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Effective field theory approach to Casimir interactions on soft matter surfaces

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We utilize an effective field theory approach to calculate Casimir interactions between objects bound to thermally fluctuating fluid surfaces or interfaces. This approach circumvents the complicated constraints imposed by such objects on the functional integration measure by reverting to a point particle representation. To capture the finite size effects, we perturb the Hamiltonian by $\Delta H$ that encapsulate the particles' response to external fields. $\Delta H$ is systematically expanded in a series of terms, each of which scales homogeneously in the two power counting parameters: $\lambda \equiv R/r$, the ratio of the typical object size ($R$) to the typical distance between them ($r$), and $\delta \equiv k_B T / k$, where $k$ is the modulus characterizing the surface energy. The coefficients of the terms in $\Delta H$ correspond to generalized polarizabilities and thus the formalism applies to rigid as well as deformable objects. Singularities induced by the point particle description can be dealt with using standard renormalization techniques. We first illustrate and verify our approach by re-deriving known pair forces between circular objects bound to films or membranes. To demonstrate its efficiency and versatility, we then derive a number of new results: The triplet interactions present in these systems, a higher order correction to the film interaction, and general scaling laws for the leading order interaction valid for objects of arbitrary shape and internal flexibility.

We will assume that the bare surface Hamiltonian $H$ is a quadratic functional of some field $\phi(r)$:

$$H[\phi] = \frac{1}{2} k \int d^3 r \phi \hat{K} \phi,$$  \hspace{1cm} (1)

where $k$ is a generalized modulus and the kernel $\hat{K}$ defines the physics of the problem. For films $\hat{K} = -\nabla^2$ while for membranes $\hat{K} = (\nabla^2)^2$. In both cases $\phi(r)$ is the surface height in Monge parametrization \[13\].

The idea behind the EFT formalism is to treat the objects as point particles and to recapture their internal structure through additional terms $\Delta H = \sum_a C_a O_a$ in the Hamiltonian, where the scalars $O_a$ are polynomial in the field and its derivatives and $a$ labels the particles. The coefficients $C_a$ are chosen to reproduce the long wavelength physics, in analogy with block spin renormalization, and are fixed by a matching calculation. In principle one must add all terms which are consistent with the underlying theory. On dimensional grounds the coefficients of these new terms will scale with powers of the particle size, $R$, such that the limit $R \to 0$ is well defined. Let us for the moment assume that the new terms obey a shift symmetry $\Delta H(\phi) = \Delta H(\phi + h)$, where $h$ is a constant. This symmetry eliminates boundary conditions in which the object is pinned to a fixed height. Violating this symmetry leads to long wavelength fluctuations which can lead to pathologies. Given this restriction, it is clear that we may truncate the sum in a derivative expansion, as each derivative will scale as $\lambda \equiv R/r$. Quadratic terms in $\phi$ generate multipoles when the particles are subjected to external fields, hence we may interpret their coefficients as polarizabilities. Terms which are higher order in the fields will not have such a simple interpretation. Such terms are suppressed by powers of $\delta \equiv k_B T / k$. To see this, we may simply rescale the field $\phi \to \sqrt{\delta} \phi$. In this way the leading term in the partition function has a

Objects which constrain a fluctuating field experience a Casimir interaction \[1\]. The underlying fluctuations can be either quantum mechanical or thermal in origin \[2\]. In this letter we will be interested in forces induced by thermal fluctuations between particles bound to surfaces characterized by surface tension (films) \[3, 4\] or bending rigidity (membranes) \[6–9\].

The non-trivial aspect of such calculations tends to arise from the constraints that the extended objects impose on the partition sum. This issue is usually dealt with by pinning the field to the surface of the objects through delta-functions in the integration measure \[3\]. A clear exposition of this method, applied to compact objects in the physics of the problem. For films $\hat{K} = -\nabla^2$ while for membranes $\hat{K} = (\nabla^2)^2$. In both cases $\phi(r)$ is the surface height in Monge parametrization \[13\].

