote by $[u]_{u_p}$ its Lipschitz semi-norm.

We will see that the directions of propagation of singularities are related to the geometry of the subdifferential du(x) at the starting point x. To analyze the singular directions we introduce the following sets.

DEFINITION 2.2. Let $x \in \text{fi}$ and $0 \in dBi \setminus \text{we set}$

$$du\{x,6\} = \{pe\ du(x) : \langle p,0\rangle = \frac{du}{\partial a}(x) = \max_{q \in \hat{a}u\{x\}} \langle q,\theta\rangle\},$$

$$\mathbf{V.u}(\mathbf{x},0) = \{ \lim_{n \to \infty} Vu(x_h) : x_h \in S^{\bullet}(u) \setminus \{x\}, \ x_h \twoheadrightarrow x, \ \frac{x_h - x}{|x_h - x|} \to \theta \}.$$

The collection $\{0tt(x,0) : ^G dBi\}$ consists of all the exposed faces of the convex set du(x). The following theorem is the basis of our singularity propagation argument (see Theorem 2.2 and Theorem 2.3).

THEOREM 2.1. Let $x \in £2,p \in \mathbb{R}^n$ and sequences $x^{\wedge} - x, du(xh) \ 3 \ Ph \longrightarrow P \ be$ given. Suppose that

$$\lim_{x \to +\infty} \frac{Xh X}{|X_h - X|} = 0$$

Tien, p G du(x,6). In particular,

$$V+u(x,6) \subset du(x,8)$$
.

Conversely, for any $p \in \partial u(x, \theta)$ there are sequences $x_h \to x$ satisfying (2.1), and $\partial u(x_h) \ni p_h \to p$.

Proof. We have to show that $\partial_* u(x,\theta) = \partial u(x,\theta)$, where

$$\partial_* u(x,\theta) = \big\{ \lim_{h \to +\infty} p_h : p_h \in \partial u(x_h), \, x_h \neq x, \, x_h \to x, \, \frac{x_h - x}{|x_h - x|} \to \theta \big\}.$$

Let p_h , x_h be as in the definition of $\partial_* u(x, \theta)$ and set

$$t_h = |x_h - x|, \qquad p = \lim_{h \to +\infty} p_h.$$

We know, by the upper semicontinuity of $\partial u(x)$, that $p \in \partial u(x)$. We will now show that $p \in \partial u(x, \theta)$. Indeed, by the semi-convexity of u we have

$$u(x) - u(x_h) - \langle p_h, x - x_h \rangle \ge -t_h \omega(t_h).$$

Devide both sides by t_h to obtain

$$\langle p_h, \frac{x_h - x}{t_h} \rangle \ge \frac{u(x + t_h \theta) - u(x)}{t_h} + \frac{u(x_h) - u(x + t_h \theta)}{t_h} - \omega(t_h).$$

Since

$$\frac{|u(x_h) - u(x + t_h \theta)|}{t_h} \le [u]_{\text{Lip}} \left| \frac{x_h - x}{t_h} - \theta \right| \to 0,$$

by letting $h \to +\infty$ we get

$$\langle p, \theta \rangle \geq \frac{\partial u}{\partial \theta}(x).$$

Thus, $p \in \partial u(x, \theta)$ and $\partial_* u(x, \theta) \subset \partial u(x, \theta)$.

Next, we proceed to show the reverse inclusion. Let us denote by d the dimension of $\partial u(x,\theta)$. Since θ is orthogonal to $\partial u(x,\theta)$, d is strictly less than n. We may assume that d>0, the inclusion being trivial if $\partial u(x,\theta)$ is a singleton.

Since $\partial_* u(x,\theta)$ is compact, it suffices to show that $p \in \partial_* u(x,\theta)$ for any $p \in \text{Int}(\partial u(x,\theta))$, the relative interior of $\partial u(x,\theta)$.

Let θ_i , $1 \leq i \leq (n-d)$ be an orthonormal basis of $[\partial u(x,\theta)]^{\perp}$, i.e.,

$$\langle \theta_i, \theta_j \rangle = \delta_{ij}, \qquad \langle (p-q), \theta_i \rangle = 0 \quad \forall p, q \in \partial u(x, \theta).$$

We can also take θ_1 to be equal to θ . For r, t > 0 satisfying the condition $t\sqrt{1+r^2} < \text{dist}(x,\partial\Omega)$, let y(r,t) be a minimizer of the function

$$u(x+t(\theta_1+y))-t\langle p,y\rangle$$

in the compact set K_r defined by

$$K_r = \{ y \in \mathbf{R}^n : \langle y, \theta_i \rangle = 0 \ \forall i = 1, \dots, (n-d), \ |y| \le r \}.$$

