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ON CHAINS OF RELATIVELY SATURATED SUBMODELS OF A STABLE MODEL

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Abstract Let $M$ be a given model, we call $N \prec M$ relatively saturated iff for every $B \subseteq N$ of cardinality less than $|M|$ every type over $B$ which is realized in $M$ is realized in $N$. We discuss the existence of such submodels.

The following are corollaries of the existence theorems.
(1) If $M$ is of cardinality at least $2^{\omega_1}$, and fails to have the $\omega$ order property then there exists $N \prec M$ which is relatively saturated in $M$ of cardinality $2^{\omega_1}$.
(2) Let $T$ be a countable $L_{\omega_1, \omega}$ theory. If there exists an uncountable cardinal $\chi$ such that $I(\chi, T) < 2^\chi$ then every model $M \models T$ of cardinality greater or equal to $2^{\omega_1}$ has a relatively saturated submodel of cardinality $2^{\omega_1}$.

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Here we continue the study of stability inside a model (as started in [Sh2] and [Gr]) rather than of stability inside an arbitrarily large saturated model $\mathfrak{S}$ as in [Sh1]. In general a union of a chain of saturated models need not be saturated (see [AlGr]). However from Theorem III 3.11 of [Sh1] it follows that when a first order theory $T$ is stable then a union of a chain of cofinality greater or equal than $|T|^+$ of saturated models is saturated. Our goal here is to generalize this result, this is done in Theorem 2(2). This seems to be of greater interest than just a simple generalization of [Sh1] to this context. We hope that this may serve as a beginning of a classification theory for some non elementary classes.

A word on notation: Since our work will be carried out inside a given model $M$, when $A \subseteq M$ we use $S(A)$ to denote the set of types with parameters from $A$ realized in $M$. We follow the notation of [Gr]. $M$ has the $\mu$-order property iff there exists a formula $\varphi(x;y) \in L(M)$ and a subset of finite sequences of $M \{a_\alpha : \alpha < \mu\}$ satisfying:

$$\text{for every } \alpha, \beta < \mu \quad \alpha < \beta \iff M = \varphi[a_\alpha ; a_\beta].$$

Definition 1 We call $N \subseteq M \prec \kappa$ relatively saturated iff for every $B \subseteq N$ of cardinality less than $\kappa$ every type over $B$ which is realized in $M$ is realized in $N$. We denote this by $N < \kappa M$. When $N$ is $\mathfrak{N}$-relatively saturated we say that $N$ is a relatively saturated substructure of $M$. When $N$ is relatively $\kappa^+$ saturated we denote this by $N < \kappa^+ M$.

Our aim here is to find conditions on $M$, $\mathfrak{N}$, and $\kappa$ which will imply the existence of a relatively saturated submodel $N$ of the structure $M$.

Main Theorem 2 Let $M$ be a structure whose similarity type is of cardinality $\kappa$. 
(1) Suppose $M$ does not have the $oa$ order property.

(l.a) If $X$, and $j_1$ satisfy $X^+=X$ (and $j\upharpoonright X$) then $M$ has a submodel $N$ of cardinality $X$ which is $p^+$ relatively saturated.

(l.b) If $x<\text{cf}X$, $X=2^X$, and for every $i<\text{cf}X$ \(X_i^X=X_i\), then for every \(\{M'_i<r_1=i<\text{cf}X\}\) increasing chain of relatively saturated submodels such that $I\models I=\omega$: the model $M\upharpoonright U^\omega_{i<\text{cf}X}$ is relatively saturated in $M$.

(2) Suppose $M$ fail to have the $\text{a}>$ order property. Let $X$ be a cardinal such that $X^X=X$. if $\{M_j<r_1=j<i<\text{cf}X\}$ is an increasing chain of $X$-relatively saturated submodels, and $\text{cf}<\omega X$ then $U_{i<\text{cf}X}$ is a $X$-relatively saturated submodel of $M$.

(3) Let $T=\Gamma_{(0,\omega)}$, and $x=\omega_0$.\(\text{Suppose there exists an}\) uncountable cardinal $X$ such that $I(X,T)<2^X$. If $X$ satisfy $X=\omega_1$, then every model $N\models T$ of cardinality at least $X$ has a relatively saturated submodel of cardinality $X$.

It is not hard to derive the following corollary from parts (l.a),(l.b) and (3) of Theorem 2:

Corollary 3 Suppose that $M$ has a countable similarity type, and is of cardinality at least $^1\omega_1$.

(1) If $M$ does not have the $GJ$ order property then $M$ always has a relatively saturated substructure of cardinality $^1\omega_1$.

(2) If $\exists X\omega_0$ such that $\text{KX}(M)\omega_1<2^\omega$ then $M$ always has a relatively saturated substructure of cardinality $3\omega_1$.  

