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A REMARK ON THE REGULARITY OF SOLUTIONS OF MAXWELL'S EQUATIONS ON LIPSCHITZ DOMAINS

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Abstract

Let $\mathbf{u}$ be a vector field on a bounded Lipschitz domain in $\mathbb{R}^3$, and let $\mathbf{u}$ together with its divergence and curl be square integrable. If either the normal or the tangential component of $\mathbf{u}$ is square integrable over the boundary, then $\mathbf{u}$ belongs to the Sobolev space $H^1_{\text{loc}}$ on the domain. This result gives a simple explanation for known results on the compact embedding of the space of solutions of Maxwell's equations on Lipschitz domains into $L^2$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with connected Lipschitz boundary $\Gamma$. This means that $\Gamma$ can be represented locally as the graph of a Lipschitz function. For properties of Lipschitz domains, see [7], [3], [2]. In particular, $\Gamma$ has the strict cone property.

We consider real vector fields $\mathbf{u}$ on $\Omega$ satisfying in the distributional sense

$$
\mathbf{u} \in L^2(\Omega); \quad \text{div} \mathbf{u} \in L^2(\Omega); \quad \text{curl} \mathbf{u} \in L^2(\Omega).
$$

We denote the inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$.

It is well known that functions $\mathbf{u}$ satisfying (1) have boundary values $\mathbf{n} \times \mathbf{u}$ and $\mathbf{n} \cdot \mathbf{u}$ in the Sobolev space $H^{1/2}(\Gamma)$ defined in the distributional sense by the natural extension of the Green formulas

$$
\begin{align*}
\langle \text{curl} \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{w}, \text{curl} \mathbf{u} \rangle &= \langle \mathbf{n} \times \mathbf{u}, \mathbf{v} \rangle \quad (\text{2}) \\
\langle \text{div} \mathbf{u}, \phi \rangle + \langle \mathbf{w}, \text{grad} \phi \rangle &= \langle \mathbf{n} \cdot \mathbf{u}, \phi \rangle \quad (\text{3})
\end{align*}
$$

for all $\mathbf{v} \in H^1(\Omega)$.

Here $\mathbf{n}$ denotes the exterior normal vector which exists almost everywhere on $\Gamma$, and $\langle \cdot, \cdot \rangle$ is the natural duality in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ extending the $L^2(\Gamma)$ inner product.
It is known that for smooth domains (e.g., $\Gamma \in C^{1,1}$), each one of the two boundary conditions
\[ \vec{n} \times \vec{u} \in H^{1/2}(\Gamma) \quad \text{or} \quad \vec{n} \cdot \vec{u} \in H^{1/2}(\Gamma) \] (4)
implies $\vec{u} \in H^1(\Omega)$, see [2] and, for the case of homogeneous boundary conditions, [6], where one finds also a counterexample for a non-smooth domain. Such counterexamples are derived from non-smooth weak solutions $v \in H^1(\Omega)$ of the Neumann problem ($\partial_n := \vec{n} \cdot \text{grad}$ denotes the normal derivative)
\[ \Delta v = g \in L^2(\Omega); \quad \partial_n v = 0 \quad \text{on} \ \Gamma \] (5)
If $\vec{u} = \text{grad} v$, then $\vec{u}$ satisfies (1) and $\vec{n} \cdot \vec{u} = 0$ on $\Gamma$, and $\vec{u} \in H^s(\Omega)$ if and only if $v \in H^{1+s}(\Omega)$. For smooth or convex domains, one knows that $v \in H^2(\Omega)$. If $\Omega$ has a nonconvex edge of opening angle $\alpha \pi$, $\alpha > 1$, then, in general, the solution $v$ of (5) is not in $H^{1+s}(\Omega)$ for $s = 1/\alpha$, hence $\vec{u} \notin H^s(\Omega)$. This upper bound $s$ for the smoothness of $\vec{u}$ can be arbitrary close to $1/2$.