We claim that for any r > 0 there is r > 0 (depending on r) such that for t < r any minimizer y(r,t) satisfies the condition $|y(r,\pounds)| < r$. Indeed, if the claim were not true it would be possible to find r > 0 and a sequence of minimizers $y_h = y(r,th) \in K_r$ n dB_r corresponding to an infinitesimal sequence th- Passing to a subsequence, we may assume that y^{\wedge} converges to $y \in K_r$ n dB_r . Since yh is a minimizer, we have

$$u(x+t_h(\theta_1+y_h))-t_h(p,y_h)\leq u(x+t_h\theta_1).$$

Hence,

$$\frac{u(x+th(Oi+Vh))-u(x)}{-u(x+thg_1)-u(x)} < \langle p, y_h \rangle.$$

Recalling that

$$\left|\frac{u(x+t_h(\theta_1+y_h))-u(x+t_h(\theta_1+y))}{t_h}\right| \leq [u]_{Lip}|y_h-y| \to 0,$$

we obtain

(2.2)
$$\frac{\partial u}{\partial (\theta_1 + y)}(x) - \frac{\partial u}{\partial \theta_1}(x) \le \langle p, y \rangle.$$

On the other hand, since the map (-,0i) is constant on du(x,0), we have that du/ddi(x) = (p,0i). Also, since $p \in Int(du(0))$,

$$\frac{du}{\partial(\theta_1 + y)} \mathbf{r}(x) > (\mathbf{p} + \mathbf{e})/(0\mathbf{i} + \mathbf{v}) = (\mathbf{p}, 0\mathbf{i}) + (Piv) + \mathbf{cr}^2$$

for |e| sufficiently small. We thus obtain a contradiction with (2.2), and the claim is proved.

Now, let r > 0 and let r(r) > 0 be given by the claim. Returning to the definition of $y(r, \pounds)$, by the nonsmooth Lagrange multiplier rule (see for instance [6], 6.1.1) we conclude that for any $t \in]0,r(r)[$ we can find Aj(r,i) 6 R satisfying

0 e
$$t\{du(x + i\{9i + y\{r,t\})) - p\} - \sum_{i=1} \lambda_i(r,t)\theta_i$$

or, equivalently,

(2.3)
$$p + \sum_{i=1}^{n-a} \frac{\lambda_i(r,t)}{t} \theta_i \in \partial u (x + t(\theta_1 + y(r,t))).$$

Let (r_h) C]0,+oo[and $t_h \in]0,r(r/_1)[$ be two sequences converging to 0. By taking scalar products in (2.3) with 6i it is easy to see that $\langle i(rh,t_h)\rangle \langle t_h \rangle$ is not greater than $2[u]_{\underline{\mathbf{l}}_{\mathbf{p}}}$. Hence, by passing to a subsequence if necessary, we may assume that $(r_h,t_h)/th$ converges to $A^{\overline{*}}$ as $h \to +\infty$ for $i=1,\ldots,(n-d)$.

Then, by letting $h \longrightarrow +00$ in (2.3) we get

$$p + \sum_{i=1}^{n-d} \bar{\lambda}_i \theta_i \in \partial_+ u(x, \theta_1),$$

as $|y\{r_h,t_h\}| < r_h$. Moreover,

$$\lim_{h\to+\infty}\frac{\theta_1+y(r_h,t_h)}{|\theta_1+y(r_h,t_h)|}=\theta_1.$$

On the other hand, since the vectors 0^* are orthogonal to $du(x,6)_y$ all A^* are equal to 0. Thus, $p \in 5^*u(x,0i)$ and the proof of the theorem is complete.

THEOREM 2.2. Let $x \in \text{ft,0} \in \&\text{Bi}$, and an integer $m \in [1,n]$ be given. Then,

(2.4)
$$J_m(V^*Tz(x,0)) \# du(x,0) = 0 \in Tan(E^m(tz),x).$$

Moreover, < 2tz(x,0) = co(V*tz(x,0)).

REMARK 2.1. In particular, if $V^*u(x,0)$ ^ du(x,0), then 0 is a direction of propagation of the singularity at x. Moreover, (2.4) provides a lower bound on the degree of the singularity near x. Indeed, in view of definition 1.1, (2.4) implies that $0 \in T(E^m(u),x)$, where $m = m(V^*u(x,0))$. Hence, there are singular points of degree m near x, along the direction 0.

