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PRESENTED BY Emily Allen

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Irina Gheorghiciuc	May 13, 2013
MAJOR PROFESSOR	DATE
Thomas Bohman	May 13, 2014
DEPARTMENT HEAD	DATE

APPROVED BY THE COLLEGE COUNCIL

Frederik J. Gilman	May 13, 2014
DEAN	DATE

Combinatorial Interpretations of Generalizations of Catalan Numbers and Ballot Numbers

Emily Ann Allen
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Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213

Thesis Committee:
Tom Bohman (Committee Chair)
Irina Gheorghiciuc (Thesis Advisor)
James Cummings
Bogdan Ion (University of Pittsburgh)

Abstract

The super Catalan numbers $T(m,n) = (2m)!(2n)!/2m!n!(m+n)!$ are integers which generalize the Catalan numbers. Since 1874, when Eugene Catalan discovered these numbers, many mathematicians have tried to find their combinatorial interpretation. This dissertation is dedicated to this open problem.

In Chapter 1 we review known results on $T(m,n)$ and their q-analog polynomials. In Chapter 2 we give a weighted interpretation for $T(m,n)$ in terms of 2-Motzkin paths of length $m+n-2$ and a reformulation of this interpretation in terms of Dyck paths. We then convert our weighted interpretation into a conventional combinatorial interpretation for $m = 1, 2$. At the beginning of Chapter 2, we prove our weighted interpretation for $T(m,n)$ by induction. In the final section of Chapter 2 we present a constructive combinatorial proof of this result based on rooted plane trees.

In Chapter 3 we introduce two q-analog super Catalan numbers. We also define the q-Ballot number and provide its combinatorial interpretation. Using our q-Ballot number, we give an identity for one of the q-analog super Catalan numbers and use it to interpret a q-analog super Catalan number in the case $m = 2$.

In Chapter 4 we review problems left open and discuss their difficulties. This includes the unimodality of some of the q-analog polynomials and the conventional combinatorial interpretation of the super Catalan numbers and their q-analogs for higher values of m.

Contents

Chapter 1

Introduction

1.1 Super Catalan Numbers

The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

have over 66 combinatorial interpretations [29]. For example, they count the number of lattice paths from $(0,0)$ to (n,n) with unit up steps and unit right steps which never go below the line $y = x$ (see Fig. 1.1), the number of binary parenthesizations of a string of $n+1$ letters, the number of full binary trees with $n+1$ leaves, and the number of 123-avoiding permutations of $[n]$.

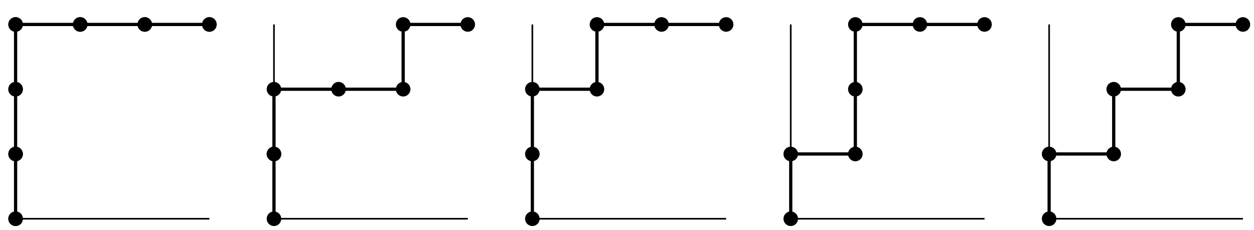


Figure 1.1: The five lattice paths from $(0,0)$ to $(3,3)$ with unit up and right steps.

As early as 1874 E. Catalan [8] observed that the numbers

$$S(m,n) = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}} = \frac{(2m)!(2n)!}{m!n!(m+n)!}$$

are integers with $S(1,n) = 2C_n$. In 1875, P. Bachmann [5] gave an algebraic proof showing $S(m,n)$ is an integer. The same year J. Bourguet proved that for $k \geq 2$ the numbers

$$S(m_1, m_2, \ldots, m_k) = \frac{(km_1)!(km_2)!...(km_k)!}{m_1!m_2!...m_k!(m_1 + m_2 + ... + m_k)!}$$

are integers [15]. The sequence $S(m,n)$ had seemingly been forgotten until 1910, when H.C. Feemster asked for a proof that $S(m,n)$ is an integer in The American Mathematical Monthly. The next year, G.E. Wahlin rediscovered Bachmann's proof and provided his own proof [16].

Here is a simple proof that was given in [1]. By the Landau criterion [26], the exact power of a prime p diving $n!$ is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

To prove that $S(m,n)$ is an integer, we will show that, for every prime p, the power of p which divides $m!n!(m+n)!$ is at most the power of p which divides $(2m)!(2n)!$. Equivalently,

$$\sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor + \left\lfloor \frac{n}{p^i} \right\rfloor + \left\lfloor \frac{m+n}{p^i} \right\rfloor \leq \sum_{i=1}^{\infty} \left\lfloor \frac{2m}{p^i} \right\rfloor + \left\lfloor \frac{2n}{p^i} \right\rfloor. \tag{1.1}$$

Since for $x, y \in \mathbb{R}$, $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$, Eq. 1.1 follows.

Interest in the sequence $S(m,n)$ in the modern era was reignited by Gessel [22]. He noted that, except for $S(0,0)$, the numbers $S(m,n)$ are even. In [23] Gessel and Xin refer to

$$T(m,n) = \frac{S(m,n)}{2} = \frac{(2m)!(2n)!}{2[m!n!(m+n)!]}$$

as the super Catalan numbers. Interpretations of $T(m,n)$ have been given for several values of m.

Pippenger and Schleich [28] interpreted $T(2,n)$ as the number of cubic plane trees in which every edge is drawn at an angle that is a multiple of $\pi/3$ from a reference line. In [30], Schaeffer defined a blossom tree to be a planted binary plane tree with n inner vertices of degree 3, $n+2$ leaves including the root vertex, and an associated orientation at the root (clockwise or counter-clockwise). He gave a combinatorial interpretation of $T(2,n)$ in terms of balanced blossom trees. Schaeffer's proof is related to combinatorial proofs of enumerating various families of planar trees.

Define a **Dyck path** of length $2n$ to be a path from $(0,0)$ to $(2n,0)$ in which the allowable steps are diagonally up and diagonally down and which never goes below the x-axis. Denote the set of Dyck paths of length $2n$ by $\mathcal{C}_n$. There is a natural bijection between the set of Dyck paths of length $2n$ and the set of lattice paths from $(0,0)$ to (n,n) with unit up steps and unit right steps which never go below the line $y=x$. Given a lattice path, rotate it 45 degrees clockwise to obtain a Dyck path. Define the **height** of a Dyck path π, denoted by $h(\pi)$, to be the maximum y-coordinate π obtains. Gessel and Xin gave the following combinatorial interpretation of $T(2,n)$ in [23].

Theorem 1 (Gessel, Xin)**.** *For $n \geq 1$, the number $T(2,n)$ counts the ordered pairs of Dyck paths (π,ρ) of total length $2n$ with $|h(\pi)-h(\rho)| \leq 1$. Here π and ρ are allowed to be the empty path. The height of the empty path is zero.*

The proof of Theorem 1 uses an inclusion-exclusion argument and an identity attributed to Dan Rubenstein

$$S(m+1,n) = 4S(m,n) - S(m,n+1). \tag{1.2}$$

In [23] Gessel and Xin also pointed out some properties that the Dyck paths counted by $T(3,n)$ have to satisfy.

Several identities on super Catalan numbers were provided by Gessel in [22]. Among them

$$S(m,n) = \sum_k 2^{n-m-2k}\binom{n-m}{2k}S(m,k) \qquad (n \geq m). \tag{1.3}$$

A combinatorial interpretation of this identity is known for $0 \leq m \leq 2$. When $m=1$, this identity becomes $C_{n+1} = \sum_k 2^{n-2k}\binom{n}{2k}C_k$, known as Touchard's Identity [31]. Callan [7] interpreted Eq. 1.3 for $m=2$ by using Pippenger's and Schleich's interpretation of $T(2,n)$ in terms of cubic plane trees [28].

The following identity, due to von Szily [34], has been helpful to many authors working on super Catalan numbers

$$S(m,n) = \sum_k (-1)^k \binom{2m}{m+k}\binom{2n}{n-k}. \tag{1.4}$$

Chen and Wang [10] used it to derive

$$S(m,m+p) = \sum_k (-1)^k \binom{2m}{m-k}\binom{2p}{p+2k} \qquad (p \geq 0). \tag{1.5}$$

For $p \leq 3$, this leads to

$$S(m,m+p) = \binom{2m}{m}\binom{2p}{p} - \binom{2m}{m-1}\binom{2p}{p+2} - \binom{2m}{m+1}\binom{2p}{p-2}, \tag{1.6}$$

which implies that

$$S(m,m) = \binom{2m}{m}, \qquad S(m,m+1) = 2\binom{2m}{m}, \tag{1.7}$$

$$S(m,m+2) = 2(2m+3)C_m, \qquad S(m,m+3) = 4(2m+5)C_m. \tag{1.8}$$

Chen and Wang used Eq. 1.6 to provide an interpretation of $S(m,m+p)$ for $p \leq 3$. They gave an interpretation of $S(m,m+4)$ using Eq. 1.5 and an inclusion-exclusion argument.

We point out that the generalization of Eqs. 1.7 and 1.8 is

$$S(m,m+p) = \frac{2^{2r-p}\prod_{i=\lceil p/2\rceil}^{p-1}(2m+2i+1)}{\prod_{i=1}^{r}(2i-1)} S(m,r), \text{ where } r = \lfloor p/2 \rfloor,$$

which is not helpful in providing a combinatorial interpretation since the coefficients of $S(m,r)$ are not integers.

Georgiadis, Munemasa and Tanaka [21] noted that

$$S(m,n) = (-1)^m K_{m+n}^{2m+2n}(2m) \tag{1.9}$$

where $K_j^d(x)$ is the Krawtchouk polynomial [11] defined by

$$K_j^d(x) = \sum_{h=0}^{j}(-1)^h \binom{x}{h}\binom{d-x}{j-h}.$$

They substituted $h = m+k$ into von Szily's identity (Eq. 1.4) to obtain

$$S(m,n) = (-1)^m \sum_{h=0}^{m+n}(-1)^h \binom{2m}{h}\binom{2n}{m+n-h},$$

which has the following combinatorial interpretation.

Theorem 2 (Georgiadis, Munemasa and Tanaka)**.** *Let $\mathcal{L}_{m+n}$ be the set of lattice paths from $(0,0)$ to $(m+n,m+n)$ with unit right steps and unit up steps. Given $\pi \in \mathcal{L}_{m+n}$, let $h_{2m}(\pi)$ denote the y-coordinate of the end of the $2m^{th}$ step of π. Then*

$$S(m,n) = (-1)^m \sum_{\pi \in \mathcal{L}_{m+n}} (-1)^{h_{2m}(\pi)}.$$

Georgiadis, Munemasa, and Tanaka point out that transforming their weighted interpretation into a conventional interpretation is difficult. In fact, they were able to do it only for $m = 1$.

In Chapter 2, we give several weighted interpretations of $T(m, n)$ including an interpretation in terms of Dyck paths of length $2m+2n-2$. Since the lattice paths in Theorem 2 are not Dyck paths, our interpretation is different from the one by Georgiadis, Munemasa and Tanaka. The advantage of our interpretation is that it will be used to obtain conventional interpretations of both $T(1, n)$ and $T(2, n)$. After giving an initial proof of the weighted interpretation of $T(m, n)$ found in Theorem 3, we will return to this theorem at the end of Chapter 2 to give a combinatorial construction involving rooted plane trees. The combinatorial construction is based on Proposition 1, which is a new identity for the super Catalan number.

1.2 Super Catalan Polynomials

An interpretation of $T(m, n)$ may rest on their q-analog generalization, as it can provide additional information about the objects counted by these integers. In this section we will review some important q-analog polynomials. We will use the standard q-notation

$$[r]_q = 1 + q + \cdots + q^{r-1}$$

$$[n]!_q = \prod_{r=1}^{n} [r]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

Let $\mathfrak{S}(m, n)$ denote the set of lattice paths π from $(0, 0)$ to $(m+n, m-n)$ with m steps diagonally *up* (e.g. from $(0, 0)$ to $(1, 1)$) and n steps diagonally *down* (e.g. from $(0, 0)$ to $(1, -1)$). We naturally associate π with a sequence $\pi = \pi_1 \ldots \pi_{m+n}$ with m zeros and n ones where zeros denote *up* steps and ones denote *down* steps. For a lattice path $\pi \in \mathfrak{S}(m, n)$ we define the **inversion number** $\mathrm{inv}(\pi)$, **descent set** $D(\pi)$, the **major index** $\mathrm{maj}(\pi)$, and the **descent index** $\mathrm{des}(\pi)$ to be

$$\mathrm{inv}(\pi) = |\{(i, j) : i < j, \pi_i > \pi_j\}|$$

$$D(\pi) = \{i : \pi_i > \pi_{i+1}, 1 \le i \le m+n-1\}$$

$$\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i$$

$$\operatorname{des}(\pi) = |D(\pi)|.$$

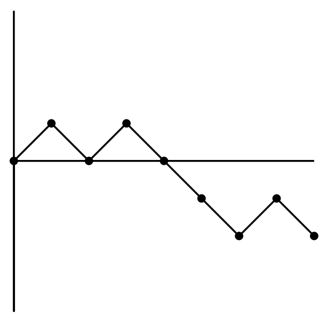


Figure 1.2: $\pi = 01011101 \in \mathfrak{S}(3,5)$ with $\operatorname{inv}(\pi) = 5, D(\pi) = \{2,6\}, \operatorname{maj}(\pi) = 8, \operatorname{des}(\pi) = 2$.

