A Linear-time Algorithm to Compute a MAD Tree of an Interval Graph

Elias Dahlhaus
Department of Computer Science
University of Bonn
Bonn, Germany

Peter Dankelmann*
School of Mathematical and Statistical Sciences
University of Natal
Durban, South Africa

R. Ravi†
Graduate School of Industrial Administration
Carnegie Mellon University
Pittsburg (PA), USA

Abstract

The average distance of a connected graph G is the average of the distances between all pairs of vertices of G. We present a linear time algorithm that determines, for a given interval graph G, a spanning tree of G with minimum average distance (MAD tree). Such a tree is sometimes referred to as a minimum routing cost spanning tree.

Keywords: Graph Algorithms, Interval Graphs, Spanning Tree.

1 Introduction

The *average distance* $\mu(G)$ of a finite, connected graph $G = (V, E)$ is the average over all unordered pairs of vertices of the distances,

$$\mu(G) = \binom{|V|}{2}^{-1} \sum_{\{u,v\} \subset V(G)} d_G(u, v),$$

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where $d_G(u,v)$ denotes the distance between the vertices u and v. A *minimum average distance spanning tree* of G (*MAD tree* in short) is a spanning tree of G of minimum average distance. MAD trees, also referred to as minimum routing cost spanning trees, are of interest in the design of communication networks [6]. One is interested in designing a tree subnetwork of a given network, such that on average, one can reach every node from every other node as fast as possible. In general, the problem of finding a MAD tree is NP-hard [6]. A polynomial-time approximation scheme is due to [1]. Hence it is natural to ask for which restricted graph classes a MAD tree can be found in polynomial time. In [3], an algorithm is exhibited that computes a MAD tree of a given distance-hereditary graph in linear time. In [5] it is shown that a MAD tree of a given outerplanar graph can be found in polynomial time.

In this paper, we show that for a given interval graph G a MAD tree can be computed in time $O(|E|)$. If the interval representation of G is known and the left and right boundaries of the intervals are ordered, then a MAD tree of G can be found in time $O(|V|)$. In section 2, we present some structural results on MAD trees. In section 3, we sketch the algorithm, and in section 4, we give some concluding remarks.

2 Structure of MAD Trees in Interval Graphs

We always assume that the graph under consideration is connected. The *distance* of a vertex v, $d_G(v)$, is defined as $\sum_{x\in V} d_G(v,x)$. The *total distance* of a graph H denoted $d(H)$ is the sum of all pairwise distances between nodes in H, i.e. $d(H) = \sum_{\{u,v\}\subset V(H)} d_H(u,v)$.

A *median vertex* of G is a vertex c for which $d_G(c)$ is minimum. The *eccentricity* of a vertex v is defined as $ex_G(v) = \max_{w\in V} d_G(v,w)$.

The neighbourhood of a vertex v in G is defined in two ways: the *open* version $N_G(v) = \{u : uv \in E\}$ and the *closed* version $N_G[v] = \{v\} \cup N_G(v)$.

The following lemma applies not only to interval graphs but to all connected graphs. Part (i) improves on a result in [2], which states that, if T is a MAD tree of a given connected graph G, then there exists a vertex c in T such that every path in T starting at c is induced in G. We now prove that c can be chosen to be a median vertex of T.

Lemma 1 *(i) If T is a MAD tree of G and c is a median vertex of T then every T-path starting at c is an induced path in G (i.e. has no diagonals in G).*
(ii) T and c can be chosen such that there is no vertex $c' \neq c$ such that $N_G[c]$ is strictly contained in $N_G[c']$.

Proof: (i) Let $P = v_1, \ldots, v_k$ be a path in T with $c = v_1$. Suppose P is not induced in G. Then there are i and j, such that $i < j-1$ and $v_iv_j \in E(G)$. Let T_1 and T_2 be the connected components of $T - v_{j-1}v_j$ containing v_{j-1} and v_j, respectively. It is known (see [2]) that $d_T(c) \leq d_T(v_i) < d_T(v_{j-1})$. Since the vertices in T_2 are further away (in T) from v_i than from v_{j-1}, we obtain

$$d_{T_1}(v_i) < d_{T_1}(v_{j-1}).$$

Consider the tree

$$T' = T - v_{j-1}v_j + v_iv_j.$$

Since the distances between any two vertices are the same in T and T', unless one vertex is in T_1 and the other vertex is in T_2, we have

$$d(T') - d(T) = |V(T_2)|(d_{T_1}(v_i) - d_{T_1}(v_{j-1})) < 0,$$

contradicting the minimality of $d(T)$. Hence P is an induced path in G.