Proof of Theorem 2.2. Let p G $du\{x,6\} \setminus J_m(V,tz(x,0))$. We argue by contradiction. So, suppose that $0 \land T(E^m(tt),x)$. By Theorem 2.1, there are a sequence $(x_h) \cap T(x)$, and vectors $x_h \cap T(x)$ and

$$\lim_{/\mathbf{i}^{-*}+\mathbf{00}} ph = \mathbf{p}, \quad \lim_{/\mathbf{i}^{-*}+\mathbf{00}} Xh = \mathbf{x}, \quad \lim_{/\mathbf{i}^{-*}+\mathbf{c}^{*}} \mathbf{j}^{\frac{\mathbf{x} \cdot \mathbf{y}_{i} \cdot \cdots \cdot \mathbf{x}}{\mathbf{y}_{i} \cdot \cdots \cdot \mathbf{x}}} \mathbf{r} = \mathbf{0}.$$

By our assumption, for h large enough x^{\wedge} does not belong to $S^{m}(u)$. Hence, the dimension of $^{\wedge}(x^{\wedge})$ does not exceed m-1. By Theorem 1.1(3), there are vectors $Pi,h \in V_{< i}i6(x/i)$ and non negative real numbers $A^{\wedge}h$ such that

(2.5)
$$p_h = {}^{\wedge} \lambda_{i,h} p_{i,h}, \qquad \sum_{i,h}^{m} \lambda_{i,h} = 1.$$

By passing to a subsequence, we may assume that for any i the m-tuples $\lambda_{i,h}$ converge as $h \longrightarrow +00$ to A^* and pi,h converge to p^* as $h \longrightarrow +cx$). Since $p^* G ^*(x_h)$ a diagonal argument shows that $p^* \in V^*(x,0)$. Now, let $i \longrightarrow +00$ in (2.5) to obtain

$$p = \sum_{i=1}^{m} \lambda_i p_i, \qquad \sum_{i=1}^{m} \lambda_i = 1.$$

Hence, $p \in J_m(V \ll t(x, ^))$ and this contradiction proves (2.4).

Finally, a similar argument (with m = n+1) shows that each vector $p \in du(x, 6)$ is the convex combination of at most (n + 1) points of V*tx(x, 0).

Note that (2.4) implies that x is only a cluster point of $E^m(u)$. However, we will show that, under suitable assumptions, there is a whole continuum of singular points near x, whose size can be estimated from below.

Let 5 be any plane in \mathbb{R}^n passing through the origin, and let *ns* be the orthogonal projection on 5. For any 7 > 0 we denote by $C_{\nu}(S)$ the cone

$$C_{y}(S) = \{xeR^{n}: |irs(*)| \leq I|\pi_{S^{\perp}}(x)|\}.$$

We note that $C_7(5)$ D 5^X and $C_{\nu}(S)$ approaches 5^X as 7 -> 0+.

THEOREM 2.3. Let $S^k(u)$ with $\underline{J} < k < \underline{n-1}$ begiven. Set m = m(V * ti(*) J - Tien,

(2.6)
$$T(5T*,z) D [^(x)]^{*1}$$
.

Inaddition, we have

(2.7, Ihninf
$$\ll - *(* \bullet (\ll) \text{ n B M } 0 [x + C_{\gamma}(S)]) > 1$$

for any 7 > 0, where 5 is the k-plane parallel to du(x) and containing 0.

Proof Observe that c?i/(z,0) equals du(x) and V>u(x,5) C V*?x(x) for any $\theta \in [\partial u(x)]^{\perp}$. Hence, (2.6) follows from the previous theorem.

In order to simplify our proof of (2.7), we assume that x = 0. Since $E^m(u) = Q$ if m = 0, we may also assume that m > 0. Let us denote by S-¹ the unit sphere in S^{\perp} .

Let us pick a vector p in the set $du(0) \mid /_{m}(V^*u(0))$, which is not empty. For any 2 6 S 1 and any r, t > 0 we denote by y(r,t,z) a minimizer of the function u(tz + ty) - t(p,y) in the set

$$K_r = \{yeS: |y| < r\}.$$

We claim that for every r > 0 there is r(r) > 0 such that for any $t \in]0, r(r)[$ and any $z \in S^x$ any minimizer y(r,t,z) belongs to the (essential) interior of K_r . This claim can be proved as in Theorem 2.1. Indeed, suppose that the claim is not true. Then, there exist r > 0 and a sequence of minimizers $y^* = j/(r, t^*, z_h)$ $e \in K_rD$ dB_r corresponding to a sequence $th \longrightarrow 0$. Passing to a subsequence, we may assume that yh converges to $y \in K_r$ in dB_r and zh converges to $z \in S^x$. Since yh is a minimizer, we infer

$$u\{t_h z_h + 1/^{\wedge}\}$$
 - $^{\wedge}(\mathbf{p}, \mathbf{j}/\mathbf{f}\mathbf{c}) \leq u(t_h z_h)$.

