Extensions of Lo's semiparametric bound for European call options

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Abstract

Computing semiparametric bounds for option prices is a widely studied pricing technique. In contrast to parametric pricing techniques, such as Monte-Carlo simulations, semiparametric pricing techniques do not require strong assumptions about the underlying asset price distribution. We extend classical results in this area in two main directions. First, we derive closed-form semiparametric bounds for the payoff of a European call option, given up to third-order moment information on the underlying asset price. We analyze how these bounds tighten the corresponding bounds, when only second-order moment (i.e., mean and variance) information is provided. Second, we derive closed-form semiparametric bounds for the risk associated to the expected payoff of a European call option, when the mean and the variance of the underlying asset price are given. Applications of these results to other areas such as inventory and supply chain management are also discussed.

1 Introduction

Going back to the seminal work by Merton [19], many authors have studied the problem of finding bounds for option prices under incomplete market conditions or an incomplete knowledge of the distribution of the price of the underlying assets (see, e.g., [3, 4, 5, 8, 9, 15, 16, 18, 21, 22, 23] and the references therein). Here, we study bounds on the expected payoff of a European call option and its associated risk, given only information on the moments of the underlying asset price at maturity. Such type of bounds are called *semiparametric bounds*.

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Unlike *parametric* pricing techniques, such as Monte-Carlo simulation (cf. [10]), semiparametric pricing techniques do not require strong assumptions about the underlying asset price distribution. The interest in computing semiparametric bounds for option prices stems mainly from this fact. In particular, semiparametric bounds are used to set bounds for option prices under the risk-neutral measure pricing theory, and to examine the relationship between option prices and the true, as opposed to risk-neutral, distribution of the underlying asset (see, e.g., [8, 18]).

The first, now classical, results in this area were derived by Lo [18] and Grundy [8]. Lo [18] gave a closed-form upper bound on the payoff of a European call option when second-order moment (i.e., mean and variance) information about the asset price at maturity is available (see Theorem 1 in Section 2.1). Grundy [8] derived similar upper bounds when only n-th order information is given. Also, Grundy [8] gave a simple lower bound on the expected payoff of a European call option when only the mean of the underlying asset price at maturity is known. This initial work has been followed by further results in this area, such as the ones developed by: Bertsimas and Popescu [3]; Boyle and Lin [4]; D'Aspremont and El Ghaoui [5]; De la Peña, Ibragimov, and Jordan [12]; Popescu [22]; and Zuluaga and Peña [27]; to name a few recent ones.

The goal of this paper is to extend the results of Lo and Grundy in two main directions. First (Section 2), we derive closed-form semiparametric bounds for the payoff of a European call option, given up to third-order moment information on the underlying asset price (see Theorems 3 through 5). We analyze how these bounds tighten the corresponding bounds, when only up to second-order moment information is provided (Section 2.2). For example, we show that third-order moment information gives tighter bounds on the payoff of a European call option when the option is *close* to being *at the money.* This is precisely the region where the option pricing problem is more interesting. Furthermore, we show that the magnitude of this tightening depends on the relationship between the second and third moments of the underlying asset price. In particular, we prove that if a special relationship between the second and third moments holds, then third-order bounds completely determine the expected payoff of the option (Theorem 3). Second (Section 3), we derive closed-form semiparametric bounds for the risk associated to the expected payoff of a European call option, when up to second-order moment information on the underlying asset price is given (see Theorems 7 through 10). For exposition purposes, Sections 2 and 3 concentrate on the statements and interpretation of our main results, paying special attention to the bounds' values. Most of the proofs, which rely on convex duality (cf. [24]), are deferred to the Appendix. Therein we also provide information on the optimal solutions that attain our bounds' values.

The computation of semiparametric bounds is a classical probability problem (cf. Karlin and Studden [11], and Zuluaga and Peña [27]). As a consequence, many related results come from areas other than finance, such as, inventory theory and stochastic programming. For example, consider the work of: Bertsimas and Natarajan [2], Dokov and Morton [6], Gallego and Moon [7], Scarf [25], and the references therein. Conversely, our results have applications in these areas. In particular, here we discuss applications in inventory and supply chain management (see Sections 2.3 and 3.3).