Let S_n be the set of permutations of $[n]$. The following two well known combinatorial interpretations of $[n]!_q$ and $\begin{bmatrix} n+m \\ n \end{bmatrix}_q$ are due to MacMahon, see [6] and [4]:

$$[n]!_q = \sum_{\pi \in S_n} q^{\operatorname{inv}(\pi)}$$

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \sum_{\pi \in \mathfrak{S}(n,m)} q^{\operatorname{maj}(\pi)}.$$

Let $(x)_n = (1-x)(1-qx)\dots(1-q^{n-1}x)$. Furlinger and Hofbauer [19] defined q-Catalan numbers $c_n(\lambda) = c_n(\lambda; q)$ by

$$z = \sum_{n=1}^{\infty} \frac{c_n(\lambda; q) z^n}{q^{\binom{n}{2}} (-q^{-n} z)_n (-q^{\lambda} z)_n}. \tag{1.10}$$

They showed that

$$c_n(1) = \frac{[2n]!_q}{[n]!_q [n+1]!_q}, \qquad c_n(0) = \frac{[2]_q [n]_q [2n-1]_q!}{[n]!_q [n+1]!_q},$$

and

$$c_n(\lambda) = \sum_{\pi \in \mathcal{C}_n} q^{\operatorname{maj}(\pi) + (\lambda - 1)\operatorname{des}(\pi)}.$$

In [1], the q-analog of $S(m,n)$ was defined by

$$S_q(m,n) = \frac{[2m]!_q[2n]!_q}{[m]!_q[n]!_q[m+n]!_q}$$

and it was proved that $S_q(m,n)$ is a polynomial with symmetric coefficients. It was conjectured that $S_q(m,n)$ is unimodal and has non-negative integer coefficients. Below are some values of $S_q(m,n)$:

$$\begin{aligned}
&S_q(0,0) = 1\\
&S_q(0,1) = q+1\\
&S_q(0,2) = q^4+q^3+2q^2+q+1\\
&S_q(1,1) = q+1\\
&S_q(1,2) = q^3+q^2+q+1\\
&S_q(1,3) = q^7+q^6+q^5+2q^4+2q^3+q^2+q+1\\
&S_q(2,2) = q^4+q^3+2q^2+q+1\\
&S_q(2,3) = q^7+q^6+2q^5+2q^4+2q^3+2q^2+q+1\\
&S_q(2,4) = q^{12}+q^{11}+2q^{10}+2q^9+3q^8+3q^7+4q^6+3q^5+3q^4+2q^3+2q^2+q+1.
\end{aligned}$$

These polynomials are related to the q-binomial coefficient as well as to Furlinger's and Hofbauer's q-Catalan numbers:

$$S_q(0,n) = {\begin{bmatrix}2n\\ n\end{bmatrix}}_q, \qquad \frac{S_q(1,n)}{1+q} = c_n(1), \qquad \frac{S_q(1,n)}{1+q^n} = c_n(0).$$

Warnaar and Zudilin [35] discovered the identity

$$S_q(m,m+p) = \sum_{k\ge 0} S_q(m,k) \sum_{j=k}^{p-k} q^{k(m+k)+j(m+k)} {\begin{bmatrix}p\\ 2k\end{bmatrix}}_q {\begin{bmatrix}p-2k\\ j-k\end{bmatrix}}_q \qquad (p \ge 0). \tag{1.11}$$

They used Eq. 1.11, the fact that the coefficients of $S_q(0,n)$ are non-negative, and $S_q(m,n) = S_q(n,m)$ to conclude that $S_q(m,n)$ has non-negative integer coefficients.

Warnaar and Zudilin [35] also proved a q-analog of von Szily's identity

$$q^{mn} S_q(m,n) = \sum_k (-1)^k q^{\binom{k}{2}} {\begin{bmatrix}2m\\ m+k\end{bmatrix}}_q {\begin{bmatrix}2n\\ n+k\end{bmatrix}}_q. \tag{1.12}$$

In Chapter 3 we will define two q-analog super Catalan polynomials and provide a q-analog of Rubenstein's Identity (Eq. 1.2). We will define a q-Ballot number and use it to develop a new identity for the super Catalan polynomial, which will then lead to a conventional combinatorial interpretation for the super Catalan polynomial when $m = 2$. Finally we will give combinatorial proofs of various identities involving both the q- binomial coefficient and Furlinger's and Hofbauer's q-Catalan number $c_n(1)$.

Chapter 2

Super Catalan Numbers

2.1 A Super Catalan Identity

In this section we introduce a new identity for the super Catalan number. The identity in Proposition 1 generalizes an instance of Rubenstein's Identity (see Eq. 1.2), $T(2,n) = 4C_n - C_{n+1}$. This identity plays a crucial role in the weighted combinatorial interpretation of the super Catalan number given in Theorem 3, whose original combinatorial proof is given in Section 2.4.

Although a strictly algebraic proof of Proposition 1 is relatively easy, we present a combinatorial proof. By giving a combinatorial proof, we gain a better understanding of the objects which may be counted by $T(m,n)$ and the techniques used to manipulate them. In the combinatorial proof of Theorem 3 we see methods similar to the proof of Proposition 1 reappearing.

Proposition 1. *Let $m, n \geq 0$, then*

$$T(m+1,n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} 2^{2m-2k} C_{n+k}. \tag{2.1}$$

Proof. Among the identities that Gessel provides in [22] is

$$S(m,n) = \sum_{k=0}^{m} (-1)^k \binom{2n+2k}{n+k} \binom{m}{k} 2^{2m-2k}. \tag{2.2}$$

The binomial coefficient $\binom{2n+2k}{n+k}$ counts the number of paths of length $2n+2k$ which begin at the origin, have unit steps diagonally *up* and diagonally *down*, and never go below the x-axis [17]. Let $\mathcal{P}_k = \mathcal{P}_k(m,n)$ denote the set of paths of length $2m+2n$ which begin at the origin, have unit steps diagonally *up* and diagonally *down*, and remain above or on the x-axis for at least the first $2n+2k$ steps. Then $\binom{2n+2k}{n+k}2^{2m-2k} = |\mathcal{P}_k(m,n)|$. Define the weight of a path $\pi \in \mathcal{P}_k$ to be $w_k(\pi) = (-1)^k\binom{m}{k}$. We can write Eq. 2.2 as

$$S(m,n) = \sum_{k=0}^{m} \sum_{\pi \in \mathcal{P}_k} w_k(\pi).$$

Let $\mathcal{Q}_j = \mathcal{Q}_j(m,n)$ denote the set of paths of length $2m+2n$ which begin at the origin, have unit steps diagonally *up* and diagonally *down*, and go below the x-axis for the first time at exactly the $(2n+2j+1)^{st}$ step. Given a path $\pi \in \mathcal{Q}_j$, we have $\pi \in \mathcal{P}_k$ for $0 \le k \le j$ with $\sum_{k=0}^{j} w_k(\pi) = (-1)^j\binom{m-1}{j}$. Hence,

$$S(m,n) = \sum_{j=0}^{m} \sum_{\pi \in \mathcal{Q}_j} (-1)^j \binom{m-1}{j}$$

A path in $\mathcal{Q}_j$ can be decomposed into a Dyck path of length $2n+2j$, followed by a *down* step, followed by a path of length $2m+2j-1$. Thus $|\mathcal{Q}_j| = C_{n+j}2^{2m-2j-1}$. Hence,

$$S(m,n) = \sum_{j=0}^{m} (-1)^j \binom{m-1}{j} 2^{2m-2j-1} C_{n+j}.$$

And thus,

$$T(m+1,n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} 2^{2m-2k} C_{n+k}.$$

$\square$

2.2 A Weighted Interpretation

A **2-Motzkin path** of length n is a sequence of n steps, starting at the origin and ending at the point $(n,0)$, which never goes below the x-axis, where the allowable steps are diagonally *up*, diagonally *down*, and *level*. The *level* steps are either *straight* or *wavy*. Define $\mathcal{M}_n$ to be the set of all 2-Motzkin paths of length n. Note that a Dyck path of length $2n$ is a 2-Motzkin path of length $2n$ with no *level* steps. There is a well-known bijection between 2-Motzkin paths of length $n-1$ and Dyck paths of length $2n$ [12]. Given a 2-Motzkin path, read the steps from left to right and do the following replacements: replace an *up* step with two *up* steps, a *down* step with two *down* steps, a *straight* step with an *up* step followed by a *down* step, and a *wavy* step with a *down* step followed by an *up* step. The resulting path may touch level -1, thus, in addition, add an *up* step to the beginning of the resulting path and a *down* step to the end to obtain a Dyck path. See Fig. 2.1.

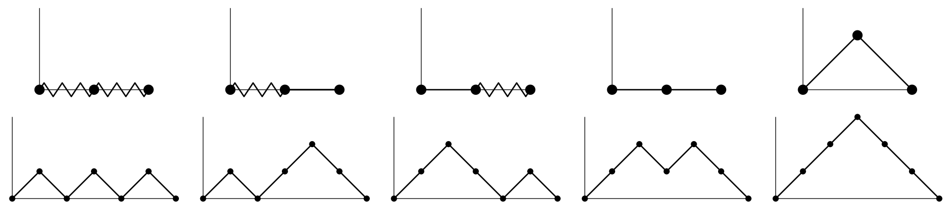


Figure 2.1: The bijection between 2-Motzkin paths of length 2 and Dyck paths of length 6.

Define the **level** of a point on a path to be its y-coordinate. For a fixed $m \geq 0$, we call a 2-Motzkin path π positive if the m^{th} step begins on an even level, otherwise π is negative. Let $P(m,n)$ be the number of positive 2-Motzkin paths of length $m+n-2$, and $N(m,n)$ be the number of negative 2-Motzkin paths of length $m+n-2$.

Theorem 3. *For $m, n \geq 1$, the super Catalan number $T(m,n)$ counts the number of positive 2-Motzkin paths minus the number of negative 2-Motzkin paths. That is,*

$$T(m,n) = P(m,n) - N(m,n).$$

Proof. Given a 2-Motzkin path π of length $m+n-2$, define the weight of π to be 1 if π is positive and -1 if π is negative.

Let $F(m,n)$ be the sum of the weights of all 2-Motzkin paths of length $m+n-2$, that is, $F(m,n) = P(m,n) - N(m,n)$. To prove $F(m,n) = T(m,n)$, we will check the initial condition

$$F(1,n) = C_n,$$

and the recurrence given by Rubenstein's Identity (Eq. 1.2),

$$4F(m,n) = F(m+1,n) + F(m,n+1).$$

For $m = 1$, the weight of any 2-Motzkin path of length n is 1 because the first step always starts at the level $y = 0$. Hence $F(1,n) = C_n$, giving the number of 2-Motzkin paths of length $n-1$.

Next we consider the sum of the weights counted by $F(m,n+1) + F(m+1,n)$. If a 2-Motzkin path of length $m+n-1$ has an *up* or *down* step at step m, it will be counted once as a positive path and once as a negative path, and will not contribute to this sum.

Paths of length $m+n-1$ with a *level* step at step m will be counted twice. Let π be such a 2-Motzkin path. By contracting the m^{th} step in π, we obtain a 2-Motzkin path of length $m+n-2$; furthermore, every 2-Motzkin path of length $m+n-2$ can be obtained by contracting exactly two 2-Motzkin paths of length $m+n-1$, one with a *wavy* step at step m and one with a *straight* step at step m.

Thus the sum of the weights counted by $F(m,n+1)+F(m+1,n)$ is twice the sum of the weights of 2-Motzkin paths of length $m+n-1$ with *level* steps at step m; which is four times the sum of the weights of 2-Motzkin paths of length $m+n-2$, that is, $4F(m,n)$. $\square$

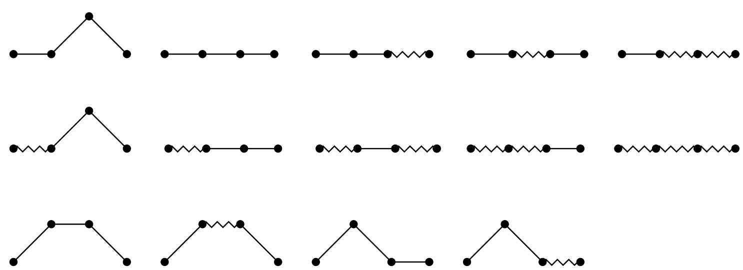


Figure 2.2: When $m=2$, there are ten positive 2-Motzkin paths and four negative 2-Motzkin paths of length 3. $T(2,3)=P(2,3)-N(2,3)=6$.

In the Section 2.2 we will use Theorem 3 to derive a conventional interpretation of $T(2,n)$. Theorem 3 can be used to prove combinatorially that $T(m,n)=T(n,m)$. Let π be a path of length $m+n-2$ counted by $T(m,n)$. Consider the reverse of a path, denoted $\text{rev}(\pi)$, to be that path read from right to left. Since the m^{th} step of π and the n^{th} step of the reverse of π start at the same point, mapping a path to its reverse is a weight preserving involution between the 2-Motzkin paths counted by $T(m,n)$ and the 2-Motzkin paths counted by $T(n,m)$.

We can eliminate certain paths π and $\text{rev}(\pi)$ from the set of objects counted by $T(m,n)$. If π is a path counted by $P(m,n)$ and the m^{th} step of $\text{rev}(\pi)$ begins on an odd level, then $\text{rev}(\pi)$ is counted by $N(m,n)$. Similarly, if π is a path counted by $N(m,n)$ and the m^{th} step of $\text{rev}(\pi)$ begins on an even level, then $\text{rev}(\pi)$ is counted by $P(m,n)$. Hence $T(m,n)$ counts the 2-Motzkin paths in which both the m^{th} and n^{th} steps begin on even levels minus the 2-Motzkin paths in which both the m^{th} and n^{th} steps begin on odd levels. We will refer to the 2-Motzkin paths in which both the m^{th} and n^{th} steps begin on even (resp. odd) levels as doubly positive (resp. negative).

Theorem 4. *Let $P'(m,n)$ (resp. $N'(m,n)$) be the number of doubly positive (resp. negative) 2-Motzkin paths of length $m+n-2$. For $m,n \geq 1$,*

$$T(m,n) = P'(m,n) - N'(m,n).$$

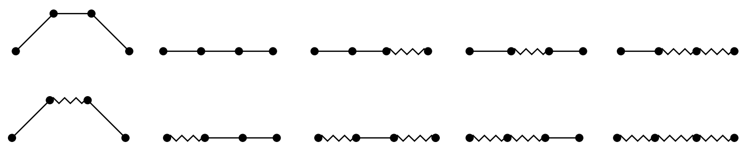


Figure 2.3: When $m = 2$ and $n = 3$, there are eight doubly positive 2-Motzkin paths and two doubly negative 2-Motzkin paths of length 3. $T(2,3) = P'(2,3) - N'(2,3) = 6$.

This theorem truly captures the symmetry $T(m,n) = T(n,m)$. Although Theorem 4 counts the super Catalan number using fewer objects than Theorem 3, we use Theorem 3 to obtain a conventional interpretation for $T(2,n)$ because the objects in Theorem 3 are more easily described and understood.

We can reformulate the result in Theorem 3 in terms of Dyck paths. Let $\mathfrak{S}_+(m,n)$ be the subset of paths that have m *up* steps, n *down* steps, and never go below the x-axis. These paths are called **ballot paths**. Here $\mathfrak{S}_+(n,n) = \mathcal{C}_n$ is the set of Dyck paths of length $2n$.

By Theorem 3 and the canonical bijection between 2-Motzkin paths and Dyck paths, $P(m,n)$ counts the number of Dyck paths of length $2m+2n-2$ whose $2m-1^{st}$ step ends on level 1 (mod 4), and $N(m,n)$ counts the number of Dyck paths of length $2m+2n-2$ whose $2m-1^{st}$ step ends on level 3 (mod 4). Let $B(n,r)$ be the number of ballot paths that start at the origin, end at the point $(2n-1, 2r-1)$, and do not go below the x-axis. It is well known that $B(n,r) = \frac{r}{n}\binom{2n}{n+r}$.

Corollary 1. *Reformulating Theorem 3, we obtain the equivalent equations:*

$$T(m,n) = \sum_{r\geq 1} (-1)^{r-1} B(m,r)B(n,r) \tag{2.3}$$

and

$$T(m,n) = \sum_{r\geq 1} (-1)^{r-1} \frac{r^2}{nm} \binom{2m}{m+r}\binom{2n}{n+r}. \tag{2.4}$$

Eq. 2.4 is a new identity for the super Catalan number $T(m,n)$. An algebraic proof of this identity can be obtained by substituting $q = 1$ in Theorem 7. We provide a q-analog of Eqs. 2.3 and 2.4 with proofs in Section 3.2.

2.3 Combinatorial Techniques

Our goal in this section is to derive a similar result to Gessel's and Xin's Theorem 1 using our Theorem 3 and some direct Dyck paths subtraction techniques that will be easier to generalize for larger values of m. We already were able to generalize Theorem 5 to super Catalan polynomials in Chapter 3.

