An FPTAS for Minimizing the Product of Two Non-negative Linear Cost Functions

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Abstract

We consider a quadratic programming (QP) problem (Π) of the form $\min x^T Cx$ subject to $Ax \geq b$ where $C \in \mathbb{R}_+^{n \times n}, rank(C) = 1$ and $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. We present an FPTAS for this problem by reformulating the QP (Π) as a parameterized LP and "rounding" the optimal solution. Furthermore, our algorithm returns an extreme point solution of the polytope. Therefore, our results apply directly to 0-1 problems for which the convex hull of feasible integer solutions is known such as spanning tree, matchings and sub-modular flows. They also apply to problems for which the convex hull of the dominant of the feasible integer solutions is known such as s, t-shortest paths and s, t-min-cuts. For the above discrete problems, the quadratic program Π models the problem of obtaining an integer solution that minimizes the *product* of two linear non-negative cost functions.

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1 Introduction

In this paper, we consider the following special case of the non-convex quadratic programming (QP) problem (Π).

$$\begin{aligned} \min\ & x^T Cx \\ & Ax \geq b \end{aligned}$$

where $C \in \mathbb{R}_+^{n \times n}, rank(C) = 1$ and $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and let $P = \{x \in \mathbb{R}^n | Ax \geq b\}$. Since a general rank-1 matrix is not positive semi-definite, $x^T Cx$ is not convex in general. Therefore, Π is a non-convex QP problem. Since $rank(C) = 1, C = c_1 c_2^T$ for some $c_1, c_2 \in \mathbb{R}^n$. Furthermore, since $C \in \mathbb{R}_+^{n \times n}$ both $c_1, c_2 \in \mathbb{R}_+^n$. Therefore, the objective function $x^T Cx$ can be written as a product of two non-negative linear functions,

$$x^T (c_1 c_2^T) x = (c_1^T x) \cdot (c_2^T x)$$

and the problem Π models the problem of minimizing the product of two non-negative linear cost functions over a polyhedral set. This problem is in general non-convex and is known to be NP-hard [9].

We would like to note that an FPTAS for this problem is already known due to Kern and Woeginger [7]. However, our work is independent of [7] and our algorithm differs significantly and gives an interesting alternate approach to solve the problem with a reduced running time. The algorithm presented in [7] does a parametric search for the possible values of the objective function in powers of $(1 + \epsilon)$ for a fixed $\epsilon > 0$. For each possible objective function value (say λ), the authors solve a set of $O(\log_{(1+\epsilon)} \det A)$ linear programs. Based on the optimum values of these linear programs, they are able to distinguish whether $\lambda \leq \mathsf{OPT}$ or $\lambda > \mathsf{OPT} \cdot (1 + \epsilon)$, where $\mathsf{OPT}$is the objective value of the optimal solution. Hence, the total number of LPs solved by their method is $O(\log_{(1+\epsilon)} \frac{U}{L} \cdot \log_{(1+\epsilon)} \det A) \simeq O(\frac{\log \frac{U}{L} \cdot \log \det A}{\epsilon^2})$ where U and L are the upper and lower bounds on the value of the objective function respectively.

On the other hand, our algorithm does a parametric search over the possible values of one of the cost functions. Furthermore, for each possible value of the cost function (say B) we solve a single linear program and then obtain an extreme point x of the polytope such that $c_1(x) \cdot c_2(x) \leq z^* \cdot B$ where z^* is the optimum value of the linear program. Therefore, our algorithm has an improved running time of $O(\log_{(1+\epsilon)} \frac{u}{l}) \simeq O(\frac{\log \frac{u}{l}}{\epsilon})$ LP calls.

1.1 Our Contributions

We give a polynomial time $(1 + \epsilon)$-approximation algorithm for minimizing the problem Π for any fixed $\epsilon > 0$. The following theorem is the main contribution of this paper.

Theorem 1.1 *Given a rank-1 matrix $C \in \mathbb{R}_+^{n \times n}$, a polytope P and $\epsilon > 0$, there is a polynomial time $(1 + \epsilon)$-approximation algorithm $\mathcal{A}$ for the problem Π to minimize*

$$\min_{x \in P} x^T Cx$$

Furthermore, $\mathcal{A}$ returns a solution that is an extreme point of P.