Hence,

$$\frac{u(t_h z_h + t_h y_h) - \operatorname{tt}(0) \quad u(t_h z_h) - u(0)}{t_h} < (p, y_h).$$

Recalling that

$$\left| \frac{u\{t_h z_h + t_h y_h\} - u(t_h z + t_h y)}{t_h} \right| \leq [u]_{\text{Lip}} (|z_h - z| + |y_h - y|) \to 0,$$

and

$$\left|\frac{u(t_h z_h) - u\{t_h z\}}{th}\right| \leq [u]_{\operatorname{Lip}}|z_h - z| \to 0,$$

we obtain

(2.8)
$$\frac{du}{\partial(z+y)}(0) - \frac{\partial u}{\partial z}(0) \le \langle p, y \rangle.$$

On the other hand, since the map (\bullet, z) is constant on \bullet)u(O), we have that

$$\frac{du}{Tz}(0) = \langle p, z \rangle.$$

Also, since $p \in Int(\dot{c}hz(0))$,

$$\frac{du}{d(z+y)}(0) \ge (p+\epsilon y, z+y) = O(v^z) + \langle p, y \rangle + \epsilon r^2$$

for |e| sufficiently small. We thus obtain a contradiction with (2.8), and the claim is proved.

Next, we claim that there is 6 > 0 such that if r < 6 and $t < \inf\{r(r), 5\}$, then for any $z \in S^{-1}$, any minimizer 2/(r,t,z) satisfies the condition

$$tz + ty(r, t, z) \in \Sigma^{m}(u).$$

Indeed, let us assume that the claim is not true. Then, by the variational argument used in the proof of Theorem 2.1, we construct a sequence of minimizers $y_h = y(rhith,Zh) \in K_{rh}$ corresponding to sequences r&, $th \longrightarrow 0$ and real constants $A^*,i,...,^*h_v n-k$ such that

$$(2.9) p_h := p + \sum_{i=1}^{n-fc} \lambda_{h,i} \theta_i \in \partial u(t_h z_h + t_h y_h),$$

$$(2.10) t_h z_h + t_h y_h \notin \Sigma^m(u),$$

and

$$\lim z_h = 2 6 S^1$$
, $\lim A_{/1i} = A_i e R Vi = 1,... An - k$.

Passing to the limit as h = * + oo in (2.9) we get

$$p+\sum_{i=1}^{\text{n-fc}}\lambda_i\theta_i\in\partial u(0).$$

Hence $A^* = 0$ for any $i = 1, \ldots$, (n - fc) and p_h converges to p as $h \to +00$. Moreover, by (2.10) and Theorem 1.1(3) each vector p_h belongs to the convex hull of at most m vectors of V+u(xh). Repeating the argument of Theorem 2.2 we obtain a set $A \subset V^*u(0)$ consisting of at most m points, such that $p \in co(A)$. Hence, $p \in I_m(V^*u(0))$, and this contradiction proves the second claim.

Finally, let 6>0 be given by the second claim. For any fixed 7>0 let $r<\inf\{7,\pounds\}$. Then,

$$\Sigma^m(u)\cap C_{\gamma}(S)\cap B_{\rho}\supset \big\{tz+ty(r,t,z):z\in \mathbf{S}^{\perp},\ 0\leq t<\frac{\rho}{\sqrt{1+r^2}}\big\}$$

provided $p < y/\overline{1 + r^2}$ inf $\{r(r), 6\}$. Since $7Ts \pm$ does not increase the Hausdorff measure (see for instance [17], Proposition 3.5), by the inclusion

TT_xX (E^m(tx) n C₇(5) n B_p) D {ze S[±] : |z| <
$$\frac{\rho}{\sqrt{1+r^2}}$$
}

we infer

$$\liminf_{\rho\to 0^+}\frac{\pi^{n-n}(\Sigma^{n}(u)\cap D_\rho\cap D_\gamma(S))}{\omega_{n-k}\rho^{n-k}} + r^2\big)^{(k-n)/2}.$$

By letting r - *0, we complete the proof.