For a path $\pi \in \mathcal{C}_n$, let X be the last, from left to right, level one point up to and including the right-most maximum R on π. Let $h_-(\pi)$ denote the maximum height of π until and including point X, and $h_+(\pi)$ denote the maximum height of π after and including point X. Obviously $h_-(\pi) \leq h_+(\pi) = h(\pi)$. Let Ω_n denote the set of $\pi \in \mathcal{C}_n$ such that $h_+(\pi) \leq h_-(\pi) + 2$.

Theorem 5. *Let $n \geq 1$. The super Catalan number $T(2,n)$ counts Dyck paths π of length $2n$ such that $h_+(\pi) \leq h_-(\pi) + 2$, the path of height one counted twice. That is,*

$$T(2,n) = |\Omega_n| + 1.$$

Proof. Let $\mathcal{A}_n$ denote the set of Dyck paths of length $2n$ that start with *up, down, up*, $\mathcal{B}_n$ denote the set of Dyck paths of length $2n$ that start with *up, up, down*, and $\mathcal{N}_n$ denote the set of Dyck paths of length $2n$ that start with *up, up, up*.

By Theorem 3, $T(2,n) = P(2,n) - N(2,n)$, where $P(2,n)$ is the number of 2-Motzkin paths of length n that start with a level step, and $N(2,n)$ is the number of 2-Motzkin paths of length n that start with an up step. The canonical bijection between 2-Motzkin paths and Dyck paths leads to the following interpretation:

$$T(2,n) = |\mathcal{A}_{n+1}| + |\mathcal{B}_{n+1}| - |\mathcal{N}_{n+1}|.$$

By contracting the second and third steps in the paths in $\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$ we get twice $\mathcal{C}_n$, so $|\mathcal{A}_{n+1}| = |\mathcal{B}_{n+1}| = C_n$.

We consider all paths π in $\mathcal{N}_{n+1}$ that do not attain level one between the third step of π and the right-most maximum point R on π. The set of all such paths will be denoted by $\mathcal{N}^*_{n+1}$. Let $\mathcal{N}^{**}_{n+1} = \mathcal{N}_{n+1} - \mathcal{N}^*_{n+1}$. Then

$$T(2,n) = 2|\mathcal{C}_n| - |\mathcal{N}^*_{n+1}| - |\mathcal{N}^{**}_{n+1}|.$$

First we establish an injection f from $\mathcal{N}^*_{n+1}$ to $\mathcal{C}_n$. For $\pi \in \mathcal{N}^*_{n+1}$, let RQ be the *down* step that follows the right-most maximum point R of π. We define $f(\pi)$ to be the path obtained by removing the second and third steps in π, both of which are *up* steps, and then substituting the *down* step RQ by an *up* step. See Figure 2.4. Since π does not attain level one between its third step and R, $f(\pi)$ is a Dyck path of length $2n$. Note that Q is the left-most maximum on $f(\pi)$. Also, since at least two *up* steps precede Q on $f(\pi)$, the height of $f(\pi)$ is at least two. Thus the Dyck path of height one and length $2n$ is not in the image of f.

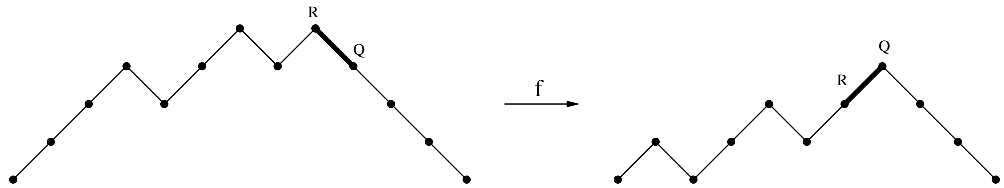


Figure 2.4: f removes the 2^{nd} and 3^{rd} steps, substitutes the *down* step RQ by an *up* step.

We will show that f is a bijection between $\mathcal{N}^*_{n+1}$ and $\mathcal{C}_n - \{\pi \in \mathcal{C}_n : h(\pi) = 1\}$. Let ρ be in $\mathcal{C}_n$ of height $h(\rho) > 1$. Let Q be the left-most maximum on ρ and RQ be the *up* step that precedes Q. Insert two *up* steps after the first step of ρ, then substitute the *up* step RQ by a *down* step, which makes R the right-most maximum of the resulting path π. The path π is in $\mathcal{N}^*_{n+1}$ and $f(\pi) = \rho$.

It follows that $|\mathcal{C}_n| - |\mathcal{N}^*_{n+1}|$ counts only one path, the Dyck path of length $2n$ and height one.

Next we establish an injection g from $\mathcal{N}^{**}_{n+1}$ to $\mathcal{C}_n$. A path π in $\mathcal{N}^{**}_{n+1}$ attains level one between its third step and the right-most maximum point R on π. Let Y be the first point between the third step of π and R at which π attains level one. The segment XY that consists of two *down* steps precedes Y. We remove the second and third steps of π and substitute the two *down* steps XY by two *up* steps. See Figure 2.5. The resulting path is a ballot path of length $2n$ that ends at level two. From left to right, X is the last level one point on this ballot path. The maximum level that this path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4.

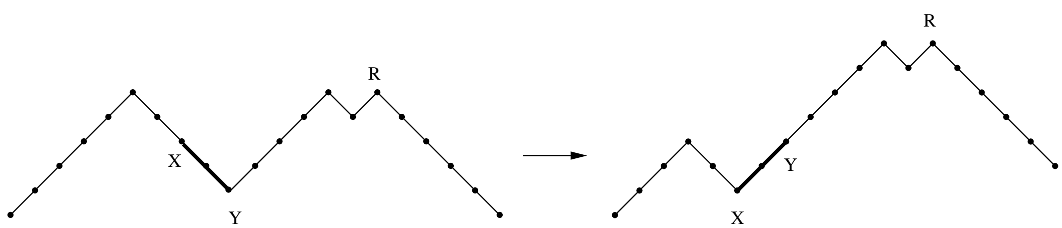


Figure 2.5: First part of g action is removing the 2^{nd} and 3^{rd} steps, substituting the two *down* steps XY by two *up* steps.

Let L be the left-most maximum point of this ballot path and ML be the *up* step that precedes L. Substitute the *up* step ML by a *down* step. See Figure 2.6. The resulting path $g(\pi)$ is in $\mathcal{C}_n$ and M is its right-most maximum. Note that X is the last level one point on $g(\pi)$ before its right-most maximum M and $h_+(g(\pi)) \geq h_-(g(\pi)) + 3$.

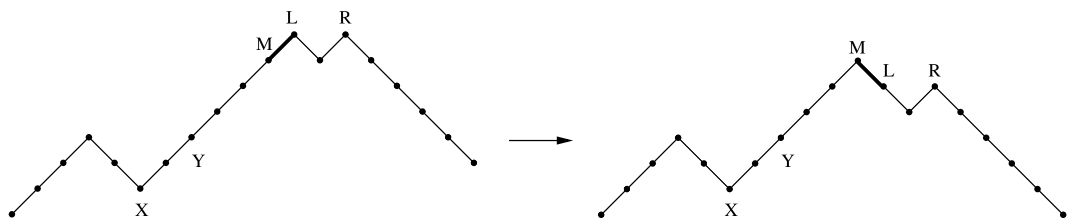


Figure 2.6: Second part of g action is substituting the *up* step ML with a *down* steps.

We will show that g is a bijection between $\mathcal{N}^{**}_{n+1}$ and $\mathcal{C}_n - \Omega_n$. Let ρ be in $\mathcal{C}_n$ and $h_+(\rho) \geq h_-(\rho) + 3$. Let M be the right-most maximum on ρ and ML be the *down* step that follows M. Substitute the *down* step ML by an *up* step. The result is a ballot path of length $2n$ that ends at level two. Note that L is the left-most maximum on this ballot path. Let R denote the right-most maximum on this ballot path. From left to right, X is the last level one point on this ballot path. The maximum level that this path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4. Since X is the last level one point, it is followed by the segment XY that consists of two *up* steps. Next we insert two *up* steps after the first step of this ballot path and then substitute the two *up* steps XY by two *down* steps. The resulting path is a Dyck path of length $2n+2$, we denote it by π. Point Y is the first level one point after the third step of π. Note that the maximum level that this Dyck path reaches after Y is at least the maximum level that this Dyck path reaches up to and including Y, which means that the right-most maximum R is to the right of Y. If follows that $\pi \in \mathcal{N}^{**}_{n+1}$ and $g(\pi) = \rho$.

Thus $|\mathcal{C}_n| - |\mathcal{N}^{**}_{n+1}|$ counts Dyck paths π that satisfy $h_+(\pi) \leq h_-(\pi) + 2$.

□

We will now show a simple bijection between the objects described in Theorem 5 and Theorem 1.

Let π be a Dyck path of length $2n$ and height $h(\pi) > 1$, such that $h_+(\pi) \leq h_-(\pi) + 2$. Let R be the right-most maximum of π. Note that X is followed by an *up* step XY and R is followed by a *down* step RL. Substitute the *up* step XY with a down step, substitute the *down* step RL with an *up* step. See Figure 2.7. As a result, the portion of π between Y and R will be lowered by two levels. Since π does not attain level one between Y and R, the resulting path is a Dyck path with point Y on level zero.

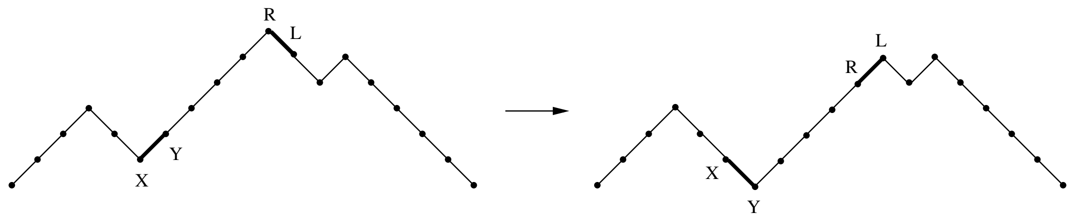


Figure 2.7: From Dyck paths described in Theorem 5 to pairs of Dyck path described in Theorem 1.

Note that Y separates this Dyck path into a pair of Dyck paths (ρ, σ). The height of ρ is $h_{-}(\pi)$, the height of σ is $h_{+}(\pi) - 1$. Thus $|h(\rho) - h(\sigma)| \leq 1$. Since L is the left-most maximum on σ, this mapping is reversible. Theorem 5 counts the Dyck path τ of height one twice. This corresponds to the pairs (τ, ϵ) and (ϵ, τ) in Theorem 1, where ϵ is the empty path.

2.4 A Combinatorial Proof of Theorem 3

In this section we derive Theorem 3 from Eq. 2.1 combinatorially. In doing this, we introduce a family of weighted rooted plane trees which are counted by the super Catalan numbers. This new family of objects is an alternative to Dyck paths and could be useful for interpreting super Catalan numbers in future works.

A **rooted plane tree**, T, is a finite set of vertices such that one specifically designated vertex is called the root of T and the remaining vertices (excluding the root) are put into an ordered partition $(T_1, \ldots, T_m)$ of $m \geq 0$ pairwise disjoint, non-empty sets $T_1, \ldots, T_m$ each of which is a plane tree [29]. The ordering of the sub-trees $(T_1, \ldots, T_m)$ is depicted by drawing them from left to right. These are **unlabeled** rooted plane trees in the sense that the vertices are regarded as indistinguishable, but the sub-trees at any vertex are linearly ordered. However we will often give them the canonical pre-order labeling for the purpose of referring to them. There is a standard bijection, called the glove bijection, between rooted plane trees on $n+1$ vertices and Dyck paths of length $2n$ [13]. Traverse the rooted plane tree in pre-order. Every time an edge is traversed for the first time, draw a *up* step, when it is traversed the second time, draw a *down* step. See Fig. 2.8.

Deutsch and Shapiro [14] provide a bijection between rooted plane trees on $n+2$ vertices and 2-Motzkin paths of length n. Given a rooted plane tree, for non-root vertices with degree ≥ 3, call the left-most out-edge a **left** edge and the right-most out-edge a **right** edge. For non-root vertices with degree 2 call the unique out-edge **lonely**. Call all remaining edges

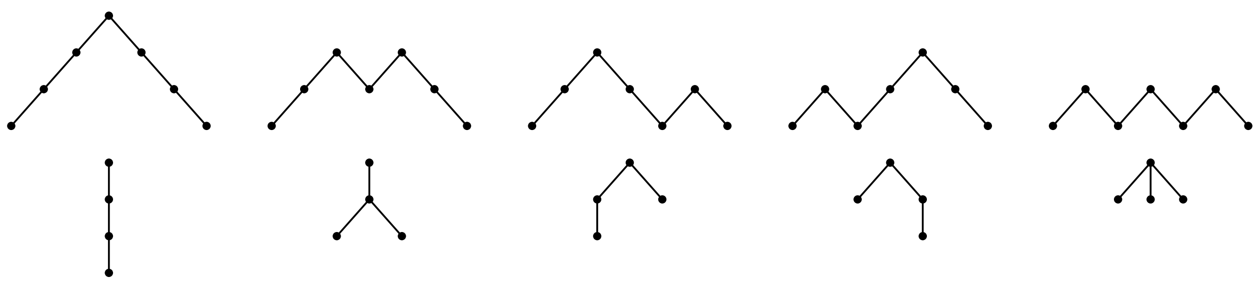


Figure 2.8: The glove bijection between rooted plane trees on 4 vertices and Dyck paths of length 6.

redundant. To construct a 2-Motzkin path, traverse the tree in pre-order, and the first time an edge is encountered, draw the following for each type of edge: nothing for the first edge, an *up* (resp. *down*) step for every left (resp. *right*) edge, and a *straight* (resp. *wavy*) *level* step for every lonely (resp. redundant) edge.

Note that the bijection of Deutsch and Shapiro is NOT the composition of the glove bijection with the bijection previously mentioned between Dyck paths and 2-Motzkin paths. See Fig. 2.9.