Recall that a point $x \in P$ is an extreme point of P if and only if x can not be expressed as a convex combination of any set of points (not including x) in P. It is well known [7] that the minimum of $x^T Cx$ is achieved at an extreme point of the polyhedral set. We will present a proof of this lemma for the sake of completeness.

Lemma 1.2 *[7] Let $extr(P)$ denote the set of extreme points of P. Then*

$$\min_{x \in extr(P)} (c_1^T x) \cdot (c_2^T x) = \min_{x \in P} (c_1^T x) \cdot (c_2^T x)$$

Since our algorithm obtains an extreme point approximate solution for the problem Π, we show application of our algorithm to the problem of minimizing a rank-1 quadratic objective over a set of 0-1 points when the description of the convex hull or the dominant of the 0-1 points is known.

Corollary 1.3 *Let $S \subset \{0,1\}^n$, $c_1 \in \mathbb{R}^n_+$, $c_2 \in \mathbb{R}^n_+$. If we can optimize over either the convex hull of S or the dominant of S (i.e. $dom(S) = \{x \in \{0,1\}^n | \exists x' \in S, x \geq x'\}$) in polynomial time, then there is an FPTAS for the problem*

$$\min_{x \in S} (c_1^T x) \cdot (c_2^T x)$$

1.2 Applications

Minimum Product Spanning Tree problem: Given an undirected graph $G = (V, E)$, cost functions $c_1 : E \to \mathbb{R}_+$ and $c_2 : E \to \mathbb{R}_+$, the goal is to find a spanning tree T that minimizes $c_1(T) \cdot c_2(T))$. Note that for any subset $E' \subset E, c_i(E') = \sum_{e \in E'} c_i(e)$. The convex hull of all spanning trees is known (see Edmonds [1]).

Minimum Product Matching problem: Given an undirected graph $G = (V, E)$, cost functions $c_1 : E \to \mathbb{R}_+$ and $c_2 : E \to \mathbb{R}_+$, the goal is to find a perfect matching M that minimizes $c_1(M) \cdot c_2(M))$. The convex hull of all perfect matchings is known (see Edmonds [1]).

Minimum Product Submodular Flows: Given a directed graph $D = (V, A)$, cost functions $c_1 : A \to \mathbb{R}_+$ and $c_2 : A \to \mathbb{R}_+$ and a submodular function $f : C \to \mathbb{Z}$ such that for all $S, T \subset V$,

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

A submodular flow $x \in \mathbb{Z}^{|A|}$ is such that

$$\sum_{a \in \delta^+(U)} x(a) - \sum_{a \in \delta^-(U)} x(a) \leq f(U) \forall U \subset V$$

The goal is to find a submodular flow x that minimizes $(c_1^T x) \cdot (c_2^T x)$. The convex hull of submodular flows is known due to Edmonds and Giles [2]. For many applications of submodular flows such as directed spanning trees, matroid bases and orientations, as well as a linear description of all feasible solutions for it, please see [12].

Minimum Product s,t-Min-Cut. Given an undirected graph $G = (V, E)$, vertices $s, t \in V$, cost functions $c_1 : E \to \mathbb{R}_+$ and $c_2 : E \to \mathbb{R}_+$, the goal is to find a cut $(S, \bar{S})$ such that $s \in S, t \notin S$ that minimizes $c_1(\delta(S)) \cdot c_2(\delta(S)))$. For any $S \subset V$, $\delta(S) = \{e = (u, v) \in E | u \in S, v \notin S\}$. We show that the convex hull of the dominant of feasible s, t-cuts is known.

Let $\mathcal{C} = \{x \in \{0,1\}^{|E|} | x$ is an incidence vector of minimal $s, t - cut\}$ and $dom(\mathcal{C}) = \{x \in \{0,1\}^{|E|} | \exists x' \in \mathcal{C}, x \geq x'\}$. While we do not know a linear description of the convex hull of $\mathcal{C}$, a description of the convex hull of $dom(\mathcal{C})$ is available [3].

Minimum Product s,t-Path. Given an undirected graph $G = (V, E)$, vertices $s, t \in V$, cost functions $c_1 : E \to \mathbb{R}_+$ and $c_2 : E \to \mathbb{R}_+$, the goal is to find a path P between s and t that minimizes $c_1(P) \cdot c_2(P)$.

Let $\mathcal{P} = \{x \in \{0,1\}^{|E|} | x$ is an incidence vector of $s, t - path\}$ and $dom(\mathcal{P})$ be defined analogously. As in the case of s,t-cuts, a linear formulation of the convex hull of $dom(\mathcal{P})$ is known.