REMARK 2.2. By (1.1) and Theorem 1.1(5) we infer that $Ti^{n+k}(S^i(u)) = 0$ for any $i \ge k+1$. Hence, (2.7) can be written in the equivalent form: for any $x \in S^k(u)$

$$\liminf_{\rho\to\infty} \frac{\mathbf{f}}{\sum_{i=m}^{\infty}} \frac{(-\mathbf{M}^{\wedge}(-\mathbf{n}\mathbf{B},(\mathbf{x})\mathbf{n}[\mathbf{x}+\mathbf{c}_{7}(5)])}{\sum_{i=m}^{\infty}} > 1,$$

where $m = m(V_{1||}ix(x))$. In particular, if $7^{(V*u(x))} 7^{du(x)}$ (i.e., m = A:), we get

$$\liminf_{\rho \to 0^+} \frac{\mathcal{H}^{n-k} \left(S^k(u) \cap B_{\rho}(x) \cap \left[x + C_{\gamma}(S) \right] \right)}{\omega_{n-k} \rho^{n-k}} \ge 1,$$

and coupling this estimate with Theorem 1.1(5) we conclude that H— $dim(S^k(u)) = (\mathbf{n-fc})$.

3. HAMILTON-JACOBI EQUATIONS

In this section we will apply the general results on the singularities of semiconvex functions to solutions of the Hamilton-Jacobi-Bellman equation

(3.1)
$$F(y,u(y),\nabla u(y))=0, \quad y\in\Omega$$

where $Q \subset \mathbb{R}^{\wedge}$ is an open domain. We will assume that

(3.2)
$$F: flx Rx R^{\wedge} - R$$
 is continuous;

(3.3)
$$p *+ F(t/, s,p)$$
 is convex in $K^N V(j/, s) \in ft \times R$;

$$(3.4)$$
 n is semi-concave (i.e. $-u$ is semi-convex);

$$(3.5)$$
 (3.1) holds at any differentiability point of u .

We note that, for a semi-concave function u, the interesting semidifferential is the so-called superdifferential, defined as

$$d^+u(y) = \{p \in \mathbb{R}^\wedge : \underset{*_- *y}{\operatorname{limsup}} \underbrace{u(z) - u(y) - \langle p | z - y \rangle}_{|^z \sim y|} \le 0\}.$$

Equivalently, d+u(y)=-d[-u](y). Hence, $d+u(y) ^ 0$ for any $y \in Q$ and the following implication holds

(3.6)
$$du(y) \# 0 = u$$
 is differentiate at j/.

Accordingly, the definitions 1.2 and 2.2 will be modified as follows for a semi-concave function u:

$$S^{k}(u) = \{x \ 6 \ n : dim(d+u\{x)) = fc\},$$

 $E^{*}(u) = \{J^{(u)} = \{x \ 6fi : dim(0+u(x)) > fc\},$

$$\partial^+ u(x,\theta) = \{ p \in \partial^+ u(x) : \langle p,\theta \rangle = \frac{\partial u}{\partial x}(x) = \min \langle q,\theta \rangle \}.$$

REMARK 3.1. From (3.2)-(3.5) it follows that u is a viscosity solution in the sense of [8] (see also [7]). Indeed, (3.2) and (3.5) yield

(3.7)
$$F(*,u(y),p)=0 \ Vp \in V.u(j/)$$

for any y 6fi, and so (3.3) implies that

$$F(y,u(y),p) \leq 0 \quad \text{Vp } \in \#^+u(j/).$$

The converse inequality on the elements of du(y) trivially follows by (3.6).

REMARK 3.2. Semi-concavity is a natural property to expect on viscosity solutions of Hamilton-Jacobi-Bellman equations. Indeed, several existence and uniqueness results were first obtained in classes of semi-concave functions (see [15]). More recently, H-J equations have been studied in the framework of viscosity solutions (see [8] and [7]). Under suitable regularity assumptions on F and on the (Dirichlet) boundary data, viscosity solutions to (3.1) are known to be semi-concave (see [16] and [12]). Similar results are also available for viscosity solution of second order H-J equations, see [13]; hence the result of §2 apply to these equations as well. For the sake of simplicity we confine our statements to first order equations.

For any compact convex set $C \subset \mathbb{R}^n$ we denote by $\operatorname{Ext}(\mathbb{C})$ the set of extreme points of C. We say that a set $A \subset \mathbb{R}^n$ is extremal if no p 6 A can be written as a convex combination of other points of A, i.e.

$$p \not\in co(A \setminus \{p\}) \quad \mathbf{Vp} \not\in A.$$

Our terminology is motivated by the following result.

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LEMMA 3.1. Any compact extremal set A coincides with Ext(co(A)).