In this paper we employ an effective field theory (EFT) formalism, originally developed to study the gravity wave profile for inspiralling black holes \[11\], to streamline the boundary condition issue. This formalism has been utilized to derive not only new results in gravitational wave physics \[12\] but also to calculate the leading order finite size correction to the Abraham-Dirac-Lorentz radiation reaction force law in classical electrodynamics \[13\]. Both of the these applications dealt with classical non-fluctuating fields, whereas here we will generalize the formalism to allow for finite temperatures.

To illustrate the effective field theory approach to soft matter surfaces most transparently, we will focus on constraints imposed by mobile but rigid objects which pin field fluctuation modes along their circumference. Extensions towards more general types of constraints entails no change in formalism.
well defined $\delta \to 0$ limit and non-linearities are automatically suppressed by powers of $\delta$. These $\delta$-corrections will be treated in a forthcoming publication. Here we will be only concerned with the more canonical $\lambda$-corrections.

The leading order in $\lambda$ is unique and $\mathcal{O}(\lambda^2)$:

$$\Delta \mathcal{H}^{(2)} = \frac{1}{2} \sum_a C_{ij}^a \left[ \partial_i \phi(\mathbf{r}_a) \partial_j \phi(\mathbf{r}_a) \right].$$  \hspace{1cm} (2)

The tensorial polarizability $C_{ij}^a$ allows for non-axisymmetric objects, but for the sake of simplicity we will restrict to the symmetric case $C_{ij}^a = C^D_a \delta_{ij}$. The coefficient $C^D_a$, which can be fixed by treating a single object in isolation, is the isotropic dipole polarizability. The reason for this nomenclature will become clear once we match the polarizabilities are fixed by a matching procedure.

The canonical partition function of the system described by the effective Hamiltonian $\mathcal{H}_{\text{eff}} = \mathcal{H} + \Delta \mathcal{H}$, and afterwards explain how the polarizabilities are fixed by a matching procedure.

The canonical partition function of the system described by the effective Hamiltonian $\mathcal{H}_{\text{eff}}$ is given by

$$Z = \int \mathcal{D}\phi \ e^{-\beta \mathcal{H}_{\text{eff}}} = Z_0 \left( e^{-\beta \Delta \mathcal{H}} \right),$$ \hspace{1cm} (3)

where $Z_0$ is the partition function of the free Hamiltonian ($\Delta \mathcal{H} = 0$) and $\langle \cdots \rangle$ denotes the associated Gaussian average. Substituting $\Delta \mathcal{H}$ from its definition in Eqn. (2), one obtains the free energy of interaction, $\mathcal{U}$, through

$$- \beta \mathcal{U} = \log(Z/Z_0) = \log \left( e^{-\beta \Delta \mathcal{H}} \right)$$

$$= \sum_{n=1}^\infty \frac{1}{n!} \left( - \frac{\beta}{2} \sum_a C^D_a [\partial_i \phi(\mathbf{r}_a)]^2 \right)^n_c.$$ \hspace{1cm} (4)

This cumulant expansion can be represented as a series of diagrams 15, each depicting a (connected) 2n-point function of some derivatives of the field.

The expansion (4) encodes n-body contributions to the free energy from the $n^{\text{th}}$ term onwards. For a given $n$-body force there are subleading corrections stemming from higher multipoles as well as terms non-linear in the lower multipole polarizabilities.

Let us now illustrate this formalism by applying it to some cases of interest. We first consider circular particles on a surface with tension-dominated energy density, i.e. a film. The relevant differential operator is then $-\nabla^2$, with Green function $G(\mathbf{r}, \mathbf{r'}) = -\frac{1}{2\pi} \log|\mathbf{r} - \mathbf{r'}|$, and the modulus $k$ will be replaced by the more familiar $\sigma$ for surface tension. At $\mathcal{O}(\lambda^4)$ we have the diagram in Fig. 1(a), which represents the lowest order pair interaction. It Wick-contracts to