In all the above cases, by using Corollary 1.3, we can obtain an FPTAS for the corresponding product problem.

1.3 Related Work

General QPs The general quadratic programming (QP) problem is the following.

$$\min_{x \in \mathbb{R}^n} f(x) = (a^T x + x^T C x) \text{ subject to } Ax \geq b.$$

Here $a \in \mathbb{R}^n, C \in \mathbb{R}^{n \times n}, A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. It is known that the objective function f is convex if and only if the matrix C is positive semi-definite. The problem is referred to as a convex QP if the objective is convex and can be solved in polynomial time. On the other hand, if f is not convex, the problem is referred to as non-convex QP and is in general NP-hard to solve [10, 13]. The non-convex QP problem has been studied widely in literature and finds important applications in numerous fields such as portfolio analysis, VLSI design, optimal power flow and economic dispatch. The bibliography of Gould and Toint [4] is an extensive list of references in non-convex QP and its applications. The special case of non-convex problem when $rank(C) = 1$ has also been proved to be NP-hard in Matsui [9] and an FPTAS for the problem is known due to Kern and Woeginger [7] as discussed earlier although their algorithm differs significantly from the one we present in this paper.

Product Spanning Tree Given two non-negative linear cost functions c_1 and c_2 on edges in an undirected graph, the problem of finding a spanning tree that minimizes cost c_1 subject to a budget constraint on cost c_2 has been considered by Ravi and Goemans [11]. They give a bi-criteria $(1, 1+\epsilon)$-approximation for any fixed $\epsilon > 0$ i.e., the algorithm outputs a tree with optimal c_1 cost while violating the budget constraint by a factor $(1+\epsilon)$. A slight improvement to an EPTAS is available in [6]. While these algorithms can be adapted to give a PTAS (though not an FPTAS) for the minimum product spanning tree problem, they are specific to the spanning tree problem.

Product Shortest Path In the above vein, given two non-negative linear cost functions c_1 and c_2 on edges in an undirected graph, the problem of finding an s, t-path that minimizes cost c_1 subject to a budget constraint on cost c_2 has been considered by Hassin [5]. He gives a similar bi-criteria $(1, 1+\epsilon)$-approximation for any fixed $\epsilon > 0$. His method can be used to design an FPTAS for the product shortest path problem.

In general, the bi-criteria problem of minimizing a non-negative linear cost subject to a budget on a second non-negative linear cost has been addressed in [8]. Their methods give a $((1+\alpha)\rho, (1+\frac{1}{\alpha})\rho)$-approximation for the bi-criteria problem for any $\alpha > 0$ where ρ is the approximation factor for the single-criterion problem. Their methods can be adapted to give a $4\rho^2$-approximation for the product problems; examples with $\rho = 1$ include shortest path, matching and min-cut. Our results improve this to give $(1+\epsilon)$-approximation for these problems.

2 $(1+\epsilon)$-Approximation Algorithm

Let $C = c_1 c_2^T, c_1, c_2 \in \mathbb{R}^n_+$ and let $P = \{x \in \mathbb{R}^n | Ax \geq b\}$. The problem Π is the following.

$$min_{x \in P} (c_1^T x) \cdot (c_2^T x)$$

We solve the problem via a parametric approach. Consider the following parametric problem $\Pi(B)$ where B is a given parameter.

$$\min c_1^T x \tag{1}$$
$$c_2^T x \leq B \tag{2}$$
$$x \in P \tag{3}$$

Lemma 2.1 *Let x^* be an optimal solution for the problem Π and let $B = c_2^T x^*$. Then x^* is also an optimal solution for $\Pi(B)$.*

Proof: Suppose not. Let $\tilde{x}$ be an optimal solution for $\Pi(B)$. Then $c_1^T \tilde{x} < c_1^T x^*$ and $c_1^T \tilde{x} \leq B = c_2^T x^*$. Therefore, $(c_1^T \tilde{x}) \cdot (c_2^T \tilde{x}) < (c_1^T x^*) \cdot (c_2^T x^*)$ which contradicts the optimality of x^* for Π. ■

Lemma 2.2 *Let $\tilde{x}(B)$ be a basic optimal solution of $\Pi(B)$ for any $B > 0$. Then $\tilde{x}(B)$ can be written as a convex combination of at most two extreme points of polytope P.*