Proof. Let C = co(A), and let $p \in Ext(C)$. By Carathéodory's Theorem, we can represent p as a convex combination of (N+1) points $pi \in A$:

$$P = \underbrace{\prod_{i=1}^{N+l} A^* P^{*t}}_{Ai} \quad Ai > 0.$$

Since p is an extreme point of C, p = pi for any $\mathfrak{t} \in \{1, \dots, 7^{n} + 1\}$, hence p 6 A

$$p = \sum_{i=1}^{N+l} \lambda_i Pt$$
, $A <> 0$, $\sum_{i=1}^{N+l} \lambda_i = 1$.

In turn, each p^* can be represented as a convex combination of at most (N+1) points $pij \in A$:

$$p_i = \sum_{j=1}^{N+l} \lambda_{ij} p_{ij}, \quad \lambda_{ij} > 0, \quad \sum_{j=1}^{N+l} \lambda_{ij} = 1,$$

so that

$$P = \sum_{i,i=1}^{N+l} A A^{p}.$$

Since -4 is extremal, p = ptj for any t, j, hence $p = p^{\wedge} \in Ext(C)$.

The main result of this section is the following.

THEOREM 3.2. Assume (3.2), (3.3), (3.4), (3.5), and let $x \in S^k(u)$ be a singular point. Let us further assume that

$$(3.8) \{p \in \mathbb{R}^N : F(y,u(y),p) = 0\} \text{ is extremal.}$$

Then

(1) V+u(y) = Ext(d*u(y)), and if k < N the singularity propagates. Moreover $m = m(V*u(y)) \ge 1$, and

$$(3.9) T(\Sigma^{m}(u), y) \supset [\partial^{+}u(y)]^{\perp}, \liminf_{\rho \to 0^{+}} \frac{\mathcal{H}^{n-k}(\Sigma^{m}(u) \cap B_{\rho}(y))}{\omega_{n-k}\rho^{n-k}} \geq 1.$$

(2) Let $0 \in \&$ Bi and let us assume that $5^+M(J/,6)$ is not a singleton. Then, V*u(y,6) coincides with Ext(d+u(y,6)), $m = m(V*u(y,6)) \ge 1$ and $6 \in T(\Sigma^m(u), y)$.

Proof. (1) By (3.7) and (3.8), $V^*u(y)$ satisfies the hypotheses of Lemma 3.1, so that $V^*u(y) = Ext(d+u(y))$. To show (3.9), we need only to apply Theorem 2.3 to -u.

(2) As in (1), Lemma 3.1 yields $V+u(y,d)=\mathrm{Ext}(\mathrm{d}+\mathrm{u}(y,0))$. The other statements follow from Theorem 2.2 and Remark 2.1.

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REMARK 3.3. The extremality condition (3.8) cannot be dropped. In fact, let N=2 and $u(y_uy_2)=-y/yj+y*<>_2$ as in example 2.1. Then, u is concave in \mathbb{R}^2 , and has an isolated singularity at (0,0). Moreover, it is a viscosity solution of the equation

$$\sqrt{y_2^2 u_{y_1}^2 + \frac{1}{4} u_{y_2}^2} = |y_2|.$$

REMARK 3.4. The condition (3.8) is trivially satisfied if

$$p \gg F(y, s, p)$$
 is strictly convex in R^{\(\delta\)} $V(y, s)$ G ft x R.

Theorem 3.2 also applies to nonstationary H-J equations with strictly convex Hamiltonian. Infact, let N = n + 1, $y = (\pounds, x)$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and $p = (pt, p_x) \in \mathbb{R}$ $x \in \mathbb{R}^n$. Let

$$H((t,x),s,p_x):\Omega\times\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}$$

be a continuous function, strictly convex in p_x . Then,

$$F(y,s,p) = p_t + H((t,x),s,p_x)$$

satisfies (3.2) and (3.3), and any semi-convave function $u: ft \longrightarrow \mathbb{R}$ satisfying (3.5) is a viscosity solution of the equation

$$(3.10) u_t + H(t, x, u, \nabla u) = 0.$$

Finally, for any $y \in ft$ and any $s \in \mathbb{R}$

$$Z(y,s) = \{ (p_u p_x) \ G \ R \ x \ R^n : p < + H(y,s,p_x) = 0 \}$$

is extremal, because of the strict convexity of H. Indeed, let

$$Z(y,s)\ni p=\sum_{i=1}^{N+1}\lambda_i p_i$$

with $pi \ e \ Z(y, 5)$, Aj > 0 and $S \wedge {}^{1}A^{*} = 1^{*} \wedge d {}^{*et \ us \ s}h^{\circ w}$ that p = p for any i. Since

$$p_t + H(y,s,p_x) < 53(A_tp_{it} + A^{(j/5,p_{ix})}\dot{J} = 0$$

unless $p_x = pi_X$ for any i G $\{1, \ldots, TV + 1\}$, we have

$$pa = -H(y,s,p_{ix}) = -H(y,s,p_{x}) = p_{t}$$
 Vi G {1,...,JV + 1}

and, in particular, p = Pi for any i.