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Remarks

(1) There is a natural example showing that $\exists_{\omega_1}$ in Corollary 3 cannot be replaced by $\exists_{\omega}$, namely the assumption in the Main Theorem that $\text{cf}\lambda>|\text{Th}(M)| (=\kappa)$ is essential. For this see Theorem 2 of [AlGr] when $\text{Th}(M)$ is stable but not superstable.

(2) Corollary 3 can easily be generalized to uncountable similarity types.

(3) Let $\psi\in L_{\omega_1,\omega}^\lambda$ have countable similarity type. If $\psi$ has a model of cardinality $\exists_{\lambda^+}$ and there exists an uncountable $\chi$ such that $l(\chi,\text{Th}_{\omega_1,\omega}(M))<2^\chi$ then by Theorem 1.6(1) of [GrSh] there is no $\varphi(\chi;y)\in L_{\lambda^+,\omega}$ which has the $\exists_\delta$ order property for some $\delta<\lambda^+$. This can be used to repeat the relevant parts of [Gr] which are needed here, and by proving a version of the main Theorem we can get submodels which are strongly relatively saturated submodels i.e. the types can include also formulas from some countable fragment of $L_{\lambda^+,\omega}$.

(4) The last remark can be generalized to uncountable similarity types. Suppose $\psi\in L_{\kappa^+,\omega}$ when $L$ is of cardinality $\ll\kappa$. In this case in order to apply Theorem 4.2 of [GrSh] we should start with a model of cardinality $\exists_{\kappa^+}$ (for definition of $\delta(\lambda,\kappa)$ see Definition 4.1(1) of [GrSh]), and the fragment of $L_{\lambda^+,\omega}$ can be of cardinality $\kappa$.

We will make use of the following

Lemma 4 Suppose $B$ is a set of cardinality $\kappa$. Let $N\supseteq B$ satisfy $N<_{\kappa^+}\text{M}$, and let $C\subseteq M$ contain $N$. If $p_1,p_2\in S(C)$ both do not split over $B$, and $p_1|N=p_2|N$ then $p_1=p_2$.

Proof of 4 Follows easily from Exercise 1.2.3 of [Sh1].
Proof of the main Theorem (I.a) Since $\lambda^\mu=\lambda$ by Lemma 6(1) of [Gr] $M$ is stable in $\lambda$. Define by induction on $i<\mu^+$ $M_i$ increasing and continuous chain of submodels of $M$ of cardinality $\lambda$, such that $M_{i+1}$ contains realizations to every type from $S(M_i)$. Clearly (by regularity of $\mu^+$) $N:=\bigcup_{i<\kappa^+}M_i$ is as required.

(I.b) Let $\lambda_i$, $M'_i$, and $M'$ be as in the assumption. We will prove that $M'$ is a relatively saturated submodel of $M$. Let $A\subseteq M'$ be a given set such that $|A|\leq\lambda_i$, and suppose $p\in S(A)$, we will find below a finite sequence (in $M'$) realizing this type (in fact $\lambda^+$ many elements).

Let $a\in M$ be an element such that $p=tp(a,A)$. Define $p':=tp(a,M')$, since $cf\lambda>\kappa$ and $M'$ is a union of relatively saturated models we have that $M'<_{\kappa}M$. By Lemma 7(1) of [Gr] there exists $B\subseteq M'$ of cardinality at most $\kappa$ such that $p'$ does not split over $B$.

Since $cf\lambda>\kappa$ there exists $i<cf\lambda$ such that $B\subseteq M'_i$. Let $\lambda:=\lambda_i$, $N^*::=M'_i+1$. Define $\{N_i<N^*: i<\lambda^+\}$ increasing and continuous, and $\{a_i\in N_{i+1}: i<\lambda^+\}$ such that

(i) $N_0\supseteq B$,
(ii) $|N_i|=\lambda$,
(iii) $N_{i+1}<_{\kappa}N^*$,
(iv) $tp(a_i,N_i)=tp(a_i|N_i|)$, and
(v) $N_{i+1}\supseteq N_i\cup\{a_i\}$.

The construction can be carried out since $\lambda^\kappa=\lambda$, the relative saturation of $N^*$, and Lemma 6(1) in [Gr]. Let $N:=\bigcup_{i<\lambda^+}N_i$. (It is easy to check that $I:=\{a_i: i<\lambda^+\}$ is an indiscernible sequence over $N_i$.)
Notation 5 Let \( CQM, \) and \( I \subseteq M \) be a set of finite sequences (all of the same length).

\[
\text{Av}(I,C) := \{ \langle P(x;c) : c \in C, \text{there exists } J \subseteq I \text{ of cardinality less than the cardinality of } I \text{ satisfying: } \forall j \text{ there exists a sequence } a \in J \text{ such that } \forall a \in J = \exists \text{ Mt} \leq P[a;c] \}.
\]

Claim 6 For \( I \) as above, and every \( C \) of cardinality \( <X \), \( \text{Av}(I,C) \) is a complete type, realized by an element of \( I \).