Proof: If the constraint $c_2^T x \leq B$ is not tight for $\tilde{x}(B)$, then clearly $\tilde{x}(B)$ is an extreme point of P and the claim holds. Recall $P = \{x | Ax \geq b\}$. Let $A_T = \{a_i | a_i \cdot \tilde{x} = b_i\}$. Since, $\tilde{x}(B)$ is a basic optimal solution of $\Pi(B)$ and only one other constraint except those corresponding to A_T is tight for $\tilde{x}(B)$, $rank(A_T) \geq n-1$. Therefore, $\tilde{x}(B) \in F$ where F is a face of dimension at most one in polytope P. Any point in a face of dimension one can be expressed as a convex combination of two extreme points. Therefore, there exist $x^1, x^2 \in extr(P)$ such that $\tilde{x}(B) = \alpha \cdot x^1 + (1-\alpha) \cdot x^2$ for some $0 \leq \alpha \leq 1$. Note that x^1 and x^2 may not be feasible for the problem $\Pi(B)$. ■

Lemma 2.3 *Let $\tilde{x}(B)$ be a basic optimal solution for $\Pi(B)$ for some $B > 0$. There exists an extreme point $x \in extr(P)$ such that*

$$(c_1^T x) \cdot (c_2^T x) \quad \leq \quad (c_1^T \tilde{x}(B)) \cdot B \tag{4}$$

$$\tag{5}$$

Proof: From Lemma 2.2, we know that there exist two extreme points $x^1, x^2 \in extr(P)$ such that $\tilde{x}(B) = \alpha \cdot x^1 + (1-\alpha) \cdot x^2$ for some $0 \leq \alpha \leq 1$. Let $a_i = c_1^T x^i$ and $b_i = c_2^T x^i, i = 1, 2$. We consider the following two cases.

Case 1: Suppose $a_1 = a_2$. Then either $b_1 \leq b_2$ or $b_2 \leq b_1$. Let us assume $b_1 \leq b_2$ (the other case is symmetric). Clearly, $c_1^T x^1 = c_1^T \tilde{x}(B) = a_1$ and $c_2^T x^1 \leq c_2^T \tilde{x}(B)$ and the inequality 4 holds.

Case 2: $a_1 < a_2$ ($a_1 > a_2$ is symmetric). We can claim that $b_1 > b_2$ without loss of generality. If $b_1 \leq b_2$, then $a_1 = c_1^T x^1 \leq c_1^T \tilde{x}(B)$ and $b_1 = c_2^T x^1 \leq c_2^T \tilde{x}(B)$ and the inequality 4 holds in this case for x^1. Now,

$$c_1^T \tilde{x}(B) = \alpha \cdot a_1 + (1-\alpha) \cdot a_2 \tag{6}$$

$$c_2^T \tilde{x}(B) = \alpha \cdot b_1 + (1-\alpha) \cdot b_2 \tag{7}$$

$$\tag{8}$$

Either $a_1 b_1$ or $a_2 b_2$ is less than or equal to $\alpha \cdot a_1 b_1 + (1-\alpha) \cdot a_2 b_2$ (say $a_1 b_1$). Then,

$$\begin{aligned}
&\alpha \cdot a_1 b_1 + (1-\alpha) \cdot a_2 b_2 - (c_1^T \tilde{x}(B)) \cdot (c_2^T \tilde{x}(B)) \\
&= \alpha \cdot a_1 b_1 + (1-\alpha) \cdot a_2 b_2 - (\alpha \cdot a_1 + (1-\alpha) \cdot a_2) \cdot (\alpha \cdot b_1 + (1-\alpha) \cdot b_2) \\
&= \alpha(1-\alpha)(a_1 b_1 + a_2 b_2 - a_1 b_2 - a_2 b_1) \\
&= \alpha(1-\alpha)(a_1 - a_2)(b_1 - b_2) \\
&\leq 0
\end{aligned}$$

The last inequality follows because $a_1 < a_2$ and $b_1 > b_2$. Therefore,

$$a_1 b_1 \leq \alpha \cdot a_1 b_1 + (1-\alpha) \cdot a_2 b_2 \leq (c_1^T \tilde{x}(B)) \cdot (c_2^T \tilde{x}(B))$$

■

Since we do not know the value of parameter B, we try different powers of $(1+\epsilon)$ for a fixed $\epsilon > 0$. The algorithm can now be stated as follows.