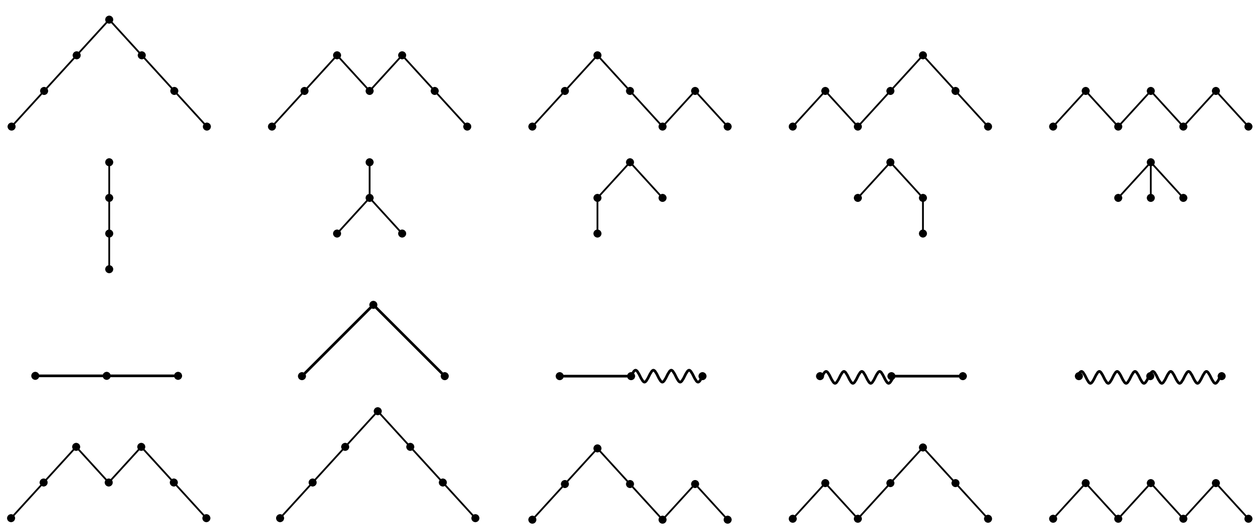
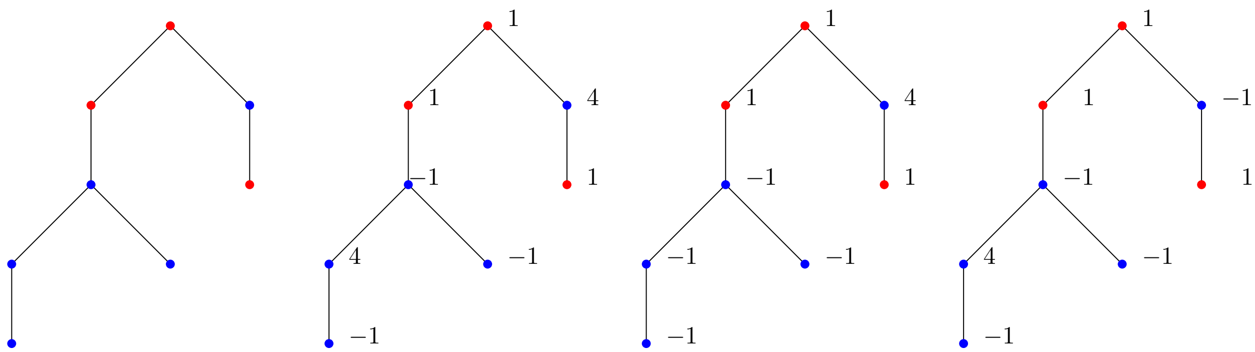


Figure 2.9: The standard glove bijection followed by Deutsch's and Shapiro's bijection followed by the standard bijection between Motzkin paths and Dyck paths.

Let $\mathcal{T}(m,n)$ be the set of rooted plane trees on $m+n+1$ vertices with the vertices given the canonical pre-order labeling, the first n vertices colored red, the last vertex colored red, and the remaining m vertices colored blue. Let $b(T)$ be the number of blue vertices of T which have degree 2. Since the Catalan number C_n counts the number of rooted plane trees on $n+1$ vertices, $|\mathcal{T}(m,n)| = C_{m+n}$.

Given $T \in \mathcal{T}(m,n)$, let $\Omega_k = \Omega_k(T)$ be the set of weight functions on the vertices of T which assign the weight 1 to each red vertex, the weight 4 to exactly k of the blue vertices of degree 2, and the weight -1 to each of the remaining blue vertices. See Fig. 2.10. Clearly there are $|\Omega_k| = \binom{b(T)}{k}$ possible weight functions. Given a weight function σ, define $w_\sigma(T)$ to be the product of the weights of the vertices of T assigned by σ. For $\sigma \in \Omega_k$, obviously $w_\sigma(T) = (-1)^{m-k}4^k$. We wish to calculate

$$\sum_k \sum_{T \in \mathcal{T}(m,n)} \sum_{\sigma \in \Omega_k(T)} w_\sigma(T).$$



$T \in \mathcal{T}(5,2)$ with $b(T) = 2$ | $\sigma \in \Omega_2(T)$, $w_\sigma(T) = -16$ | $\sigma_1 \in \Omega_1(T)$, $w_{\sigma_1}(T) = 4$ | $\sigma_2 \in \Omega_1(T)$, $w_{\sigma_2}(T) = 4$

Figure 2.10: An example of T, $\Omega_1(T)$, and $\Omega_2(T)$.

For fixed m, n, k, let $R \in \mathcal{T}(m-k, n)$ and $\rho \in \Omega_0(R)$. Label the edges of R in canonical pre-order $e_1, \ldots, e_{m+n-k}$. Define $\Theta_k(R)$ to be the set of pairs (T, σ) which can be obtained by inserting k vertices of degree 2 and weight 4 into any of the edges $e_n, \ldots, e_{m+n-k}$. Obviously $|\Theta_k(R)| = \binom{m}{m-k}$ since there are $\binom{m}{m-k}$ ways to distribute k indistinguishable vertices among $m-k+1$ distinguishable edges. See Fig. 2.11. Furthermore, given $T \in \mathcal{T}(m,n)$ and $\sigma \in \Omega_k(T)$, let R be the tree obtained by contracting the vertices of T which are assigned a weight of 4 in σ. Then $R \in \mathcal{T}(m-k, n)$ with $T \in \Theta_k(R)$. Finally, we recall that

$|\mathcal{T}(m-k,n)| = C_{m+n-k}$. Hence,

$$\begin{aligned}
\sum_k \sum_{T\in\mathcal{T}(m,n)} \sum_{\sigma\in\Omega_k(T)} w_\sigma(T) &= \sum_k \sum_{R\in\mathcal{T}(m-k,n)} \sum_{(T,\sigma)\in\Theta_k(R)} w_\sigma(T) \\
&= \sum_k \sum_{R\in\mathcal{T}(m-k,n)} \sum_{(T,\sigma)\in\Theta(R)} (-1)^{m-k}4^k \\
&= \sum_k \sum_{R\in\mathcal{T}(m-k,n)} \binom{m}{m-k}(-1)^{m-k}4^k \\
&= \sum_k \binom{m}{m-k}(-1)^{m-k}4^k C_{m+n-k} = T(m+1,n).
\end{aligned}$$

The last step of the equality above follows from Proposition 1.

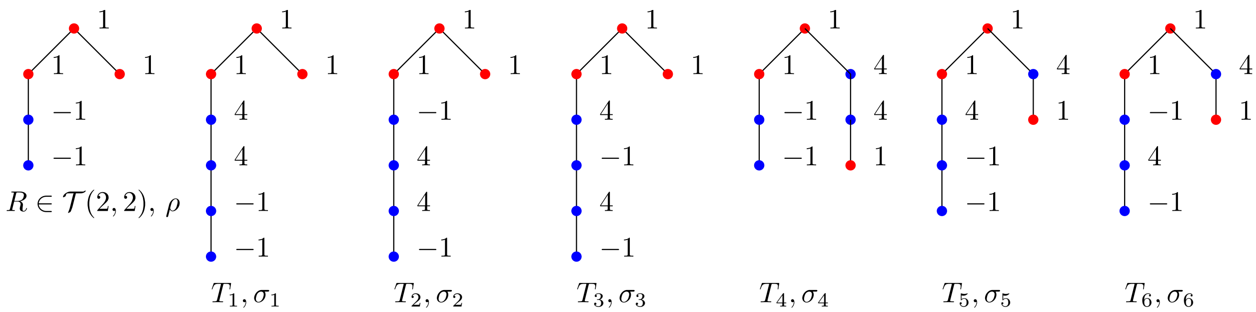


Figure 2.11: An example of R and $\Theta_2(R)$.

On the other hand, fixing $T \in \mathcal{T}(m,n)$, the total sum of the weights of T is

$$\sum_k \sum_{\sigma\in\Omega_k(T)} w_\sigma(T) = \sum_k \binom{b(T)}{k}(-1)^{m-k}4^k = (-1)^{m-b(T)}3^{b(T)}.$$

Given $T \in \mathcal{T}(m,n)$, let $\mu(T)$ be the weight function on the vertices of T which assigns the weight 1 to each red vertex, the weight 3 to each blue vertex of degree 2, and the weight -1 to each of the remaining blue vertices. Clearly $w_{\mu(T)} = (-1)^{m-b(T)}3^{b(T)}$. It follows that

$$\sum_{T\in\mathcal{T}(m,n)} w_{\mu(T)} = T(m+1,n).$$

Given a 2-Motzkin path of length $m+n-1$, we will refer to the last m steps as the **end** of the path.

Let $\mathcal{Q} = \mathcal{Q}(m,n)$ be the set of colored 2-Motzkin paths of length $m+n-1$ in which the *straight level* edges that appear within the end of the path are three colored: green, yellow, and purple. Given $\pi \in \mathcal{Q}$, define $w(\pi) = (-1)^{m-k}$ where k is the number of *straight level* edges at the end of π. Define $w(\mathcal{Q}) = \sum_{\pi\in\mathcal{Q}} w(\pi)$. From the canonical bijection between rooted plane tress and 2-Motzkin paths, it follows that

$$w(\mathcal{Q}) = \sum_{T\in\mathcal{T}(m,n)} w_{\mu(T)} = T(m+1,n).$$

Let $\mathcal{A}$ be the subset of paths $\pi \in \mathcal{Q}$ such that the end of π has a purple *straight level* edge or a *wavy level.* Define a sign reversing involution on $\mathcal{A}$: given a path π, within the end of π, locate the first edge which is either a purple *straight level* edge or a *wavy level* edge and if it is a purple *straight level* edge (resp. *wavy level* edge) replace it with a *wavy level* edge (resp. purple *straight level* edge). It follows that $w(\mathcal{Q}) = w(\mathcal{Q}-\mathcal{A})$.

However $\mathcal{Q}-\mathcal{A}$ is the set of colored 2-Motzkin paths in which all *level* edges appearing in the end of the path are either green *straight level* or yellow *straight level* edges. We can biject $\mathcal{Q}-\mathcal{A}$ to $\mathcal{M}_{m+n-1}$. Given a path π replace all of the green *straight level* edges with *straight level* edges and replace all of the yellow *straight level* edges with *wavy level* edges.

Recall $w(\pi) = (-1)^{m-k}$ where $m-k$ is the number steps in the end of π which are either *up* or *down* edges. If within the end of π there are an even number of steps which are either *up* or *down*, then the end of the path begins on an even level. This means that the n^{th} step of π begins on an even level and $w(\pi) = 1$. If within the end of π there are an odd number of steps which are either *up* or *down*, then the end of the path begins on an odd step. This means that the n^{th} step of π begins on an odd level and $w(\pi) = 1$. Thus,

$$T(m+1,n) = P(n,m+1) - N(n,m+1).$$

Together with $T(m,n) = T(n,m)$, this implies the result of Theorem 3.

Chapter 3

Super Catalan Polynomials

We define two q-analogs of the super Catalan number

$$T_q(m,n) = \frac{S_q(m,n)}{1+q^n}$$

and

$$U_q(m,n) = \frac{S_q(m,n)}{1+q^m}.$$

Note that $T_q(m,n) = U_q(n,m)$. Observe that

$$T_q(1,n) = c_n(0) = \sum_{\pi \in \mathfrak{S}_+(n,n)} q^{\mathrm{maj}(\pi)-\mathrm{des}(\pi)}$$

and

$$U_q(1,n) = c_n(1) = \sum_{\pi \in \mathfrak{S}_+(n,n)} q^{\mathrm{maj}(\pi)},$$

where $c_n(\lambda)$ are the q-Catalan numbers defined by Furlinger and Hofbauer [19], see Section 1.2. In [1], $T_q(m,n)$ and $U_q(m,n)$ were proved to be polynomials. A more elegant proof of the fact that $T_q(m,n)$ and $U_q(m,n)$ are polynomials with integer coefficients is given in Section 3.2.

Here are some values of $T_q(m,n)$:

$$
\begin{aligned}
T_q(1,1) =& 1\\
T_q(2,1) =& q^2+1\\
T_q(3,1) =& q^6+q^4+q^3+q^2+1\\
T_q(1,2) =& q+1\\
T_q(2,2) =& q^2+q+1\\
T_q(3,2) =& q^5+q^4+q^3+q^2+q+1\\
T_q(1,3) =& q^4+q^3+q^2+q+1\\
T_q(2,3) =& q^4+q^3+2q^2+q+1\\
T_q(3,3) =& q^6+q^5+2q^4+2q^3+2q^2+q+1
\end{aligned}
$$

3.1 The q-Ballot Number

Let $\mathcal{B}(n,r)$ denote the set of paths of length $2n$ which begin at the origin with an *up* step, end at $(2n,-2r+2)$, and never go below the line $y=-2r+2$. In particular $\mathcal{B}(n,1)=\mathcal{C}_n$. There is a bijection between $\mathcal{B}(n,r)$ and the set of ballot paths $\mathfrak{S}_+(n+r-1,n-r)$ mentioned in Section 2.2. Given a path $\pi\in\mathcal{B}(n,r)$, let ρ be the path obtained from π by deleting the first step of π and reading the resulting path from right to left. Then $\rho\in\mathfrak{S}_+(n+r-1,n-r)$. Hence,

$$
|\mathcal{B}(n,r)| = \frac{r}{n}\binom{2n}{n+r}.
$$

Define the q-Ballot Number

$$
B_q(n,r) = \frac{[2n-1]!_q[2r]_q}{[n+r]_q![n-r]_q!} = \frac{1}{q^{n-r}}\left(\begin{bmatrix}2n-1\\n+r-1\end{bmatrix}_q - \begin{bmatrix}2n-1\\n+r\end{bmatrix}_q\right).
$$

A different q-analog of the Ballot Number has been defined by Chapoton and Zeng in [9].

Here are some values of $B_q(n,r)$:

$$
\begin{aligned}
B_q(1,1) &= 1\\
B_q(2,1) &= q+1\\
B_q(2,2) &= 1\\
B_q(3,1) &= q^4+q^3+q^2+q+1\\
B_q(3,2) &= q^3+q^2+q+1\\
B_q(3,3) &= 1\\
B_q(4,1) &= q^9+q^8+q^7+2q^6+2q^5+2q^4+2q^3+q^2+q+1\\
B_q(4,2) &= q^8+q^7+2q^6+2q^5+2q^4+2q^3+2q^2+q+1\\
B_q(4,3) &= q^5+q^4+q^3+q^2+q+1\\
B_q(4,4) &= 1.
\end{aligned}
$$

Lemma 1. *Let $\pi \in \mathfrak{S}(m,n)$ ending with an up step. Reflecting π over the x-axis gives a path $\rho \in \mathfrak{S}(n,m)$ ending with a down step which satisfies* $\operatorname{maj}(\pi) = \operatorname{maj}(\rho) + n$.

Proof. Given a path $\pi \in \mathfrak{S}(m,n)$ ending with an *up* step, let $D(\pi) = \{X_1, \ldots, X_\ell\}$. We let $X_0 = 0$. Define u_i and d_i to be the number of *up* and *down* steps, respectively, between indices X_i and X_{i+1}. Let ρ be the reflection of π across the x-axis. Then the descents of ρ occur exactly at indices $X_i + u_i$ for $i < \ell$. Hence,

$\operatorname{maj}(\rho) = \sum_{i=0}^{\ell-1}(X_i+u_i) = \operatorname{maj}(\pi)+m-(X_\ell+u_\ell) = \operatorname{maj}(\pi)+m-(n+m) = \operatorname{maj}(\pi)-n.$ $\square$

Theorem 6.

$$B_q(n,r) = \sum_{\pi\in\mathcal{B}(n,r)} q^{\operatorname{maj}(\pi)-\operatorname{des}(\pi)}. \tag{3.1}$$

Proof. Let $\mathfrak{S}_>$ denote the set of paths in $\mathfrak{S}(n+r-1, n-r)$ which have height strictly greater than $2r-1$, and let $\mathfrak{S}_\le$ denote the set of paths in $\mathfrak{S}(n+r-1, n-r)$ which never go above $y = 2r-1$.