Proof Let \( c \in C \) be given, consider \( q := \text{tp}(c,N) \). By Lemma 7(1) of [Gr] there exists \( B' \subseteq QN \) of cardinality at most \( x \), such that \( q \) does not split over \( B' \). There exists \( i < X^+ \) such that \( B' \subseteq QN_i \). Let \( J_c := \text{in} N_j \), we will show that if there exists a sequence \( a \in I \) such that \( \forall a \in J_c = \exists \text{ Mt} \leq P[a;c] \) then for every \( a \in J \Rightarrow \text{Mt} \leq P[a;c] \) for every \( P(x;c) \) be formula over \( C \) such that \( \text{Mt} \leq P[a;c] \) for some \( a \in J_c \). Let \( b \in J_c \) be an arbitrary sequence. We have that

\[
\text{Mt} \leq P[b;c] \iff \text{tp}(b,y) \leq \text{tp}(a,y)
\]

By the choice of \( I \), and Lemma 4 we have

\[
\text{tp}(a,N_j) = \text{tp}(b,N_j).
\]

However since \( q \) does not split over \( N_j \) certainly also \( \text{tp}(c,I) \) does not split over \( N_j \) (remember IQN). This together with (2) implies

\[
\text{tp}(b,y) \leq \text{tp}(a,y)
\]

Now (1) and (3) together imply what we wanted, namely:

\[
\text{Mt} \leq P[b;c] \iff \text{tp}(b,y) \leq \text{tp}(a,y)
\]

Since \( |I| = X^+ \) is a regular cardinal and greater than \( |C| \), \( J_i = \bigcup_{c \in C} J_c \) is as required, and \( \text{Av}(I,C) \) is realized by any element of \( I-J \).

\( \square \)

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Apply Lemma 6 to \( C=A \). Let \( \delta<\lambda^+ \) be such that \( J \) from Lemma 6 is included in \( N_\delta \). We may assume that \( N_\delta \) also contains \( B \) (the set from the beginning of the proof). Apply Lemma 6 again to \( C=A\cup N_\delta \).

Let \( \xi<\lambda^+ \) be such that the corresponding \( J \) is included in \( N_{\xi+1} \) and \( N_\delta \subseteq N_{\xi+1} \).

**Claim 7** \( \text{Av}(I,N_{\xi+1}\cup A)=\text{Av}(I,N_{\xi+1}\cup A) \).

**Proof** By Lemma 6 \( q:=\text{Av}(I,N_{\xi+1}\cup A) \) is a complete consistent (=realized in \( M \)) type over \( C:=N_{\xi+1}\cup A \). Since \( p'|C \) does not split over \( B \), and the choice of \( I \) (remember they all realize \( p'|N_{\xi+1} \)), by Lemma 4 it is enough to show that \( q \) does not split over \( B \).

Suppose \( c_{\xi}:=n_{\xi}^l \cdot a_{\xi}^l \) (\( l=1,2 \)) are such that \( \text{tp}(c_{1},B)=\text{tp}(c_{2},B) \), and \( \varphi(x;c_{1})\land\neg\varphi(x;c_{2})\in\varphi \); when \( n_{\xi}\in N_{\xi+1} \), and \( a_{\xi}^l \in A \). By the \( \kappa^+ \)-relative saturation of \( N_{\xi+1} \) there are \( a_{\xi}^l \in N_{\xi+1} \) such that \( \text{tp}(a_{\xi}^l, B\cup n_{\xi+1}\cup n_{\xi+2})=\text{tp}(a_{\xi}^l, B\cup n_{\xi+2}) \). Hence there are \( c_{\xi}'_{\xi}:=n_{\xi}^l \cdot a_{\xi}^l \in N_{\xi+1} \) realizing the same type over \( B \) such that \( \varphi(x;c_{1}')\land\neg\varphi(x;c_{2}')\in\varphi \)\( |N_{\xi+1} \), a contradiction to the fact that \( q|N_{\xi+1} \) does not split over \( B \).

This completes the proof of (1.b).

(2) Similar to (1.b).

(3) By [Sh3] (see also [GrSh]) If \( \exists \chi>\kappa \) such that \( I(\chi,T)<\aleph_1 \) then there exists a limit ordinal \( \delta<\omega_1 \) such that if \( M=\Gamma \) then \( M \) fail to have the \( \exists_\delta \)-order property. By our assumption on \( \lambda \) we have that \( \lambda=\omega_\delta \), we can now use Lemma 7(2) of [Gr] and repeat the argument of (1.b).

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REFERENCES