$$- \beta \mathcal{U}^{(4)} = \frac{1}{4} \sum_{a,b} C^D_a C^D_b \left[ \partial_i \partial_j G(\mathbf{r}_a, \mathbf{r}_b) \right]^2$$ \hspace{1cm} (5)

and evaluates to the pair potential

$$- \beta U^{(4)}_{(ab)} = \frac{C^D_a C^D_b}{4\pi^2 \sigma^4 r_{ab}^4},$$ \hspace{1cm} (6)

where we define $r_{ab} = \mathbf{r}_a - \mathbf{r}_b$ and denote by $\{\ldots\}$ an n-tuplet of distinct particles. Self energies corresponding to self-links in diagrams, such as the one in Fig. 1(c), lead to divergent contributions to the interaction energy. However, for derivative interactions these divergences are all power like and can be absorbed into $C^D_a$. These divergences carry no physical information, as there is no non-trivial renormalization group flow, and thus effectively we may set these diagrams to zero.

Let us now consider contributions beyond $\mathcal{O}(\lambda^4)$ to the two body interaction, for which there are two possible sources: Higher order corrections to $\Delta \mathcal{H}$ itself and higher order terms in the cumulant expansion. The next order term in $\Delta \mathcal{H}$, which is not subleading in $\delta$, is given by

$$\Delta \mathcal{H}^{(4)} = \frac{1}{2} \sum_a C_{ij}^a \left[ \partial_i \partial_j \phi(\mathbf{r}_a) \right]^2.$$ \hspace{1cm} (7)

It corresponds to a quadrupole polarizability and is $\mathcal{O}(\lambda^4)$. In principle there is also a term involving $\partial_i \partial_j \phi(\mathbf{r}_a)$, but it can be removed by a redefinition of $\phi$. \textit{O}, owing to the fact that $\partial_i \partial_j \phi = 0$ is the Euler-Lagrange equation for the problem. Inserting (2) + (4) into the cumulant expansion (4) generates a quadrupole-dipole interaction at $\mathcal{O}(\lambda^6)$, given by

$$- \beta \mathcal{U}^{(6)}_{(ab)} = \frac{2(C^Q_a C^D_b + C^Q_a C^D_b)}{\pi^2 \sigma^4 r_{ab}^6}.$$ \hspace{1cm} (8)

At $\mathcal{O}(\lambda^6)$ one might additionally expect a contribution involving three dipole interactions in the cumulant expansion. However these vanish, since such terms necessarily involve self-energies, c.f. diagram 1(c).

At $\mathcal{O}(\lambda^8)$ we expect the quadrupole-quadrupole interaction, arising from $[\partial^4 G(\mathbf{r}_a, \mathbf{r}_b)]^2$, but also a non-linear dipole-dipole term, proportional to four dipole polarizabilities and stemming from the 4th cumulant, and diagram 1(e), with $[\partial^2 G(\mathbf{r}_a, \mathbf{r}_b)]^4$:

$$- \beta \mathcal{U}^{(8)}_{(ab)} = \frac{36 C^Q_a C^Q_b}{\pi^2 \sigma^8 r_{ab}^8} + \frac{(C^D_a)^2 (C^D_b)^2}{32 \pi^4 \sigma^8} \frac{1}{r_{ab}^8}.$$ \hspace{1cm} (9)

![FIG. 1: Relevant Feynman diagrams for the calculations in this letter. Solid lines correspond to propagators, as usual, while dashed lines represent world lines of particles.](image-url)
As long as the objects resist curvature, $C^Q_a \neq 0$. However, in addition to vertical translations the objects can also tilt, their ability to align with local gradients in $\phi$ implies $C^D_a = 0$ and the lowest order interaction is $O(\lambda^8)$. While the precise value of $C^Q_a$ might be hard to calculate for arbitrarily shaped rigid or elastic objects, Eqsns. (6,8,9) nevertheless show that on films these will always interact with an asymptotic $r^{-8}$ potential, thus generalizing a finding obtained in Ref. [3] for ellipsoidal objects. If the objects are identical, $(C^Q)^2 > 0$ implies they attract.