(ii) If there is a vertex c' with $N_G[c] \subset N_G[c']$ and $N_G[c] \neq N_G[c']$, then joining all T-neighbours of c to c' instead of c yields a spanning tree T' with $d(T') \leq d(T)$, in which c is an end vertex and c' is a median vertex. QED

From now on we assume that G is an interval graph. Each vertex v corresponds to an interval $I(v) = [l(v), r(v)]$, such that two vertices v and w are adjacent if and only if $I(v) \cap I(w) \neq \emptyset$.

Proposition 1 *Let $v_0, \ldots, v_k$ be an induced path of the interval graph G. Then*

1. *$l(v_1) < l(v_2) < \ldots < l(v_k)$ and $r(v_0) < \ldots < r(v_{k-1})$ or*
2. *$r(v_1) > \ldots > r(v_k)$ and $l(v_0) > \ldots > l(v_{k-1})$.*

In the first case, we call such a path an R-path, in the second case, we call it an L-path.

Consequently, each path of the MAD tree T of G starting at a median vertex c of T is an L-path or an R-path.

For a vertex v of G let $h(v)$ be a neighbour x of v such that $r(x)$ is maximum; If $r(v) = \max_{w \in V} r(w)$, we say $h(v)$ is undefined. Similarly, let $k(v)$ be a neighbour y of v such that $l(y)$ is minimum. Again, if $l(v) = \min_{w \in V} l(w)$, we say that $k(v)$ is undefined. We also define $h^0(v) = k^0(v) = v$ for all v. For $i \geq 2$, $h^i(v)$ is defined as $h(h^{i-1}(v))$. Analogously, we define $k^i(v)$.

A component of a graph is *trivial* if it contains only one vertex, otherwise it is called nontrivial.

For a given vertex v of G and an integer $i \geq 2$ let $V_i^R(v)$ ($V_i^L(v)$) be the set of all vertices at distance i from v, whose intervals lie completely to the right (left) of the interval of v. We also let $V_1^R(v) = V_1^L(v) = N_G(v)$.

Theorem 1 *If G is an interval graph then there is a MAD tree T of G with a median vertex c, such that for each i, $1 \leq i \leq ex_G(c)$,*

$$(*) \begin{cases} \text{each } v \in V_i^R(c) \text{ is adjacent in } T \text{ to } h^{i-1}(c) \\ \text{each } v \in V_i^L(c) \text{ is adjacent in } T \text{ to } k^{i-1}(c). \end{cases}$$

Proof. Let T be a MAD tree of G and let c be a median vertex of T, where c is chosen according to Lemma 1(ii).

By Lemma 1(i), each G-neighbour v of c is also in T adjacent to c, since otherwise the $c - v$ path in T would have a diagonal. Hence $(*)$ holds for $i = 1$.

Suppose that $(*)$ does not hold for some i. Let i be the smallest such number. For the rest of this proof, we omit the reference to the median vertex c in the notation for V^R's.

We first show that only one vertex in V_{i-1}^R has T-neighbours in V_i^R. Suppose that there exist vertices $v_i, v_i' \in V_i^R$, $v_{i-1} \neq v_{i-1}' \in V_{i-1}^R$ such that $v_i v_{i-1}, v_i' v_{i-1}' \in E(T)$. Since, by the minimality of i, v_{i-1} and v_{i-1}' are adjacent in T to $h^{i-2}(c)$, we have $v_i \neq v_i'$ (Else there would be a cycle in T). Without loss of generality, we may assume that $r(v_{i-1}) \geq r(v_{i-1}')$. Let B be the component of $T - v_{i-1}v_i$ not containing c and let B' be the component of $T - h^{i-2}(c)v_{i-1}'$ not containing c. Moreover let

$$child(v_{i-1}') = \{w \in B' \mid v_{i-1}'w \in E_T\}.$$

Note that $child(v_{i-1}')$ is exactly the set of children of v_{i-1}' when T is rooted at c. Now $child(v_{i-1}') \subseteq N_G(v_{i-1})$ since every child of v_{i-1}' is on an R-path from c and $r(v_{i-1}) \geq r(v_{i-1}')$. Therefore,

$$T' = T - \{v_{i-1}'w \mid w \in child(v_{i-1}')\} + \{v_{i-1}w \mid w \in child(v_{i-1}')\}$$

is a spanning tree of G. Since the distance between any two vertices are the same in T and T', unless one of the vertices is in B and the other one is in B', the difference between the total distances is