More generally, the same argument of Theorem 3.2 shows that singularities propagate in the direction 6 if $d \sim v(y, 6)$ is not a singleton and if the restriction of $F(yi^u(y)i^m)^{-1/2} + w(j/0)$ is strictly convex, so that $m(V^*u(y,0)) \geq 1$.

REMARK 3.5. In Theorem 3.2(1) it is necessary to assume that x is not a singularity of degree N. In fact, u(y) = -|y| is a solution of the eikonal equation $|Vu(y)|^2$ -1 = 0, and the singularity in the origin does not propagate.

However, propagation of singularities of any degree has been proved for nonstationary H-J equations with strictly convex Hamiltonian (see [4]). Due to the special structure of the equation it has been shown in [5] that for any singularity y there is at least a direction $6 \in dBi$ such that du(y, 6) is not a singleton. Note that, once the existence of such a direction has been proved, the propagation of the singularity would follow by Theorem 3.2(2).

In [5] it is also shown that viscosity solutions of (3.10) with strictly convex H are such that any $p \in V^*tt(y)$ is exposed, i.e., there exists $\delta \in dB \setminus t$ such that $d+u(y,0) = \{p\}$. This condition is stronger than extremality.

REMARK 3.6. We note that the lower bound in Theorem 3.2 on the maximum degree of the singularity near y depends only on the geometry of d+u(y). To illustrate this phenomenon, we now discuss three examples. In the first example the subdifferential d+u(y) is a triangle in \mathbb{R}^3 and the singularity propagates in singularities of degree two, as implied by Theorem 3.2.

In the second example we show that a singularity y of degree k may well propagate in singularities of degree m < k when $m\{V^*u(y)\}$ < k.

Finally, the third example shows that Theorem 3.2 provides only a sufficient condition for the propagation of singularities of high degree.

EXAMPLE 3.1. Let $fi = R^3$ and let

$$u(t,x,z)=\min\{t,x,z\}.$$

Then, u is a viscosity solution of the equation -Ut + H(Vu) = 0, where

$$H(p_x,p_z) = (p \ll -p_z)^2 + 2\{p_x + p_z - I\}^2 - 1$$

is strictly convex. We note that $S^2(u)$ is equal to the line spanned by (1,1,1) and

$$\nabla_{\bullet} u(s, s, s) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \qquad \forall s \in \mathbf{R}.$$

In this case $m(V^*u(0,0,0)) = 2$. We note that $S^l(u)$ consists of three halfplanes intersecting each other in the above singular line, with directions orthogonal to the triangle generated by $V^*u(0,0,0)$. This example describes the typical situation analyzed in Theorem 3.2.

EXAMPLE 3.2. Let $u : \mathbb{R}^3$ -» \mathbb{R} be the function

$$u(t,x,z) = -\sqrt{x^2 + (|z| + t^2)^2}.$$

The equality

$$V\overline{a^2 + /3^2} = \sup\{aa + bp : a \ge 0, b \ge 0, a^2 + b^2 < 1\}$$
 $a, /3 \ge 0$

implies that $y/(p^2+i>^2)$ is a convex function whenever cp and V are non negative convex functions. In particular, u is a concave function. The origin belongs to $S^2\{u\}$ and

$$9^+u(0,0,0) = \{0\}xBi, \qquad m\{\{0\}\ x\ dBi\} = I.$$

The singularity in the origin propagates in singularities of degree 1. In fact, the origin is the only point in $S^2(u)$, $S^1(u) = \{(t, x, 0) : t \neq 0\}$ and

$$\partial^+ u(t,x,0) = \left\{ \left(\frac{-2t^3}{u(t,x,0)}, \frac{-x}{u(t,x,0)}, \frac{t^2 \rho}{u(t,x,0)} \right) : |\rho| \le 1 \right\} \qquad \forall (t,x,0) \in S^1(u).$$

Finally, we note that

$$\nabla u(t,x,z) = \left(\frac{-2t\left(|z|+t^2\right)}{u(t,x,z)}, \frac{-x}{u(t,x,z)}, \frac{-y\left(|z|+t^2\right)}{|z|u(t,x,z)}\right) \qquad \forall (x,z,t) \in S^0(u),$$

so that u is a solution of the equation (3.1) with

$$F(t, x, z, p_t, p_x, p_z) = -p_t + |p_x|^2 + |p_z|^2 - 1 + \frac{2t(|z| + t^2)}{\sqrt{x^2 + (|z| + t^2)^2}}.$$

The function F satisfies (3.2), (3.3) and the extremality condition (3.8).