Regularity theorems for (1), (4) have applications in the numerical approximation of the Stokes problem [2] and in the analysis of initial-boundary value problems for Maxwell’s equations [6]. The compact embedding into $L^2(\Omega)$ of the space of solutions of the time-harmonic Maxwell equations is needed for the principle of limiting absorption. This compact embedding result was shown by Weck [10] for a class of piecewise smooth domains and by Weber [9] and Picard [8] for general Lipschitz domains. In these proofs, no regularity result for the solution $\vec{u}$ was used or obtained. See Leis’ book [6] for a discussion.

In this note, we use the result by Dahlberg, Jerison, and Kenig [4], [5] on the $H^{3/2}$ regularity for solutions of the Dirichlet and Neumann problems with $L^2$ data in potential theory (see Lemma 1 below). Together with arguments similar to those described by Girault and Raviart [2], this yields $\vec{u} \in H^{1/2}(\Omega)$ (Theorem 2). The compact embedding in $L^2$ is an obvious consequence of this regularity. If instead of Lemma 1, one uses only the more elementary tools from [1], one obtains $H^{3/2-\epsilon}$ regularity for solutions of the Dirichlet and Neumann problems in potential theory and, consequently $\vec{u} \in H^{1/2-\epsilon}(\Omega)$ for any $\epsilon > 0$. This kind of regularity is also known for the case of an open manifold $\Gamma$ (screen problem). It suffices, of course, for the compact embedding result.

The proof of the following result can be found in [4].

**Lemma 1.** (Dahlberg-Jerison-Kenig) Let $v \in H^1(\Omega)$ satisfy $\Delta v = 0$ in $\Omega$. Then the two conditions

\[ \begin{align*}
(i) \quad v |_\Gamma & \in H^1(\Gamma) \quad \text{and} \\
(ii) \quad \partial_n v |_\Gamma & \in L^2(\Gamma)
\end{align*} \]

are equivalent. They imply $v \in H^{3/2}(\Omega)$.  

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Remarks.

a.) The first assertion in the Lemma goes back to Nečas [7].

b.) There are accompanying norm estimates, viz.
There exist constants $C_1, C_2, C_3$, independent of $v$ such that

$$C_1 \| \partial_n v \|_{L^2(\Gamma)} \leq \| \vec{n} \times \text{grad} v \|_{L^2(\Gamma)} \leq C_2 \| \partial_n v \|_{L^2(\Gamma)},$$

$$\| v \|_{H^{3/2}(\Omega)} \leq C_3 \| v \|_{H^1(\Gamma)}.$$  

c.) The boundary values are attained in a stronger sense than the distributional sense (2), (3), namely pointwise almost everywhere in the sense of nontangential maximal functions in $L^2(\Gamma)$.

Theorem 2. Let $\tilde{u}$ satisfy the conditions (1) in $\Omega$ and either

$$\vec{n} \times \tilde{u} \in L^2(\Gamma)$$

or

$$\vec{n} \cdot \tilde{u} \in L^2(\Gamma).$$

Then $\tilde{u} \in H^{1/2}(\Omega)$.

If (1) is satisfied, then the two conditions (6) and (7) are equivalent.

Proof. The proof follows the lines of [2]. It is presented in detail to make sure that it is valid for Lipschitz domains.

Let $\tilde{f} := \text{curl} \, \tilde{u} \in L^2(\Gamma)$. Then $\text{div} \, \tilde{f} = 0$ in $\Omega$.

According to [2, Ch. I, Thm 3.4] there exists $\tilde{w} \in H^1(\Omega)$ with

$$\text{curl} \, \tilde{w} = \tilde{f}, \quad \text{div} \, \tilde{w} = 0 \quad \text{in} \, \Omega.$$  

The construction of $\tilde{w}$ is as follows:

Choose a ball $\mathcal{O}$ containing $\overline{\Omega}$ in its interior and solve in $\mathcal{O} \setminus \overline{\Omega}$ the Neumann problem: $\chi \in H^1(\mathcal{O} \setminus \overline{\Omega})$ with

$$\Delta \chi = 0 \quad \text{in} \, \mathcal{O} \setminus \overline{\Omega}; \quad \partial_n \chi = \vec{n} \cdot \tilde{f} \quad \text{on} \, \Gamma; \quad \partial_n \chi = 0 \quad \text{on} \, \partial \mathcal{O}.$$  

Note that $\vec{n} \cdot \tilde{f} \in H^{-1/2}(\Gamma)$ satisfies the solvability condition $\langle \vec{n} \cdot \tilde{f}, 1 \rangle = 0$ because $\text{div} \, \tilde{f} = 0$ in $\Omega$.