We will define a bijection $\psi : \mathfrak{S}_> \to \mathfrak{S}(n+r, n-r-1)$ which preserves the major index. Given a path $\pi \in \mathfrak{S}_>$, let R be the right-most highest point on π. Since π has height strictly greater than $2r-1$ and ends at level $2r-1$, the point R is not the last point on π. Let RL be the *down* step following R. Define $\psi(\pi)$ to be the path obtained from π by changing the *down* step RL into an *up* step. Note that $\psi(\pi) \in \mathfrak{S}(n+r, n-r-1)$ and L is the left-most highest point on $\psi(\pi)$. To see that ψ is a bijection from $\mathfrak{S}_>$ to $\mathfrak{S}(n+r, n-r-1)$, given a path ρ in $\mathfrak{S}(n+r, n-r-1)$, locate the left-most highest point L on ρ and change the *up* step preceding it into a *down* step to obtain π. Since ρ has height at least $2r+1$, the path π will have height at least $2r$. Therefore $\pi \in \mathfrak{S}_>$ and $\psi(\pi) = \rho$. The bijection ψ preserves the major index because the descent sets of π and $\psi(\pi)$ are the same. It follows that

$$\begin{bmatrix} 2n-1 \\ n+r-1 \end{bmatrix}_q - \begin{bmatrix} 2n-1 \\ n+r \end{bmatrix}_q = \sum_{\pi\in\mathfrak{S}(n+r-1,n-r)} q^{\operatorname{maj}(\pi)} - \sum_{\pi\in\mathfrak{S}(n+r,n-r-1)} q^{\operatorname{maj}(\pi)} = \sum_{\pi\in\mathfrak{S}_\le} q^{\operatorname{maj}(\pi)}.$$

We will define a bijection $\varphi : \mathfrak{S}_{\leq} \to \mathcal{B}(n,r)$. Let $\pi \in \mathfrak{S}_{\leq}$. Define $\varphi(\pi)$ to be the path obtained from π by reflecting π across the x-axis and then adding an *up* step to the beginning of the path. This is clearly a bijection from $\mathfrak{S}_{\leq}$ to $\mathcal{B}(n,r)$. By Lemma 1, reflecting π across the x-axis causes the major index to decrease by $n-r$. Adding an *up* step to the beginning of the path increases all descents by 1, hence the major index of the reflection of π equals the major minus descent index of $\varphi(\pi)$. It follows that

$$\frac{1}{q^{n-r}}\left(\begin{bmatrix}2n-1\\n+r-1\end{bmatrix}_q - \begin{bmatrix}2n-1\\n+r\end{bmatrix}_q\right) = \sum_{\pi\in\mathfrak{S}_{\leq}} q^{\mathrm{maj}(\pi)-(n-r)} = \sum_{\pi\in\mathcal{B}(n,r)} q^{\mathrm{maj}(\pi)-\mathrm{des}(\pi)}.$$

□

Consider a different q-analog Ballot number

$$B_q^{(2)}(n,r) = \frac{1+q^n}{1+q^r}B_q(n,r) = \frac{[2n]_q![r]_q}{[n]_q[n+r]!_q[n-r]!_q}.$$

The generating function $B_q^2(n,r)$ will appear in the next section in Theorem 7. Here are some values of $B_q^{(2)}(n,r)$:

$$\begin{aligned}
&B_q^{(2)}(1,1) = 1\\
&B_q^{(2)}(2,1) = q^2+1\\
&B_q^{(2)}(2,2) = 1\\
&B_q^{(2)}(3,1) = q^6+q^4+q^3+q^2+1\\
&B_q^{(2)}(3,2) = q^4+q^3+q+1\\
&B_q^{(2)}(3,3) = 1\\
&B_q^{(2)}(4,1) = q^{12}+q^{10}+q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+1\\
&B_q^{(2)}(4,2) = q^{10}+q^9+q^8+q^7+2q^6+2q^5+2q^4+q^3+q^2+q+1\\
&B_q^{(2)}(4,3) = q^6+q^5+q^4+q^2+q+1\\
&B_q^{(2)}(4,4) = 1.
\end{aligned}$$

The following identity holds:

$$B_q^{(2)}(n,r) = \begin{bmatrix}2n-1\\n+r-1\end{bmatrix}_q - q^r\begin{bmatrix}2n-1\\n+r\end{bmatrix}_q. \tag{3.2}$$

This demonstrates that $B_q^{(2)}(n,r)$ is a polynomial with integer coefficients. We do not know a combinatorial interpretation of $B_q^{(2)}(n,r)$.

It is interesting to notice that

$$B_q(n,1) = \frac{S_q(1,n)}{1+q^n} = T_q(1,n) \qquad \text{and} \qquad B_q^{(2)}(n,1) = \frac{S_q(1,n)}{1+q} = U_q(1,n).$$

Because of this relation it might be tempting to think that $B_q^2(n,r)$ is the generating function for the major index of $\mathcal{B}(n,r)$. This is not the case. For example,

$$B_q^{(2)}(3,2) \neq \sum_{\pi \in \mathcal{B}(3,2)} q^{\mathrm{maj}(\pi)}.$$

Here are some values of the function $B_q^{(3)}(n,r) = \sum_{\pi \in \mathcal{B}(n,r)} q^{\mathrm{maj}(\pi)}$:

$$\begin{aligned}
&B_q^{(3)}(1,1) = 1\\
&B_q^{(3)}(2,1) = q^2+1\\
&B_q^{(3)}(2,2) = 1\\
&B_q^{(3)}(3,1) = q^6+q^4+q^3+q^2+1\\
&B_q^{(3)}(3,2) = q^4+q^3+q^2+1\\
&B_q^{(3)}(3,3) = 1\\
&B_q^{(3)}(4,1) = q^{12}+q^{10}+q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+1\\
&B_q^{(3)}(4,2) = q^{10}+q^9+2q^8+q^7+2q^6+2q^5+2q^4+q^3+q^2+1\\
&B_q^{(3)}(4,3) = q^6+q^5+q^4+q^3+q^2+1\\
&B_q^{(3)}(4,4) = 1.
\end{aligned}$$

We do not know an explicit formula for $B_q^3(n,r)$.

3.2 q-analogs of Two Important Super Catalan Identities

In this section we provide two identities for $T_q(m,n)$. The following is a q-analog of Rubenstein's Identity (Eq. 1.2).

Proposition 2.

$$(1+q^n)(1+q^{n-m})T_q(m,n) = q^{n-m}T_q(n,m+1) + T_q(m,n+1). \tag{3.3}$$

Proof.

$$
\begin{aligned}
&(1+q^n)(1+q^{n-m})T_q(m,n) - T_q(m,n+1)\\
&\quad = (1+q^n)(1+q^{n-m})\frac{S_q(m,n)}{1+q^n} - \frac{S_q(m,n+1)}{1+q^{n+1}}\\
&\quad = \frac{(1+q^{n-m})[2n]_q![2m]_q!}{[n]_q![m]_q![n+m]_q!} - \frac{[2n+2]_q![2m]_q!}{(1+q^{n+1})[n+1]_q![m]_q![n+m+1]_q!}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{[2n]_q![2m]_q!}{[n]_q![m]_q![n+m+1]_q!}\right)\left((1+q^{n-m})[n+m+1]_q - \frac{[2n+2]_q[2n+1]_q}{(1+q^{n+1})[n+1]_q}\right)\\
&= \left(\frac{[2n]_q![2m]_q!}{[n]_q![m]_q![n+m+1]_q!}\right)\left((1+q^{n-m})[n+m+1]_q - [2n+1]_q\right)\\
&= \left(\frac{[2n]_q![2m]_q!}{[n]_q![m]_q![n+m+1]_q!}\right)\left(q^{n-m} + \ldots + q^{n+m}\right)\\
&= q^{n-m}\frac{[2n]_q![2m]_q![2m+1]_q}{[n]_q![m]_q![n+m+1]_q!}\\
&= q^{n-m}\frac{[2n]_q![2m+2]_q!}{(1+q^{m+1})[n]_q![m+1]_q![n+m+1]_q!}\\
&= q^{n-m}\frac{S_q(m+1,n)}{1+q^{m+1}} = q^{n-m}T_q(n,m+1).
\end{aligned}
$$

□

We will use the next identity later in this section to generalize Eq. 2.3.

Proposition 3. *For* $1 \le k \le m$*,*

$$
q^{(n-1)(k-1)}T_q(m,n) = \sum_{r=1}^{k}(-1)^{r-1}q^{\binom{r-1}{2}}\begin{bmatrix}k-1\\r-1\end{bmatrix}_q \frac{[2n-1]!_q[2m]!_q[m-k+r]_q![m+2r-k]_q}{[n-r]!_q[n+m+r-k]!_q[m]!_q[m+r]!_q}. \tag{3.4}
$$

Proof. We will proceed by induction on k. When $k=1$, Eq. 3.4 is

$$
T_q(m,n) = \frac{[2n-1]!_q[2m]!_q[m]!_q[m+1]_q}{[n-1]!_q[n+m]!_q[m]!_q[m+1]!_q},
$$

which is the super Catalan polynomial.

Assume Eq. 3.4 holds for some $k \geq 1$, then

$$
\begin{aligned}
&q^{(n-1)k} T_q(m,n) = \\
&= q^{n-1} \sum_{r=1}^{k} (-1)^{r-1} q^{\binom{r-1}{2}} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q \frac{[2n-1]!_q [2m]!_q [m-k+r]_q! [m+2r-k]_q}{[n-r]!_q [n+m+r-k]!_q [m]!_q [m+r]!_q} \\
&= q^{n-1} \sum_{r=1}^{k} (-1)^{r-1} q^{\binom{r-1}{2}} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q \frac{[2n-1]!_q [2m]!_q [m-k+r]_q! ([n+m+r-k]_q - [n-r]_q)}{q^{n-r} [n-r]!_q [n+m+r-k]!_q [m]!_q [m+r]!_q} \\
&= \sum_{r=1}^{k} (-1)^{r-1} q^{\binom{r-1}{2}} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q \frac{q^{r-1} [2n-1]!_q [2m]!_q [m-k+r]_q!}{[n-r]!_q [n+m+r-k-1]!_q [m]!_q [m+r]!_q} \\
&\quad - \sum_{r=1}^{k} (-1)^{r-1} q^{\binom{r-1}{2}} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q \frac{q^{r-1} [2n-1]!_q [2m]!_q [m-k+r]_q!}{[n-r-1]!_q [n+m+r-k]!_q [m]!_q [m+r]!_q} \\
&= \frac{[2n-1]!_q [2m]!_q [m-k+1]!_q}{[n-1]!_q [n+m-k]!_q [m]!_q [m+1]!_q} \\
&\quad + \sum_{j=2}^{k} (-1)^{j-1} q^{\binom{j-1}{2}} q^{j-1} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \frac{[2n-1]!_q [2m]!_q [m-k+j-1]_q! [m-k+j]_q}{[n-j]!_q [n+m+j-k-1]!_q [m]!_q [m+j]!_q} \\
&\quad + \sum_{j=2}^{k} (-1)^{j-1} q^{\binom{j-1}{2}} \begin{bmatrix} k-1 \\ j-2 \end{bmatrix}_q \frac{[2n-1]!_q [2m]!_q [m-k+j-1]_q! [m+j]_q}{[n-j]!_q [n+m+j-k-1]!_q [m]!_q [m+j]!_q} \\
&\quad + (-1)^k q^{\binom{k}{2}} \frac{[2n-1]!_q [2m]!_q [m]!_q}{[n-k-1]!_q [n+m]!_q [m]!_q [m+k]!_q} \\
&= \frac{[2n-1]!_q [2m]!_q [m-k]!_q [m+2-(k+1)]_q}{[n-1]!_q [n+m+1-(k+1)]!_q [m]!_q [m+1]!_q} \\
&\quad + \sum_{j=2}^{k} (-1)^{j-1} q^{\binom{j-1}{2}} \begin{bmatrix} k \\ j-1 \end{bmatrix}_q \frac{[2n-1]!_q [2m]!_q [m-k+j-1]_q! [m+j-k]_q}{[n-j]!_q [n+m+j-k-1]!_q [m]!_q [m+j]!_q} \\
&\quad + \sum_{j=2}^{k} (-1)^{j-1} q^{\binom{j-1}{2}} q^{m+j-k} \begin{bmatrix} k-1 \\ j-2 \end{bmatrix}_q \frac{[2n-1]!_q [2m]!_q [m-k+j-1]_q! [k]_q}{[n-j]!_q [n+m+j-k-1]!_q [m]!_q [m+j]!_q} \\
&\quad + (-1)^k q^{\binom{k}{2}} \frac{[2n-1]!_q [2m]!_q [m]!_q [m+2(k+1)-(k+1)]_q}{[n-(k+1)]!_q [n+m+k+1-(k+1)]!_q [m]!_q [m+k+1]!_q}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{[2n-1]!_q[2m]!_q[m-k]!_q[m+2-(k+1)]_q}{[n-1]!_q[n+m+1-(k+1)]!_q[m]!_q[m+1]!_q} \\
&\quad + \sum_{j=2}^{k}(-1)^{j-1}q^{\binom{j-1}{2}}\begin{bmatrix}k\\ j-1\end{bmatrix}_q \frac{[2n-1]!_q[2m]!_q[m-k+j-1]_q![m+j-k]_q}{[n-j]!_q[n+m+j-k-1]!_q[m]!_q[m+j]!_q} \\
&\quad + \sum_{j=2}^{k}(-1)^{j-1}q^{\binom{j-1}{2}}q^{m+j-k}\begin{bmatrix}k\\ j-1\end{bmatrix}_q \frac{[2n-1]!_q[2m]!_q[m-k+j-1]_q![j-1]_q}{[n-j]!_q[n+m+j-k-1]!_q[m]!_q[m+j]!_q} \\
&\quad + (-1)^k q^{\binom{k}{2}} \frac{[2n-1]!_q[2m]!_q[m]!_q[m+2(k+1)-(k+1)]_q}{[n-(k+1)]!_q[n+m+k+1-(k+1)]!_q[m]!_q[m+k+1]!_q} \\
&= \frac{[2n-1]!_q[2m]!_q[m-k]!_q[m+2-(k+1)]_q}{[n-1]!_q[n+m+1-(k+1)]!_q[m]!_q[m+1]!_q} \\
&\quad + \sum_{j=2}^{k}(-1)^{j-1}q^{\binom{j-1}{2}}\begin{bmatrix}k\\ j-1\end{bmatrix}_q \frac{[2n-1]!_q[2m]!_q[m-k+j-1]_q![m+2j-k-1]_q}{[n-j]!_q[n+m+j-k-1]!_q[m]!_q[m+j]!_q} \\
&\quad + (-1)^k q^{\binom{k}{2}} \frac{[2n-1]!_q[2m]!_q[m]!_q[m+2(k+1)-(k+1)]_q}{[n-(k+1)]!_q[n+m+k+1-(k+1)]!_q[m]!_q[m+k+1]!_q} \\
&= \sum_{j=1}^{k+1}(-1)^{j-1}q^{\binom{j-1}{2}}\begin{bmatrix}k\\ j-1\end{bmatrix}_q \frac{[2n-1]!_q[2m]!_q[m-(k+1)+j]_q![m+2j-(k+1)]_q}{[n-j]!_q[n+m+j-(k+1)]!_q[m]!_q[m+j]!_q}.
\end{aligned}
$$

□

We remark that depending on k, Eq. 3.4 is not necessarily the sum of polynomials. We are interested in Eq. 3.4 when $k = m$ and will show that it provides an expression for $q^{(n-1)(m-1)}T_q(m,n)$ which is the sum of polynomials. The following is a generalization of Eq. 2.3.