The lowest order 3-body interaction arises from three dipole insertions. Henceforth using the shorthand notation $G^{ab} = G(r_a, r_b)$ and denoting partial derivatives by subscripts, the relevant interaction is

$$-\beta U_{(6)}^{(abc)} = -\frac{C^D_a C^D_b C^D_c}{\sigma^3} G_{ij} G_{jk} G_{ka}, \quad (10)$$

which corresponds to diagram (h) and yields a triplet interaction that scales as $(r_{ab} r_{bc} r_{ca})^{-3}$. However, owing to the symmetries of the tensor $G_{ij} = (\delta_{ij} - 2\hat{r}_i \hat{r}_j)/(2\pi r^2)$, $\text{Tr}[G^n] = 0$ if $n$ is odd. Thus the leading dipole contribution to any $n$-body interaction vanishes for odd $n$. The first non-vanishing 3-body interaction therefore arises at $O(\lambda^8)$, from terms which are quadratic in one of the dipole polarizabilities, diagram (d):

$$-\beta U_{(8)}^{(abc)} = \frac{C^D_a C^D_b C^D_c}{16\pi^3 \sigma^4} \left( \frac{C^D_b}{r_{ab}^4 r_{ac}^4} + \frac{C^D_b}{r_{ba}^4 r_{bc}^4} + \frac{C^D_c}{r_{ca}^4 r_{cb}^4} \right). \quad (11)$$

This term drives triplets to attract, irrespective of their relative placement, as long as $C^D > 0$. A possible dipole-quadrupole-dipole interaction also scales as $O(\lambda^8)$, but again the symmetries of $G_{ij}$ and $G_{ijkl}$ force it to vanish.

So far we have calculated the forces in terms of a set of polarizability coefficients. These are fixed by calculating an observable in the full (finite-sized particle) theory, expanding it in powers of $\lambda$, and choosing the polarizabilities such that the EFT reproduces the result to the appropriate order in $\lambda$. The point to emphasize is that we are free to fix these coefficients by any means we choose. Thus it behooves us to choose as simple a setting as possible: we will place a single particle in a simple stationary external field where we can easily calculate its response.

To illustrate the procedure, let us match for $C_a$ in the case of a rigid horizontal inclusion on a film. Consider placing the point particle in a background field such that the total field is given by $\delta \phi(r) + \phi_{bg}(r)$, where $\delta \phi(r)$ is the induced field generated by the polarization of the inclusion. Terms linear in $\delta \phi$ in the Hamiltonian correspond to induced point sources

$$\rho_a(r) = -C^D_a \partial_i \left[ \delta(r - r_a) \partial_i \phi_{bg}(r) \right] \quad (12)$$

in the effective theory. The field emitted by an induced source is thus given by

$$\phi_a(r) = -\frac{C^D_a}{k} \partial_i^{(a)} G(r, r_a) \delta_{ij} \partial_j \phi_{bg}(r_a). \quad (13)$$

We may pick any background field we wish, but clearly it is simplest to choose the lowest multipole field configuration necessary to generate a non-zero response. For a rigid horizontal inclusion this is a dipole field $\phi_{bg} \sim r \cos \varphi$, i.e. one of constant slope. After solving the elementary boundary value problem (BVP) in the full theory and comparing to the effective theory result, one finds $C^D = \pi R^2 \sigma$. To match for $C^Q$, we need a background with curvature, so we choose $\phi_{bg} \sim r^2 \cos(2\varphi)$. Repeating the exercise we just performed, after appropriately generalizing (12) to account for the two derivative nature of (7), gives $C^Q = \frac{8}{3} R^4 \sigma$. By Eqsns. (6,8,9) this yields the pair interaction up to $O(\lambda^8)$:

$$-\beta U_{(8)}^{(ab)} = \frac{R^4}{r_{ab}^4} + \frac{R^6}{r_{ab}^2} + \left(9 + \frac{1}{2}\right) \frac{R^8}{r_{ab}^2}. \quad (14)$$