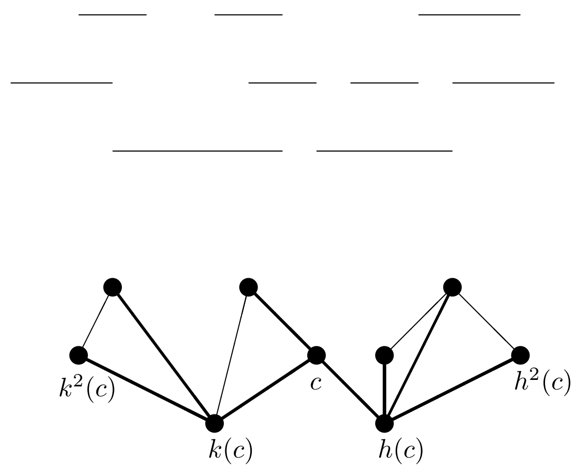
$$d(T') - d(T) = -2|V(B)|(|V(B')| - 1) < 0$$

contradicting the minimality of $d(T)$. Hence at most one vertex v in V_{i-1}^R has T-neighbours in V_i^R. Hence only one vertex in V_{i-1}^R has T-neighbours in V_i^R. Since each vertex in V_i^R that is adjacent in G to a vertex v in V_{i-1} is also adjacent in G to $h^{i-1}(c)$, we can join each vertex in V_i^R not to v, but to $h^{i-1}(c)$ without increasing the total distance of T. Analogously, we can achieve that each vertex in V_i^L is adjacent in T to $k^{i-1}(c)$. Hence T satisfies condition $(*)$. QED

Theorem 1 suggests the following polynomial time algorithm. Fix a vertex c of G, determine $h^i(c)$ and $k^i(c)$ for $i = 1, 2, \ldots$ and construct a spanning tree of G in which each vertex in $V_i^R(c)$ is adjacent to $h^{i-1}(c)$ and each vertex in $V_i^L(c)$ is adjacent to $k^{i-1}(c)$ for $i \geq 1$. Construct such a tree for each $c \in V(G)$. Among those n trees select a tree with minimum total distance. By Theorem 1, this tree is a MAD tree of G.

Theorem 2 *A MAD tree of an interval graph can be determined in polynomial time.*

A set of intervals and the corresponding interval graph G is shown below. The vertices c, $h^i(c)$ and $k^i(c)$, $i = 1, 2$, are labelled. Thick lines indicate the edges of a MAD tree T satisfying the condition $(*)$ of Theorem 1.



3 Linear-time Computation of a MAD Tree

We assume that an interval representation of $G = (V, E)$ is known and that the left borders $l(v)$ and right borders $r(v)$, $v \in V$ are sorted. As in the previous section, we assume that $h(v)$ is a neighbour x of v, such that $r(x)$ is maximum and that $k(v)$ is a neighbour y of v, such that $l(y)$ is minimum. In [7] it is shown that h and k can be determined in time linear in the number of vertices (even in logarithmic time in parallel with a linear workload).

We consider any vertex v and assume that the median vertex c is left (right) of v, i.e., that $r(c) \leq r(v)$ ($l(c) \geq l(v)$).

Let T be a MAD tree according to Theorem 1 rooted at c and let $v \neq c$ be a vertex, $v \in V_i^R(c)$, say. Consider T_v, the subtree of T rooted at v. Then either v is a leaf of T and T_v is trivial, or $v = h^i(c)$ for some i. In that case, T_v contains some G-neighbours of v, in particular $h(v) = h^{i+1}(c)$, and all vertices in $V_{i+2}^R(c) = V_2^R(v)$ (which are in T adjacent to $h^{i+1}(c) = h(v)$), $V_{i+3}^R(c) = V_3^R(v)$ (which are in T adjacent to $h^{i+2}(c) = h^2(v)$), and so on, as long as they are defined. The fact that the main part of T_v, namely $T_{h(v)}$, only depends on whether v is to the left or right of c, but not on the actual choice of c, is the key to our algorithm.

Definition 1 *Let G be a connected interval graph with h and k as defined above, and $v \in V(G)$. If $h(v)$ ($k(v)$) is not defined then let T_v^R (T_v^L) be the empty tree. If $h(v)$ is defined then let T_v^R be the tree with vertex set $\{h(v)\} \cup \bigcup_{j\geq 2} V_j^R(v)$, where a vertex $x \in V_j^R(v)$ is adjacent to $h^{j-1}(v)$. Analogously, if $k(v)$ is defined then let T_v^L be the tree with vertex set $\{k(v)\} \cup \bigcup_{j\geq 2} V_j^L(v)$, where a vertex $x \in V_j^L(v)$ is adjacent to $k^{j-1}(v)$. Note that both T_v^R and T_v^L do* not *contain v!*

Hence the tree $T = T_c$ consists of c as root, the neighbours of c in the original graph G as neighbours in T_c, and T_c^L and T_c^R appended on $k(c)$ and $h(c)$ respectively. We do not determine these trees T_v^R and T_v^L explicitly. Instead, we compute the following quantities.