Proof of Theorem 1.1: Let x^* be an optimal solution for the problem Π. There exists $j \in \mathbb{N}$ such that

$$(1+\epsilon)^{j-1} \leq c_2^T x^* < (1+\epsilon)^j.$$

Consider the problem $\Pi(B)$ for $B = (1+\epsilon)^j$ and let $\tilde{x}(B)$ be a basic optimal solution for $\Pi(B)$. Clearly, $c_1^T \tilde{x}(B) \leq c_1^T x^*$ as x^* is a feasible solution for $\Pi(B)$. From Lemma 2.3, we can find $x \in extr(P)$ such that

$$\begin{aligned}
c_1^T x \cdot c_2^T x \quad &\leq c_1^T \tilde{x}(B) \cdot B \\
&\leq c_1^T x^* \cdot B \\
&\leq c_1^T x^* \cdot c_2^T x^* (1+\epsilon)
\end{aligned}$$

Algorithm $\mathcal{A}$ for Minimizing Rank-1 QPs
Given $C = c_1 c_2^T, c_1, c_2 \in \mathbb{R}_+^n$, polytope P and $\epsilon > 0$.
Initialize $M \leftarrow \frac{\max_{x \in P} c_2^T x}{\min_{x \in P} c_2^T x}$, $N_M = \lceil \log_{1+\epsilon} M \rceil$ and $c_s \leftarrow \infty$.

1. For $j = 1, \ldots, N_M$,
   (a) Let $B = (1+\epsilon)^j$ and let $\tilde{x}(B)$ be a basic optimal solution for $\Pi(B)$.
   (b) Using Lemma 2.3 find $\hat{x}(B) \in extr(P)$ such that
   $$(c_1^T \hat{x}(B)) \cdot (c_2^T \hat{x}(B)) \leq (c_1^T \tilde{x}(B)) \cdot B.$$
   (c) If $c_s > (c_1^T \hat{x}(B)) \cdot (c_2^T \hat{x}(B))$, then
   $$\begin{aligned} x_s \leftarrow & \quad \hat{x}(B) \\ c_s \leftarrow & \quad (c_1^T \hat{x}(B)) \cdot (c_2^T \hat{x}(B)) \end{aligned}$$
2. Return the solution x_s.

Therefore, our algorithm $\mathcal{A}$ finds an extreme point of P that is a $(1+\epsilon)$-approximation for the problem. ■

Let $l = \min_{x \in P} c_2^T x$ and $u = \max_{x \in P} c_2^T x$. Then our algorithm solves $\lceil \log_{(1+\epsilon)} \frac{u}{l} \rceil$ linear programs to obtain a $(1+\epsilon)$-approximate solution (We can assume $l \neq 0$ since this case can be checked over each objective function). On the other hand, the algorithm in [7] needs to solve approximately these many linear programs for each guessed value of the optimal objective value.

Recall that the objective $(c_1^T x) \cdot (c_2^T x)$ is neither convex nor concave. However, it is known that there exists an extreme point of P that minimizes $\min_{x \in P} (c_1^T x) \cdot (c_2^T x)$ [7]. For the sake of completeness, we present a proof of this using Lemma 2.3.

Proof of Lemma 1.2: Let $\tilde{x}$ be an optimal solution for $\min_{x \in P} (c_1^T x) \cdot (c_2^T x)$. Consider $B = c_2^T \tilde{x}$ and consider the problem $\Pi(B)$. From Lemma 2.3, we have that there exists an extreme point $\hat{x} \in extr(P)$ such that $(c_1^T \hat{x}) \cdot (c_2^T \hat{x}) \leq (c_1^T \tilde{x}) \cdot B = (c_1^T \tilde{x}) \cdot (c_2^T \tilde{x})$. Therefore,

$$\min_{x \in extr(P)} (c_1^T x) \cdot (c_2^T x) = \min_{x \in P} (c_1^T x) \cdot (c_2^T x)$$

■

3 Future Work

In this paper we present an FPTAS for a special case of non-convex QP where the objective is to minimize the product of two linear non-negative cost functions and showed applications to 0-1 problems when either the convex hull of feasible integer solutions or the convex hull of the dominant of feasible integer solutions is known. It is known that this non-convex QP problem is NP-hard in general [9]. However, the complexity of the special cases of minimizing the product of two linear non-negative costs for 0-1 problems (such as shortest paths, spanning trees etc) is still open.

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