Theorem 7.

$$
q^{(n-1)(m-1)}T_q(m,n) = \sum_{r=1}^{m}(-1)^{r-1}q^{\binom{r-1}{2}}\frac{1+q^m}{1+q^r}B_q(n,r)B_q(m,r). \tag{3.5}
$$

Proof. Substituting $k = m$ in Eq. 3.4 yields

$$\begin{aligned} q^{(n-1)(m-1)} T_q(m,n) &= \sum_{r=1}^{m} (-1)^{r-1} q^{\binom{r-1}{2}} \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \frac{[2n-1]!_q [2m]!_q [r]_q! [2r]_q}{[n-r]!_q [n+r]!_q [m]!_q [m+r]!_q} \\ &= \sum_{r=1}^{m} (-1)^{r-1} q^{\binom{r-1}{2}} \frac{[2n-1]!_q [2m]!_q [r]_q [2r]_q}{[n-r]!_q [n+r]!_q [m]_q [m+r]!_q [m-r]!_q} \\ &= \sum_{r=1}^{m} (-1)^{r-1} q^{\binom{r-1}{2}} \frac{1+q^m}{1+q^r} B_q(n,r) B_q(m,r). \end{aligned}$$

□

We will use Theorem 7 in Section 3.3 to derive an interpretation of $T_q(2,n)$.

We remind the reader, in [1], proofs that $T_q(m,n)$ and $U_q(m,n)$ are polynomials were given. Here we give a more elegant proof, that they are polynomials with integer coefficients. We remark that this result could also be derived by using Warnaar's and Zudilin's Eq. 1.11.

Theorem 8. *$T_q(m,n)$, and $U_q(m,n)$ are polynomials with integer coefficients.*

Proof. Recall that Warnaar and Zudilin [35] provided a proof that $S_q(m,n)$ has integer coefficients. From Theorem 7,

$$S_q(m,n) = \frac{1}{q^{(n-1)(m-1)}} (1+q^n) \sum_{r=1}^{m} (-1)^{r-1} q^{\binom{r-1}{2}} \frac{1+q^m}{1+q^r} B_q(n,r) B_q(m,r).$$

Because $S_q(m,n)$ has integer coefficients and q does not divide $(1+q^n)$, we must have that $q^{(n-1)(m-1)}$ divides

$$\sum_{r=1}^{m} (-1)^{r-1} q^{\binom{r-1}{2}} \frac{1+q^m}{1+q^r} B_q(n,r) B_q(m,r).$$

Thus,

$$T_q(m,n) = \frac{1}{q^{(n-1)(m-1)}} \sum_{r=1}^{m} (-1)^{r-1} q^{\binom{r-1}{2}} \frac{1+q^m}{1+q^r} B_q(n,r) B_q(m,r)$$

has integer coefficients. □

We can rewrite Eq. 3.5 as

$$q^{(n-1)(m-1)} T_q(m,n) = \sum_{r=1}^{m} (-1)^{r-1} q^{\binom{r-1}{2}} B_q(n,r) B_q^{(2)}(m,r).$$

Because $B_q(n,r)$ and $B_q^{(2)}(m,r)$ have been show to be polynomials,

$$\sum_{r=1}^{m}(-1)^{r-1}q^{\binom{r-1}{2}}B_q(n,r)B_q^{(2)}(m,r)$$

is the sum of polynomials. A combinatorial interpretation of $B_q^{(2)}(m,r)$ would provide us with a weighted interpretation of $q^{(n-1)(m-1)}T_q(m,n)$.

3.3 A Combinatorial Interpretation of $T_q(2,n)$

In this section we provide an interpretation of $T_q(2,n)$. The following identity is from Theorem 7 when $m=2$:

$$q^{n-1}T_q(2,n) = (1+q^2)B_q(n,1) - B_q(n,2). \tag{3.6}$$

We recall, Ω_n denotes the set of $\pi \in \mathcal{C}_n$ such that $h_+(\pi) \le h_-(\pi) + 2$.

Theorem 9.

$$T_q(2,n) = q^{n-1} + q^{3-n}\sum_{\pi\in\Omega_n} q^{\mathrm{maj}(\pi)-\mathrm{des}(\pi)}. \tag{3.7}$$

Proof. By $\mathcal{B}^*(n,2)$ we denote the set of paths in $\mathcal{B}(n,2)$ which do not attain level $y=-1$ before their right-most maximum. Let $\mathcal{B}^{**}(n,2) = \mathcal{B}(n,2) - \mathcal{B}^*(n,2)$ and

$$B_q^*(n,2) = \sum_{\pi\in\mathcal{B}^*(n,2)} q^{\mathrm{maj}(\pi)-\mathrm{des}(\pi)}; \qquad B_q^{**}(n,2) = \sum_{\pi\in\mathcal{B}^{**}(n,2)} q^{\mathrm{maj}(\pi)-\mathrm{des}(\pi)}.$$

By Theorem 6 and Eq. 3.6

$$q^{n-1}T_q(2,n) = (B_q(n,1) - B_q^*(n,2)) + (q^2B_q(n,1) - B_q^{**}(n,2)).$$

First we compute $B_q(n,1) - B_q^*(n,2)$. For $\pi \in \mathcal{B}^*(n,2)$, let RQ be the *down* step that follows the right-most maximum point R of π. We define $f(\pi)$ to be the path obtained by substituting the *down* step RQ by an *up* step. See Figure 3.1. Note that $f(\pi) \in \mathcal{C}_n$ and, since at least two *up* steps precede Q on $f(\pi)$, the height of $f(\pi)$ is at least two. Also π and $f(\pi)$ have the same set of descents, thus $\mathrm{des}(\pi) = \mathrm{des}(f(\pi))$ and $\mathrm{maj}(\pi) = \mathrm{maj}(f(\pi))$. It is important to mention that Q is the left-most maximum on $f(\pi)$.

We will show that f is a bijection between $\mathcal{B}^*(n,2)$ and the set of paths ρ in $\mathcal{C}_n$ of height $h(\rho) > 1$. Let Q be the left-most maximum on ρ and RQ be the *up* step that precedes Q.

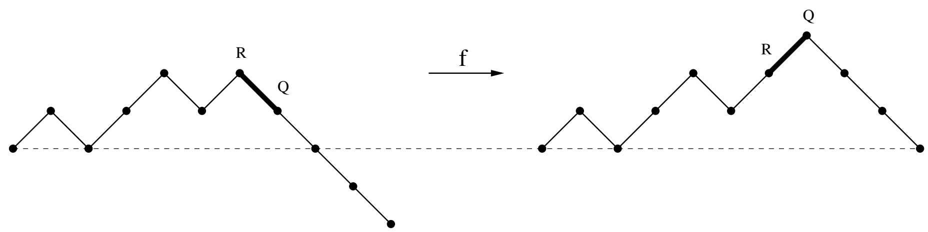


Figure 3.1: f substitutes the *down* step RQ by an *up* step

Substitute the *up* step RQ by a *down* step, which makes R the right-most maximum of the resulting path, call it π. Note that $\pi \in \mathcal{B}^*(n,2)$ and $f(\pi) = \rho$.

It follows that

$$B_q(n,1) - B_q^*(n,2) = \sum_{\substack{\pi \in \mathcal{C}_n \\ h(\pi)=1}} q^{\mathrm{maj}(\pi)-\mathrm{des}(\pi)} = q^{(n-1)^2}.$$

We define a **descent point** to be a point on a path which is preceded by a *down* step, and followed by a *up* step. We define a **down wedge sequence** to be a portion of a path that starts with a *down* step, alternates between *down* steps and *up* steps, and ends with an *up* step. See Figure 3.2.

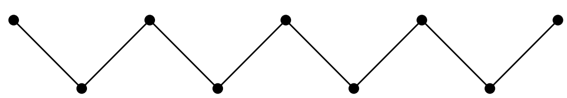


Figure 3.2: A down wedge sequence

We find a combinatorial interpretation for $q^2 B_q(n,1) - B_q^{**}(n,2)$ by establishing an injection g from $\mathcal{B}^{**}(n,2)$ to $\mathcal{C}_n$. A path π in $\mathcal{B}^{**}(n,2)$ attains $y = -1$ before its right-most maximum point R. Let N be the first point before R at which π attains $y = -1$. We consider two cases: when N is immediately followed by a *down* step, and when N is immediately followed by an *up* step.

Case one: N is immediately followed by a *down* step. Since N is the left-most point on π on level $y = -1$, N is preceded by two *down* steps and is followed by one *down* step, then one *up* step. See Figure 3.3. Let MN be the *down* step that precedes N and NY be the *down* step that follows N. Substitute MY by two *up* steps. The resulting path is a ballot path of length $2n$ that ends at level two. Rename N to be X. From left to right, X is the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4.

Let L be the left-most maximum point of this ballot path and QL be the *up* step that precedes L. Substitute the *up* step QL by a *down* step. See Figure 3.3. The resulting path $g(\pi)$ is in $\mathcal{C}_n$ and Q is its right-most maximum. Point X is the last level one point on $g(\pi)$ before its right-most maximum Q and the point on $g(\pi)$ before X is a descent. Note that $h_+(g(\pi)) \geq h_-(g(\pi)) + 3$. Also $\mathrm{des}(\pi) = \mathrm{des}(g(\pi))$ and $\mathrm{maj}(\pi) = \mathrm{maj}(g(\pi)) + 2$. Thus $\mathrm{maj}(\pi) - \mathrm{des}(\pi) = \mathrm{maj}(g(\pi)) - \mathrm{des}(g(\pi)) + 2$.

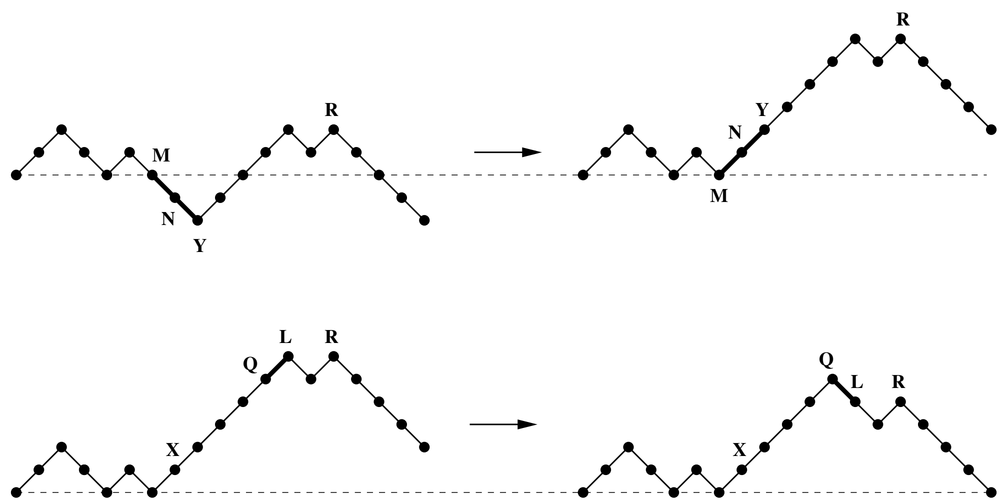


Figure 3.3: The action of g when N is followed by a *down* step

Case two: N is immediately followed by an *up* step. Since N is the left-most point on π on level $y = -1$, N is preceded by two *down* steps which form a segment we denote by XN. See Figure 3.4. Let σ be the longest, possibly empty, down wedge sequence that precedes X. Let Y be the first point of the sequence σ. Note that either Y is the second point on π or Y is preceded by a *down* step. Remove σ from its original position and insert it immediately after N. Then substitute XN by two *up* steps. The resulting path is a ballot path of length $2n$ that ends at level two. From left to right, X is the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4.

Let L be the left-most maximum point of this ballot path and QL be the *up* step that precedes L. Substitute the *up* step QL by a *down* step. See Figure 3.4. The resulting path $g(\pi)$ is in $\mathcal{C}_n$ and Q is its right-most maximum. Note that X is the last level one point on $g(\pi)$ before its right-most maximum Q and the point on $g(\pi)$ before X is NOT a descent. Also $h_+(g(\pi)) \geq h_-(g(\pi)) + 3$. If Y is the second point on the original path π, then g removes the original descent of π that occurs immediately after Y and moves the descent that originally corresponds to N one unit to the left. Thus $\text{des}(\pi) = \text{des}(g(\pi)) + 1$ and $\text{maj}(\pi) = \text{maj}(g(\pi)) + 3$. If Y is NOT the second point on the original path π and σ is not empty, then g moves the descent that originally occurs immediately after Y and the one that corresponds to N one unit to the left. If Y is NOT the second point on the original path π and σ is empty, then g moves the descent that corresponds to N two units to the left. Thus $\text{des}(\pi) = \text{des}(g(\pi))$ and $\text{maj}(\pi) = \text{maj}(g(\pi)) + 2$. In all the cases $\text{maj}(\pi) - \text{des}(\pi) = \text{maj}(g(\pi)) - \text{des}(g(\pi)) + 2$.

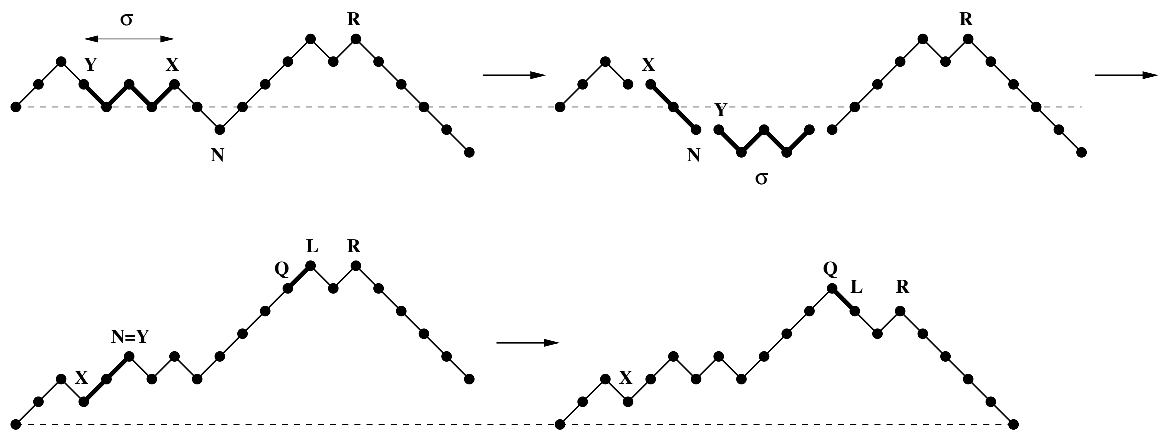


Figure 3.4: The action of g when N is followed by an *up* step

If a path ρ is in the image of g, then $h_+(\rho) \geq h_-(\rho) + 3$, thus $\rho \in \mathcal{C}_n - \Omega_n$. We will show that the image of g is $\mathcal{C}_n - \Omega_n$. Let ρ be in $\mathcal{C}_n$ and $h_+(\rho) \geq h_-(\rho) + 3$. Let Q be the right-most maximum on ρ and QL be the *down* step that follows Q. Substitute the *down* step QL by an *up* step. The result is a ballot path of length $2n$ that ends at level two. Note that L is the left-most maximum on this ballot path. Let R denote the right-most maximum on this ballot path. From left to right, let X be the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point X is less than the maximum level it reaches after and including point X by at least 4. We consider two cases: when the point before X is a descent, and when the point before X is not a descent.