1. the number of neighbours $num^R(v)$ ($num^L(v)$) of $h(v)$ ($k(v)$) that are not neighbours of v, i.e. the number of children of $h(v)$ ($k(v)$) in T_v^R ($T^L(v)$),
2. $|T_v^R|$ ($|T_v^L|$), the number of vertices of the tree T_v^R (T_v^L),
3. the total distance $t^R(v)$ ($t^L(v)$) of T_v^R (T_v^L), and
4. the distance $d^R(v)$ ($d^L(v)$) of $h(v)$ ($k(v)$) in T_v^R (T_v^L).

The numbers $num^L(v)$ and $num^R(v)$ can be determined overall in $O(n)$ time (see, e.g. [7]). For any particular v, $d^R(v)$ can be determined in $O(1)$ time if $d^R(h(v))$ and $num^R(v)$ are known. Also $t^R(v)$ can be determined in $O(1)$ time if $d^R(v)$, $num^R(v)$ and $t^R(h(v))$ are known. Analogous statements hold for the left counterparts.

In more detail, we proceed as follows.

Determine $num^R(v)$: If $h(v)$ is not defined then $num^R(v) = 0$. Otherwise, $num^R(v)$ is the number of vertices w, such that $r_v < l_w \leq r_{h(v)}$. This can be determined in overall linear time.

Determine $|T_v^R|$: If $h(v)$ is not defined, $|T_v^R| = 0$. Otherwise,

$$|T_v^R| = |T_{h(v)}^R| + num^R(v).$$

Determine $d^R(v)$: If $h(v)$ is not defined then $d^R(v) = 0$. Otherwise

$$d^R(v) = d^R(h(v)) + |T_{h(v)}^R| + num^R(v) - 1.$$

Determine the distance $t^R(v)$: If $h(v)$ is not defined then $t^R(v) = 0$. Otherwise

$$t^R(v) = t^R(h(v)) + d^R(v)$$
$$+(num^R(v) - 1)\Big((num^R(v) - 2) + 2|T^R(h(v))| + d^R(h(v))\Big).$$

Determine the total distance of T_c: First we determine the degree $\delta(c)$ of c. This can be done in overall time $O(n)$, for all c (One has to count the number of right borders between the left border l_c and the right border r_c of c and the number of intervals passing the right border of c).Then we define $n(c)$ to be the total number of neighbours of c excluding itself as well as $h(c)$ and $k(c)$ if they are defined.

The total distance of T_c is

$$(n(c))^2 + t^L(c) + t^R(c) + (1 + 2n(c) + d^L(c))|T^R(c)|$$
$$+(1 + 2n(c) + d^R(c))|T^L(c)| + 2|T_c^L||T_c^R| + (d^L(c) + d^R(c))(1 + n(c)).$$

Thus we compute the numbers $t^R(v)$ $(t^L(v))$, $d^R(v)$ $(d^L(v))$, $|T_v^L|$ $(|T_v^R|)$, and $num^R(v)$ $(num^L(v))$ in overall linear time.

We obtain the total distances and thus the average distances of the trees T_c with assumed median vertices c in overall linear time, and we only have to select one T_c with minimum average distance. We determine this particular T_c explicitly using Theorem 1. Also this can be done in linear time.

Theorem 3 *If an interval representation of an interval graph G with n vertices is given and the left and the right boundaries $l(v)$ and $r(v)$ are sorted then a MAD tree of G can be determined in $O(n)$ time.*

4 Conclusions

It remains an open problem to decide whether there exists a polynomial time algorithm to find a MAD tree of a *vertex weighted* interval graph. If w is a real valued weight function on the vertex set of G, then the average distance of G with respect to w is defined (see [4]) as

$$\binom{w(V)}{2}^{-1} \sum_{\{u,v\} \subset V(G)} w(u)w(v)d_G(x,y),$$

where $w(V)$ is the total weight of the vertices in G.
Theorem 1 does not hold (and hence the algorithm presented above does not work) if there is a weight function on the vertices of G. To see this, consider the path with four vertices $v_1, \dots, v_4$ of weight $1, 2, 1, 5$ together with a vertex v_5 of weight 0, that is adjacent to $v_1, \dots, v_4$. An interval representation of this graph is as follows: $v_1 : [0,1]$, $v_2 : [0,3]$, $v_3 : [2,5]$, $v_4 : [4,5]$, $v_5 : [0,5]$. This graph has a unique MAD tree, the path v_1, v_2, v_3, v_4, v_5. It is easy to check that T does not contain a vertex satisfying the conclusion of Theorem 1.

New techniques seem necessary to solve the edge-weighted counterpart of the MAD tree problem for Interval graphs. It would also be interesting to know whether our algorithm can be extended to the class of strongly chordal graphs.

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