The $r_{ab}^4$ (dipole-dipole) and $r_{ab}^6$ (dipole-quadrupole) interactions shown above agree with those derived in Ref. [4]. However, the $r_{ab}^{-6}$ term consists of a lowest order quadrupole-quadrupole piece (prefactor “9”, also given in Ref. [4]) plus a non-linear dipole-dipole correction (prefactor “1/2”) that has not been previously calculated. Recall that if the disc can also tilt, every term proportional to a dipole polarizability will vanish and thus only the interaction $-9 k_B T(R/r_{ab})^8$ will survive. As discussed before, a $r_{ab}^{-6}$-term will remain even for non-circular or bendable objects; only its prefactor will be different, owing to the associated BVP being slightly different.

Observe that the surface tension $\sigma$ (or generally the modulus $k$) cancels from the final result, because in every term of $-\beta U$ the number of Green functions, each accompanied by a factor $\sigma^{-1}$, always matches the number of polarizabilities, each proportional to $\sigma$. This will no longer be the case once corrections beyond lowest order in $\delta$ are included.

Now consider the case of particles embedded in a surface with a bending-dominated energy density, i.e. a membrane. The kernel of the Hamiltonian in this case is $K = \langle \nabla \rangle^2$, with the Green function $G(r, r') = \frac{1}{8\pi} |r - r'|^2 \log|r - r'|$, and the generic modulus $k$ will be replaced by the bending rigidity $\kappa$. We will assume that the particles can adjust to a constant slope background by tilting, so the first non-vanishing polarizability will be quadrupole in nature. Observe that this time, at $O(\lambda^2)$, we need to write down two distinct terms:

$$\Delta H = \frac{1}{2} \sum_a \left[ \tilde{C}^Q_a \left( \partial_i \partial_j \phi \right)^2(r_a) + \tilde{C}^{Q'}_a \left( \partial_i \partial_j \phi \right)^2(r_a) \right]. \quad (15)$$

Terms involving the Laplacian cannot be removed via a field re-definition, because $\partial_i \partial_j \phi = 0$ is no longer the Euler-Lagrange equation. Notice that due to the different kernel the quadrupole polarizabilities $\tilde{C}^Q$ and $\tilde{C}^{Q'}$ scale $\sim R^2$, and not $\sim R^4$ as in the film case.

The pair interaction follows easily from an expression
analogous to Eqn. [4]:

\[-\beta U^{(4)}_{(ab)} = \frac{C_a C_b}{2\kappa^2} \left( G_{ijkl} \right)^2 + \frac{C_a C_b}{2\kappa^2} \left( G_{ijkl} \right)^2 \cdot \frac{2}{2\kappa^2} \left( G_{ijkl} \right)^2.
\] (16)

The term \( C_a C_b \left( G_{ijkl} \right)^2 \) vanishes because \( G_{ijkl} = \delta(r_a - r_b) \) by definition of the biharmonic Green function. The other contractions are found to be \( (G_{ijkl})^2 = 4/(2\pi r_a^2) \) and \( (G_{ijkl})^2 = 2/(2\pi r_b^2). \)

The leading 3-body interaction in the membrane case stems from the third order cumulant. We have reproduced well known results and derived several new ones, pertaining to non-linear corrections, 3-body terms, and general scaling laws for the leading order interaction between objects that are arbitrarily shaped and possibly flexible. However, this formalism extends well beyond the basic cases studied here and can be applied to many further situations in which existing techniques become rather unwieldy. For instance, the corrections beyond linear order in \( k_B T/k \) follow in a straightforward manner and will be presented elsewhere. In addition, our results can be utilized to calculate the forces and torques between non-rigid objects or phase-segregated surface domains. This does not require further field-theoretical sophistication but merely the calculation of their polarizability, which for complicated objects one might even decide to extract from experiment.

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[16] For a proof see e.g. I. Z. Rothstein, “TASI lectures on effective field theories,” arXiv:hep-th/0308266.