If the point before X is a descent, let M be that descent. Let Y be the point that follows X. Since X is the last point on level one before R, XY is an *up* step. Substitute MY with two *down* steps. We will call the resulting path π. Note that π attains level $y = -1$ before its right-most maximum R and, immediately after attaining level $y = -1$ for the first time, it attains level $y = -2$. Thus π is in $\mathcal{B}^{**}(n, 2)$, falls into Case 1, and $g(\pi) = \rho$.

Next we consider the case when the point before X is NOT a descent. Note that, since X is the last level one point before R, X is followed by two *up* steps. Let XY be the segment that consists of these two *up* steps. Let σ be the longest, possibly empty, down wedge sequence that starts at Y. Note that σ is followed by an *up* step. Remove σ from its original position and insert it immediately before X, then substitute XY with two *down* steps. We will call the resulting path π. Note that π attains level $y = -1$ for the first time at Y, before its right-most maximum R, and Y is followed by an *up* step. Thus π is in $\mathcal{B}^{**}(n, 2)$, falls into Case 2, and $g(\pi) = \rho$.

It follows that

$$q^2 B_q(n,1) - B_q^{**}(n,2) = q^2 \sum_{\pi \in \Omega_n} q^{\mathrm{maj}(\pi)-\mathrm{des}(\pi)}.$$

□

3.4 Several Related Combinatorial Proofs

There are several identities regarding the q- binomial coefficient, $S_q(0,n)$ and $U_q(1,n)$ which we would like to prove combinatorially; they can easily be verified algebraically. Our main goal is to explain the relation

$$S_q(0,n) = (1 + \ldots + q^n) U_q(1,n).$$

Proposition 4.

$$\begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q = \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_q. \tag{3.8}$$

Proof. Let $\pi \in \mathfrak{S}(n, n-1)$. Let L be the leftmost highest point on π. Because π ends at level one, L is not the first point on π. Let K be the point preceding L on π. Change the *up* step KL to a *down* step to obtain a path $\rho \in \mathfrak{S}(n-1, n)$ with $\mathrm{maj}(\pi) = \mathrm{maj}(\rho)$. We see this is a bijection; given $\rho \in \mathfrak{S}(n-1, n)$, let R be the rightmost highest point on ρ. Because ρ ends at level -1, R is not the last point on ρ. Obtain π by changing the *down* step following R into an *up* step. □

Proposition 5.

$$(1 + q^n) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q = \begin{bmatrix} 2n \\ n \end{bmatrix}_q. \tag{3.9}$$

Proof. We will construct a bijection

$$\varphi : \{\pi \in \mathfrak{S}(n,n) : \mathrm{maj}(\pi) = k\} \to \{\pi \in \mathfrak{S}(n, n-1) : \mathrm{maj}(\pi) = k \text{ or } k - n\}.$$

Let $\pi \in \mathfrak{S}(n,n)$ with $\operatorname{maj}(\pi) = k$. If the last step of π is *down*, then let $\rho = \varphi(\pi)$ be the path obtained from π by deleting the last step of π. Then $\varphi(\pi) \in \mathfrak{S}(n, n-1)$ with $\operatorname{maj}(\pi) = \operatorname{maj}(\varphi(\pi))$. Otherwise, let $\rho = \varphi(\pi)$ be path obtained by reflecting π over the x-axis then deleting the last step. Then $\varphi(\pi) \in \mathfrak{S}(n, n-1)$. We have $\operatorname{maj}(\varphi(\pi)) = k - n$ by Lemma 1.

We see this is a bijection. For $\rho \in \mathfrak{S}(n, n-1)$ with $\operatorname{maj}(\rho) = k$, define π to be the path obtained from ρ by appending a *down* step to the end of ρ. Then π satisfies $\varphi(\pi) = \rho$. For $\rho \in \mathfrak{S}(n, n-1)$ with $\operatorname{maj}(\rho) = k - n$, let ρ be the path obtained from π by reflecting π over the x-axis and then appending a up step. Then ρ satisfies $\varphi(\pi) = \rho$. □

From Equations 3.8 and 3.9, we see

Corollary 2.

$$(1 + q^n)\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_q = \begin{bmatrix} 2n \\ n \end{bmatrix}_q. \tag{3.10}$$

We remind the reader that $\mathfrak{S}_+(m,n)$ is defined to be the set of paths that have m *up* steps, n *down* steps, and never go below the x-axis. Define $\mathfrak{S}_-(m,n)$ to be the set of paths that have m *up* steps, n *down* steps, and go below the x-axis. Define $\mathfrak{S}^-(m,n)$ to be the set of paths that have m *up* steps, n *down* steps, and never go above the x-axis. Define $\mathfrak{S}^+(m,n)$ to be the set of paths that have m *up* steps, n *down* steps, and go above the x-axis.

The following is a generalized version of a bijection presented in [19].

Lemma 2. *For $n \geq m$, there exists a bijection*

$$\phi : \{\pi \in \mathfrak{S}_-(n,m) : \operatorname{maj}(\pi) = k\} \to \{\pi \in \mathfrak{S}(n+1, m-1) : \operatorname{maj}(\pi) = k-1\}.$$

Proof. Let $\pi \in \mathfrak{S}_-(n,m)$. Let L be the leftmost lowest point on π. Because π goes below the x-axis, L is not the first point on π and is preceded by a *down* step. Let $\phi(\pi) = \rho$ be the path obtained from π by changing the *down* step preceding L into an *up* step. Then $\rho \in \mathfrak{S}(n+1, m-1)$ and $\operatorname{maj}(\rho) = \operatorname{maj}(\pi) - 1$. We will show that ϕ is a bijection from $\mathfrak{S}_-(n,m)$ to $\mathfrak{S}(n+1, m-1)$. Let $\rho \in \mathfrak{S}(n+1, m-1)$. Let R be the rightmost lowest point on ρ. Since $n+1 > m-1$, R is not the last point on ρ and is followed by a *up* step. Obtain π from ρ by changing the *up* step following R into a *down* step. Then π satisfies $\phi(\pi) = \rho$. □

Lemma 3. *There exists a bijection*

$$\varphi : \{\pi \in \mathfrak{S}(n+1, n-1) : \operatorname{maj}(\pi) = k\} \to \{\pi \in \mathfrak{S}^+(n,n) : \operatorname{maj}(\pi) = k\}.$$

Proof. Let $\pi \in \mathfrak{S}(n+1, n-1)$ with $\text{maj}(w) = k$. Let L be the leftmost highest point on π. Since π ends at $(2n, 2)$, L is not the first point on π and is preceded by an *up* step. Let $\varphi(\pi) = \rho$ be the path obtained from π by changing the *up* step preceding L into a *down* step. We see that $\rho \in \mathfrak{S}^+(n, n)$ because the height of ρ will be at least 1. Because the descent sets of π and ρ are equal, $\text{maj}(\pi) = \text{maj}(\rho)$. We will show that φ is a bijection from $\mathfrak{S}(n+1, n-1)$ to $\mathfrak{S}^+(n, n)$. Given $\rho \in \mathfrak{S}^+(n, n)$, let R be the rightmost highest point on ρ. Since R cannot be the last point on ρ, R must be followed by a *down* step. Define π to be the path obtained from ρ by changing the *down* step following R into a *up* step. The path π will satisfy $\varphi(\pi) = \rho$. $\square$

Corollary 3. *If we compose the bijections in Lemma 2 and Lemma 3, we have a bijection between paths which go below the x-axis and paths which go above the x-axis:*

$$\psi : \{\pi \in \mathfrak{S}_-(n, n) : \text{maj}(\pi) = k\} \to \{\pi \in \mathfrak{S}^+(n, n) : \text{maj}(\pi) = k - 1\}.$$

Proposition 6.

$$S_q(0, n) + q^{n+1} U_q(1, n) = q S_q(0, n) + U_q(1, n). \tag{3.11}$$

Equivalently,

$$(1 - q) S_q(0, n) = (1 - q^{n+1}) U_q(1, n).$$

Proof. We will construct a bijection

$$\begin{aligned}\varphi : \{\pi \in \mathfrak{S}(n, n) : \text{maj}(\pi) = k\} \cup \{\pi \in \mathfrak{S}_+(n, n) : \text{maj}(\pi) = k - n - 1\} \to \\ \{\pi \in \mathfrak{S}(n, n) : \text{maj}(\pi) = k - 1\} \cup \{\pi \in \mathfrak{S}_+(n, n) : \text{maj}(\pi) = k\}.\end{aligned}$$

For $\pi \in \mathfrak{S}(n, n)$ with $\text{maj}(\pi) = k$, if $\pi \in \mathfrak{S}_+(n, n)$ then define $\varphi(\pi) = \pi \in \mathfrak{S}_+(n, n)$ with $\text{maj}(\varphi(\pi)) = k$. Otherwise define $\varphi(\pi) = \psi(\pi)$ from Corollary 3 with $\text{maj}(\varphi(\pi)) = k - 1$.

For $\pi \in \mathfrak{S}_+(n, n)$ with $\text{maj}(\pi) = k - n - 1$, define $\varphi(\pi)$ to be the path obtained by reflecting π over the x-axis. Observe that the last step of π must be a *down* step and $\varphi(\pi) \in \mathfrak{S}^-(n, n)$. By Lemma 1, $\text{maj}(\varphi(\pi)) = k - 1$.

We will show that φ is a bijection from $\{\pi \in \mathfrak{S}(n, n) : \text{maj}(\pi) = k\} \cup \{\pi \in \mathfrak{S}_+(n, n) : \text{maj}(\pi) = k - n - 1\}$ to $\{\pi \in \mathfrak{S}(n, n) : \text{maj}(\pi) = k - 1\} \cup \{\pi \in \mathfrak{S}_+(n, n) : \text{maj}(\pi) = k\}$. Given $\rho \in \mathfrak{S}(n, n)$, if $\text{maj}(\rho) = k - 1$ and ρ lies strictly below the x-axis, then define π to be the path obtained from ρ by reflecting π over the x-axis. Then $\pi \in \mathfrak{S}_+(n, n)$ with $\text{maj}(\pi) = k - 1 - n$ and $\varphi(\pi) = \rho$. If $\text{maj}(\rho) = k - 1$ and ρ goes above the x-axis, then define $\pi = \psi^{-1}(\rho)$ from Corollary 3. Then $\pi \in \mathfrak{S}(n, n)$ with $\text{maj}(\pi) = k$ and $\varphi(\pi) = \rho$. Finally, if $\rho \in \mathfrak{S}_+(n, n)$ with $\text{maj}(\rho) = k$, then define $\pi = \rho$, so $\text{maj}(\pi) = k$ and $\varphi(\pi) = \rho$. $\square$

Our goal for the remainder of this section is to explain the relation

$$S_q(0,n) = (1 + \ldots + q^n)U_q(1,n).$$

We will fix the following functions.

Let φ be the bijection in Corollary 3. That is,

$$\varphi : \mathfrak{S}_-(n,n) \to \mathfrak{S}^+(n,n),$$

where $\varphi(\pi)$ is obtained from π by lowering the left-most highest point, and then raising the left-most lowest point. Let α be the bijection in Lemma 1 restricted to $\mathfrak{S}_+(n,n)$. That is,

$$\alpha : \mathfrak{S}_+(n,n) \to \mathfrak{S}^-(n,n)$$

reflects a Dyck path over the x-axis.

Consider the algorithm below which takes as an input $P \in \mathfrak{S}(n,n)$ and returns as an output $\mathcal{A}(P) \in \mathfrak{S}_+(n,n)$.

$k \leftarrow \text{maj}(P)$
$i \leftarrow k$
$Q \leftarrow P$
while Q is not a Dyck path or $i < k - n$ **do**
  if Q is not a Dyck path **then**
    $Q \leftarrow \varphi(Q)$
    $i \leftarrow i - 1$
  else
    $Q \leftarrow \alpha(Q)$
    $i \leftarrow i + n$
  end if
end while
$\mathcal{A}(P) \leftarrow Q$
  return $\mathcal{A}(P)$

Lemma 4. *The algorithm above terminates.*

Proof. Let $P \in \mathfrak{S}(n,n)$ be the input of the algorithm and let $\{P_j\}_{j\geq 0}$ be the list of paths which pass through the "while loop" of the above algorithm with $P_0 = P$. We will show that no path P_j can appear on this list more than once.

Throughout the algorithm $k = \text{maj}(P)$ and $i = \text{maj}(Q)$. We remark that $\text{maj}(P_{j+1}) = \text{maj}(P_j) - 1$, except when P_j is a Dyck path. If P_j is a Dyck path, then $\text{maj}(P_{j+1}) = \text{maj}(P_j) + n$. Assume, for the sake of contradiction, that a path appears twice on the list

$\{P_j\}_{j\geq 0}$. Without loss of generality, let $P = P_0$ be the path that occurs twice. Between the appearances of P there must be at least one Dyck path, otherwise we would have $\mathrm{maj}(P) = \mathrm{maj}(P) - t$, where t is number of iterations of the algorithm between the two occurrences of P. Call the last such Dyck path C. Then $\varphi^t(\alpha(C)) = P$ for some $t \geq 0$. Hence,

$$\mathrm{maj}(P) = \mathrm{maj}(C) + n - t \leq \mathrm{maj}(C) + n.$$

Then the Dyck path C satisfies $\mathrm{maj}(C) \geq \mathrm{maj}(P) - n = k - n$ and so the algorithm would have terminated at C, a contradiction.

But because $\mathrm{maj}(C)$ satisfies $\mathrm{maj}(C) \geq \mathrm{maj}(P) - n$, the algorithm would have terminated at C, a contradiction. Because the list of paths passing through the "while loop" of the above algorithm is finite, the algorithm must terminate. □

Let $\equiv$ be the relation on $\mathfrak{S}(n,n)$ defined by $P \equiv Q$ if and only if $\mathcal{A}(P) = \mathcal{A}(Q)$. We will show that $\equiv$ is an equivalence relation with C_n classes, all of size $n+1$.

Proposition 7. *If $\mathcal{A}(P) = \mathcal{A}(Q)$ for distinct paths P, Q, then* $\mathrm{maj}(P) \neq \mathrm{maj}(Q)$.

Proof. Let $\{P_j\}_{j=0}^r$ and $\{Q_j\}_{j=0}^s$ be the lists of paths which pass through the algorithm, with $P_0 = P$, $Q_0 = Q$, and $P_r = \mathcal{A}(P) = \mathcal{A}(Q) = Q_s$. Without loss of generality, assume $r \geq s$. Since φ and α are bijections, it must be that $P_{r-s} = Q$ with $r > s$ since P and Q are distinct. If there is no Dyck path on the list $\{P_j\}_{j=0}^{r-s-1}$ then $Q = \varphi^{r-s}(P)$ and

$$\mathrm{maj}(Q) = \mathrm{maj}(P_{r-s}) = \mathrm{maj}(P) - (r - s) < \mathrm{maj}(P).$$

If there is a Dyck path on the list $\{P_j\}_{j=0}^{r-s-1}$, then let C be the last such path. Then $\mathrm{maj}(C) < \mathrm{maj}(P) - n$, otherwise the algorithm would have terminated at C. We have $Q = \varphi^t(\alpha(C))$ for some $t \geq 0$. However,

$$\mathrm{maj}(Q) = \mathrm{maj}(C) + n - t < \mathrm{maj}(P) - t \leq \mathrm{maj}(P).$$

In both cases $\mathrm{maj}(Q) < \mathrm{maj}(P)$.