Define $\tilde{f}_0 := \tilde{f}$ in $\Omega$, $\tilde{f}_0 := \text{grad} \, \chi$ in $\mathcal{O} \setminus \overline{\Omega}$, $\tilde{f}_0 := 0$ in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. Then $\tilde{f}_0 \in L^2(\mathbb{R}^3)$ has compact support and satisfies $\text{div} \, \tilde{f}_0 = 0$ in $\mathbb{R}^3$. Therefore $\tilde{f}_0 = \text{curl} \, \tilde{w}$ for some $\tilde{w} \in H^1(\mathbb{R}^3)$ with $\text{div} \, \tilde{w} = 0$ in $\mathbb{R}^3$. One obtains $\tilde{w}$ for example by convolution of $\tilde{f}_0$ with a fundamental solution of the Laplace operator in $\mathbb{R}^3$ and taking the curl.
Thus (8) is satisfied. The function $\tilde{z} := \tilde{u} - \tilde{w}$ satisfies

$$\tilde{z} \in L^2(\Omega) \quad \text{and} \quad \text{curl}^* \tilde{z} = 0 \quad \text{in} \quad \Omega \quad \quad (10)$$

Since $\Omega$ is simply connected, there exists $v \in H^1(\Omega)$ with

$$\tilde{z} = \text{grad} v \quad \quad (11)$$

Then $v$ satisfies

$$Av = \text{div} u \quad \in L^2(\Omega) \quad \quad (12)$$

We can apply Lemma 1 to $v$, because by subtraction of a suitable function in $H^2(\Omega)$, we obtain a homogeneous Laplace equation from (12).

Now, since $\tilde{w} \nabla G \in L^2(\Omega)$, condition (i) in the Lemma is equivalent to

$$\tilde{n} \cdot \text{grad} v = \tilde{n} \cdot \tilde{z} = \tilde{n} \cdot \tilde{w} \quad \in L^2(\Gamma)$$

and hence to (6), and condition (ii) is equivalent to

$$\tilde{n} \cdot \text{grad} v = \tilde{n} \cdot \tilde{z} = \tilde{n} \cdot \tilde{w} \quad \in E \quad L^2(\Gamma)$$

and hence to (7). Therefore the Lemma implies that (6) and (7) are equivalent.

Also, $v \in H^{1/2}(\Omega)$ is equivalent to grad $G \in L^{1/2}(\Omega)$, hence to

$$\tilde{u} = \tilde{z} + \tilde{w} = \text{grad} v + \tilde{w} \quad \in L^{1/2}(\Omega) \quad \quad \Box$$

Remark. The accompanying norm estimates are:

There exist constants $C_1, C_2, C_3$, independent of $\tilde{u}$ such that

$$\|\tilde{n} \times \tilde{u}\|_{L^2(\Omega)} \leq C_1 (\|\tilde{u}\|_{L^2(\Omega)} + \|\text{curl}\tilde{u}\|_{L^2(\Omega)} + \|\text{div}\tilde{u}\|_{L^2(\Omega)})$$

$$\|\tilde{n} \cdot \tilde{u}\|_{L^2(\Gamma)} \leq \|	ilde{w}\|_{L^2(\Gamma)} + \|\text{curl}\tilde{u}\|_{L^2(\Gamma)}$$

$$\|\tilde{u}\|_{H^{1/2}(\Omega)} \leq C_3 (\|\tilde{w}\|_{L^2(\Omega)} + \|\text{div} \tilde{u}\|_{L^2(\Omega)} + \|\text{curl}\tilde{u}\|_{L^2(\Omega)}) \quad .$$

References


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