EXAMPLE 3.3. Let $\Omega = \mathbb{R}^3$, y = (t, x) with $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$. The function

$$u(t,x) = \begin{cases} t/2 - |x| - 1 & \text{if } |x| + 2 \ge t; \\ \\ \frac{|x|^2}{2(2-t)} & \text{if } |x| + 2 < t \end{cases}$$

is a viscosity solution of the equation $-u_t + |\nabla u|^2/2 = 0$. We note that $(2,0) \in S^3(u)$, and

$$\nabla_{\mathbf{x}} u(2,0) = \{(p_t, p_x) \in \mathbf{R} \times \mathbf{R}^2 : |p_x| \le 1, \ p_t = \frac{|p_x|^2}{2}\}.$$

Moreover, $S^2(u)$ is the halfline (t,0) with t < 2. The unit vector $\theta = (-1,0)$ belongs to $T(S^2(u), (2,0))$ even though $m(\nabla_* u((2,0),\theta)) = 1$.

REFERENCES

- [1] G. Alberti, L. Ambrosio, P. Cannarsa, On the singularities of convex functions, Manuscripta Mathematica (to appear).
- [2] L. Ambrosio, Su alcune proprietà delle funzioni convesse, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. (1992) (to appear).
- [3] J.P. AUBIN, H. FRANKOWSKA, Set-Valued Analysis, Birkhäuser, Boston, 1990.
- [4] P. CANNARSA, H.M. SONER, On the singularities of the viscosity solutions to Hamilton Jacobi Bellman equations., Indiana Univ. Math. J., 36 (1987), pp. 501-524.
- [5] P. CANNARSA, H.M. SONER, Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications, Nonlinear Anal., 13 (1989), pp. 305-323.
- [6] F.H. CLARKE, Optimization and Nonsmooth Analysis, Wiley & Sons, New York, 1983.
- [7] M.G. CRANDALL AND L.C. EVANS AND P.L. LIONS, Some properties of viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Mat. Soc., 282 (1984), pp. 487-502.
- [8] M.G. CRANDALL AND P.L. LIONS, Viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. mat. Soc., 277 (1983), pp. 1-42.

- [9] I. EKELAND, R. TEMAM, Convex Analysis and Variations! Problems, North-Holland, Amsterdam, 1976.
- [10] H. FEDERER, Geometric Measure Theory, Springer Verlag, Berlin, 1969.
- [11] W.H. FLEMING, The Cauchy problem for a nonlinear first order partial differential equation, J. Differential Equations, 5 (1969), pp. 515-530.
- [12] H. ISHII, Uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations, Indiana Univ. Math. J., 33 (1984), pp. 721-748.
- [13] H. ISHII, P.L. LIONS, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, J. Differential equations, 83 (1990), pp. 26-78.
- [14] R. JENSEN AND P.E. SOUGANIDIS, A regularity result for viscosity solutions of Hamilton-Jacobi equations in one space dimension, Trans. Amer. Math. Soc, 301 (1987), pp. 137-147.
- [15] S.N. KRUZKOV, Generalized solutions of Hamilton-Jacobi equations of eikonal type I, Math. USSR Sbornik, 27 (1975), pp. 406-446.
- [16] P.L. LIONS, Generalized solutions of Hamilton-Jacobi equations, Pitman, Boston, 1982.
- [17] F. MORGAN, Geometric Measure Theory A beginner's guide, Academic Press, Boston, 1988.
- [18] R.T. ROCKAFELLAR, Convex Analysis, Princeton University Press, Princeton, 1970.
- [19] L. SIMON, Lectures on Geometric Measure Theory Proceedings of the Centre for Mathematical Analysis, Australian National University, Camberra, 1983.
- [20] L. TONELLI, Lezioni di Analisi Matematica, vol. I, Tacchi Editrice, Pisa, 1940.
- [21] R. TSUJI, Formation of singularities for Hamilton-Jacobi equations II, J. Math. Kyoto Univ., 26 (1986), pp. 299-308.

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