□

Theorem 10.

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = (1 + \cdots + q^n) U_q(1, n).$$

Proof. The relation $\equiv$ is reflexive, symmetric, and transitive. No two Dyck paths may be in the same equivalence class because $\mathcal{A}(P) = P$ when P is a Dyck path. Each equivalence class must contain a Dyck path C because $C \equiv \mathcal{A}(C)$. It follows that $S_+(n,n)$ is a representative class for $\equiv$.

Next observe that, given an input of P, for any Q within the algorithm, $\mathrm{maj}(Q) \leq \mathrm{maj}(P)$. This is because, if $\mathrm{maj}(Q) \leq \mathrm{maj}(P)$, then either Q is reassigned to $\varphi(Q)$ so

$$\mathrm{maj}(\varphi(Q)) = \mathrm{maj}(Q) - 1 < \mathrm{maj}(P)$$

or Q is a Dyck path with $\mathrm{maj}(Q) < \mathrm{maj}(P) - n$ and Q is reassigned to $\alpha(Q)$ so

$$\mathrm{maj}(\alpha(Q)) = \mathrm{maj}(Q) + n < \mathrm{maj}(P).$$

Thus, for the algorithm to terminate at a Dyck path C, the initial path P must have major at least $\mathrm{maj}(C)$. Based on the algorithm termination condition, the initial path has major at most $\mathrm{maj}(C)+n$. By Proposition 7, no two paths with the same major may terminate at C, hence the size of each equivalence class is at most $n+1$. Since there are C_n equivalence classes, and exactly $\binom{2n}{n} = (n+1)C_n$ paths, each equivalence class must have exactly $n+1$ paths. Hence, for every Dyck path C, each major $\mathrm{maj}(C), \mathrm{maj}(C)+1, \ldots, \mathrm{maj}(C)+n$ must appear exactly once as the major of a path in the equivalence class of C. Let $[C]$ denote the equivalence class of the path C. Then

$$\begin{aligned}
\begin{bmatrix} 2n \\ n \end{bmatrix}_q &= \sum_{C \in \mathcal{C}_n} \sum_{P \in [C]} q^{\mathrm{maj}(P)} \\
&= \sum_{C \in \mathcal{C}_n} \sum_{i=0}^{n} q^{\mathrm{maj}(C)+i} \\
&= (1 + \cdots + q^n) U_q(1,n).
\end{aligned}$$

$\square$

The proof of Theorem 10 relies on partitioning a set of objects into equivalences classes and having a basic understanding of the objects in each class. This is a different method than what we had seen throughout the rest of the dissertation. It may be that $S(m,n) = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{m}}$ has an interpretation as an equivalence class of a set of objects of total size $\binom{2m}{m}\binom{2n}{n}$.

Chapter 4

Conclusion and Open Problems

4.1 Statistics of Various Lattice Paths

In this section we review results for the major statistic and the major minus descent statistic over various sets of paths. Here is a final result on the major minus descent statistic.

Theorem 11. *The generating function for the statistic* $\mathrm{maj}(w) - \mathrm{des}(w)$ *over* $w \in \mathfrak{S}(n, m)$ *is*

$$\sum_{w \in \mathfrak{S}(n,m)} q^{\mathrm{maj}(w) - \mathrm{des}(w)} = \begin{bmatrix} n+m-1 \\ n-1 \end{bmatrix}_q + \begin{bmatrix} n+m-1 \\ n \end{bmatrix}_q \tag{4.1}$$

Proof. We will demonstrate a bijection
$\varphi : \{\pi \in \mathfrak{S}(n, m) : \mathrm{maj}(\pi) - \mathrm{des}(\pi) = k\} \rightarrow$

$$\{\pi \in \mathfrak{S}(n-1, m) : \mathrm{maj}(\pi) = k\} \cup \{\pi \in \mathfrak{S}(n, m-1) : \mathrm{maj}(\pi) = k\}$$

Let $\pi \in \mathfrak{S}(n, m)$. Define $\varphi(\pi) = \rho$ to be the path obtained from π by deleting the first step of π. If the first step of π is an *up* step, then $\rho \in \mathfrak{S}(n-1, m)$. In this case,

$$\mathrm{maj}(\rho) = \sum_{i \in D(\rho)} i = \sum_{i+1 \in D(w)} i = \sum_{j \in D(w)} j - 1 = \mathrm{maj}(\pi) - \mathrm{des}(\pi)$$

Otherwise, the first step of π is a *down* step and $\rho \in \mathfrak{S}(n, m-1)$. Again, $\mathrm{maj}(\rho) = \mathrm{maj}(\pi) - \mathrm{des}(\pi)$.

To see this is a bijection, for $\rho \in \mathfrak{S}(n-1, m)$, obtain π by adding an *up* step to the beginning of ρ and for $\rho \in \mathfrak{S}(n, m-1)$, obtain π by adding a *down* step to the beginning of ρ.

Here are the previously known generating functions.

$$\sum_{w\in\mathfrak{S}(n,m)} q^{\mathrm{maj}(w)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q$$

$$\sum_{w\in\mathfrak{S}_+(n,n)} q^{\mathrm{maj}(w)} = \frac{[2n]!_q}{[n]!_q[n+1]!_q}$$

$$\sum_{w\in\mathfrak{S}_+(n,n)} q^{\mathrm{maj}(w)-\mathrm{des}(w)} = \frac{[2n-1]!_q[2]_q}{[n-1]!_q[n+1]!_q}$$

Here are the generating functions which we have provided in this dissertation. The second formula comes from reformulating Theorem 6.

$$\sum_{w\in\mathfrak{S}(n,m)} q^{\mathrm{maj}(w)-\mathrm{des}(w)} = \begin{bmatrix} n+m-1 \\ n-1 \end{bmatrix}_q + \begin{bmatrix} n+m-1 \\ n \end{bmatrix}_q$$

$$\sum_{w\in\mathfrak{S}_+(n,m)} q^{\mathrm{maj}(\mathrm{rev}(w))} = \frac{[n+m]_q![n-m+1]_q!}{[n+1]_q![m]_q!}$$

4.2 Open Problems

We have listed these problems in anticipated order of difficulty.

Question 1. *Find a combinatorial interpretation for*

$$B_q^{(2)}(n,r) = \frac{[2n]!_q[r]!_q}{[n]_q[n+r]!_q[n-r]!_q}.$$

We remind the reader that a combinatorial interpretation for $B_q^{(2)}(n,r)$ would allow us to use Theorem 7 to give a weighted interpretation of $T_q(m,n)$. To obtain an interpretation of $B_q^{(2)}(n,r)$, it may be useful to understand the relation

$$(1+q^r)B_q^{(2)}(n,r) = (1+q^n)B_q(n,r).$$

To understand this relation, we could start by understanding the relation

$$(1+q)U_q(1,n) = (1+q^n)T_q(1,n).$$

Question 2. *Find a combinatorial interpretation and a generalization of*

$$U_q(2,n) = (1+q)B_q(n,1) - qB_q(n,2). \tag{4.2}$$

This equation is similar to Eq. 3.6, which we recall is:

$$q^{n-1}T_q(2,n) = (1+q^2)B_q(n,1) - B_q(n,2).$$

Since Eq. 3.6 has been used to give a conventional combinatorial interpretation of $T_q(2,n)$, we hope that an interpretation of $U_q(2,n)$ can be derived using similar techniques. It is possible that a combinatorial interpretation of $U_q(2,n)$ would be more easily generalizable then the interpretation provided in Theorem 9. This would help in the larger goal of interpreting $T(m,n)$ for higher values of m. We provide a proof of Eq. 4.2, we have not yet attempted to generalize the proof for $U_q(m,n)$.

$$\begin{aligned}
U_q(2,n) &= \\
&= \frac{S_q(2,n)}{1+q^2} \\
&= \frac{[2n-1]_q!}{[n+2]_q![n-1]_q!}\left([3]_q(1+q)(1+q^n)\right) \\
&= \frac{[2n-1]_q!}{[n+2]_q![n-1]_q!}\big((1+q^n+q^{n+1})(1+q+q^2)+q(1+q+q^2)\big) \\
&= \frac{[2n-1]_q!}{[n+2]_q![n-1]_q!}\big((1+2q+q^2)[n+2]_q - q(1+q+q^2+q^3)[n-1]_q\big) \\
&= \frac{[2n-1]_q![2]_q(1+q)}{[n+1]_q![n-1]_q!} - q\frac{[2n-1]_q![4]_q}{[n+2]_q![n-2]!_q} \\
&= (1+q)B_q(n,1) - qB_q(n,2).
\end{aligned}$$

Question 3. *Find an explicit formula for*

$$B_q^{(3)}(n,r) = \sum_{w \in \mathcal{B}(n,r)} q^{\mathrm{maj}(w)}.$$

An explicit formula for $B_q^{(3)}(n,r)$ could help us better understand the q-ballot polynomials $B_q(n,r)$ and $B_q^{(2)}(n,r)$. There is a good chance that the techniques required to interpret the formula for $B_q^{(3)}(n,r)$ have already been developed in Chapter 3. It may be important to keep in mind the fact that $B_q^{(3)}(n,r)$ is not necessarily symmetric.

Question 4. *Find combinatorial interpretations of*

$$T(3,n) = 5B(n,1) - 4B(n,2) + B(n,3)$$

and

$$q^{2(n-1)}T_q(3,n) = (1+q^4)((1+q^2)B_q(n,1) - B_q(n,2)) - q((1+q^2)B_q(n,2) - B_q(n,3) - B_q(n,1)).$$

Ideally, having interpretations of both $T(2,n)$ and $T(3,n)$ would allow us to make a conjecture for the general case $T(m,n)$. Similarly, having interpretations of both $T_q(2,n)$ and $T_q(3,n)$ could help us make a conjecture for $T_q(m,n)$.

Note that we can use the techniques of Section 3.3 to interpret both

$$(1+q^2)B_q(n,1) - B_q(n,2)$$

and

$$(1+q^2)B_q(n,2) - B_q(n,3) - B_q(n,1).$$

Additional techniques are needed to interpret their difference. Perhaps understanding Eq. 4.2 would help us to understand the term q in the expression $q((1+q^2)B_q(n,2) - B_q(n,3) - B_q(n,1))$.

Question 5. *Find a combinatorial interpretation of the identity for $T(m,n)$ found in Eq. 2.3 or the identity for $T_q(m,n)$ found in Eq. 3.5 for $m > 3$.*

Many mathematicians have attempted to interpret $T(m,n)$. This was the inspiration for this dissertation. Based on the interpretation of $T(2,n)$, and Gessel's and Xin's work on $T(3,n)$, it is reasonable to believe that the set of objects counted by $T(m,n)$ is related to lattice paths with certain height restrictions. There is not much literature on lattice paths with certain height restrictions and this would be a significant contribution to that area. It is our belief that the best way to interpret $T(m,n)$ is through the polynomials $T_q(m,n)$ because the polynomials provide additional information about the objects counted by these integers.

Question 6. *Investigate the unimodality of the polynomials $B_q(n,r)$, $S_q(m,n)$, $T_q(m,n)$, and $U_q(m,n)$.*

The unimodality of these polynomials would provide an additional reason to believe that they have a natural combinatorial interpretation. The possible approaches to prove unimodality are discussed in the next section.

4.3 Unimodality

Conjecture 1. *$S_q(m,n)$ is unimodal for $m,n \geq 0$.*

Conjecture 2. *For $m \leq n$, $T_q(m,n)$ is unimodal. Equivalently, for $n \geq m$, $U_q(m,n)$ is unimodal. For $m > n$, $T_q(m,n)$ is unimodal with the exception of its central coefficient. Equivalently, for $n > m$, $U_q(m,n)$ is unimodal with the exception of its central coefficient.*

Conjecture 3. *$B_q(n,r)$ is unimodal for $n,r \geq 1$.*

We have verified these conjectures using Maple for $m,n \leq 10$. In general, proofs of unimodality are difficult, see [36]. The proofs of the above conjectures might be related to the proofs of unimodality of $S_q(0,n)$ and $T_q(1,n)$. A result implying the unimodality of the q-binomial coefficient was conjectured by Cayley and proved by Sylvester, cited in [25]. However, we would like to review the method Stanley uses in [33] to reprove that both $S_q(0,n)$ and $T_q(1,n)$ are unimodal.

Consider a root system to be a finite set R of vectors in a real vector space V satisfying certain axioms, cited in [24]. Given a base for the root system, $\{\alpha_1,...,\alpha_n\}$, we can represent every positive vector $\beta \in R$ uniquely as a linear combination of the roots with non-negative integer coefficients. Thus we have

$$\beta = \sum_{i=1}^{n} c_i \alpha_i \tag{4.3}$$

where c_i is a non-negative integer. We describe a root system, R, by listing the vectors $\beta \in R_+$ in the form given in Eq. (4.3). We denote the vector $\alpha_i + \alpha_{i+1} + ... + \alpha_j$ by $[i,j]$. We write $x^\beta = x_1^{c_1} x_2^{c_2} ... x_n^{c_n}$ and define

$$P(R; x_1, ..., x_n) = \prod_{\beta \in R_+} (1 - x^\beta).$$

Theorem 12 (Dynkin)**.** *Let R be a root system of rank n, and let $m_1,...,m_n$ be any positive integers. Define*

$$Q(R; m_1, ..., m_n) = \frac{P(R; q^{m_1}, q^{m_2}, ..., q^{m_n})}{P(R; q, q, ..., q)} \tag{4.4}$$

Then $Q(R; m_1,...,m_n)$ is a symmetric unimodal polynomial in the variable q with non-negative integer coefficients.

Stanley uses Theorem 12 to derive the two following results.

Corollary 4. *$S_q(0,n)$ is a symmetric unimodal polynomial in the variable q with non-negative integer coefficients.*

Proof. Let A_n be the root system defined by the vectors $[i, j]$ for $1 \leq i \leq j \leq n$ [32]. Let $R = A_n$, $m_1 = n + 1$, and $m_i = 1$ for $2 \leq i \leq n$. Then $Q(R; m_1, \ldots, m_n) = S_q(0, n)$. $\square$

Corollary 5. *$T_q(1, n)$ is a symmetric unimodal polynomial in the variable q with non-negative integer coefficients.*

Proof. Let B_n be the root system defined by the vectors $[i, j]$ for $1 \leq i \leq j \leq n$ and $[i, j-1] + 2[j, n]$ for $1 \leq i < j \leq n$ [32]. Let $R = B_n$, $m_i = 1$ for $1 \leq i \leq n-1$, and $m_n = 2$. Then $Q(R; m_1, \ldots, m_n) = T_q(1, n+1)$ $\square$

Theorem 12 allows us to describe classes of unimodal polynomials arising from Lie Algebras by means of root systems. There are four infinite families of irreducible root systems and five finite irreducible root systems [32]. Our polynomials $S_q(m, n), T_q(m, n)$, and $U_q(m, n)$ did not match these root systems for larger values of m. It is the opinion of Bogdan Ion, with whom we discussed this problem, that the proofs of the above conjectures are probably related to Lie Algebras or Super Lie Algebras.

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