It is worth mentioning that all the semiparametric bounds considered here can be numerically computed using semidefinite programming techniques (cf. [26]). This fact follows from the work of Bertsimas and Popescu [3], and other related work in the so-called area of polynomial programming (see, e.g., Lasserre [13, 14], and Zuluaga and Peña [27]). Here, however, our aim is to obtain closed-form solutions to the semiparametric bound problems being considered. Closed-form solutions are of both practical and theoretical significance, as they allow for easy computation of the bounds, and the performance of sensitivity analysis and optimization over the parameters involved in the problem.

2 Bounds on the payoff of a European call

In this section, we consider the problem of finding sharp bounds on the expected payoff of a European call option, given information on the first n moments of the underlying asset price at maturity (without making any other assumption on the distribution of the asset price). Finding the sharp upper, and the sharp lower bound for this problem can be (respectively) formulated as the following optimization problems (see, e.g., Bertsimas and Popescu [3]):

$$(\overline{P}^\sigma) \qquad \begin{array}{rll} \overline{p}(\sigma) = & \sup & \mathbb{E}_\pi((S-1)^+) \\ & \text{s.t.} & \mathbb{E}_\pi(S^i) = \sigma_i \quad \text{for } i = 0, \dots, n \\ & & \pi \quad \text{a distribution in } \mathbb{R}_+, \end{array}$$

and

$$(\underline{P}^\sigma) \qquad \begin{array}{rll} \underline{p}(\sigma) = & \inf & \mathbb{E}_\pi((S-1)^+) \\ & \text{s.t.} & \mathbb{E}_\pi(S^i) = \sigma_i \quad \text{for } i = 0, \dots, n \\ & & \pi \quad \text{a distribution in } \mathbb{R}_+. \end{array}$$

Here $\sigma_0 := 1$ and σ_i for $i = 1, \dots, n$ are the (given) non-central moments of the asset price at maturity. The random variable S represents the price of the underlying asset at maturity. These formulations assume, without loss of generality, that the strike price of the call option is \$1. Thus, the problem $(\overline{P}^\sigma)$ (resp. $(\underline{P}^\sigma)$) maximizes (minimizes) the expected payoff of a European call option ($(S-1)^+ := \max\{0, S-1\}$) over all probability distributions π in $\mathbb{R}_+$ whose moments are given by the vector $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_n)$.

In what follows, the aim is to provide closed-form solutions to $(\overline{P}^\sigma)$ and $(\underline{P}^\sigma)$ when up to third-order moment information of the underlying asset price at maturity is given (i.e., when $n = 3$). In particular, we analyze how third-order moment information tightens the corresponding bounds, when only up to second-order moment information is provided (i.e., when $n = 2$). For this purpose, we begin by discussing closed-form solution of $(\overline{P}^\sigma)$ and $(\underline{P}^\sigma)$ in this latter instance.

2.1 Bounds for up to second-order moment information

A classical result on semiparametric bounds for European options is Lo's closed-form solution of ($\overline{P}^\sigma$), when only the mean and variance of the underlying asset price at maturity is available [18]. We will represent this information by setting $\sigma = (1, \mu, \gamma\mu^2)$; that is, the mean and variance of the asset price are represented by the parameters μ and $(\gamma - 1)\mu^2$ respectively. Before stating Lo's result, Proposition 1 characterizes the feasibility of ($\overline{P}^\sigma$) and ($\underline{P}^\sigma$) when second-order moment information is available (i.e., when $n = 2$).

Proposition 1 *Let $\sigma = (1, \mu, \gamma\mu^2)$. The problems ($\overline{P}^\sigma$) and ($\underline{P}^\sigma$) are feasible if and only if $\mu \geq 0$ and $\gamma \geq 1$.*

Proposition 1 is a classical result on distributions on $\mathbb{R}^+$ (see e.g., [11, Theorem 10.1]). It is a straightforward consequence of Jensen's inequality.

Theorem 1 (Lo [18]) *Let $\sigma = (1, \mu, \gamma\mu^2)$. If $\mu \geq 0$ and $\gamma \geq 1$, then*

$$\overline{p}(\sigma) = \begin{cases} \mu - \frac{1}{\gamma} & \text{if } \mu > \frac{2}{\gamma}, \\ \frac{1}{2}((\mu - 1) + \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2}) & \text{if } \mu \leq \frac{2}{\gamma}. \end{cases}$$

Consider now the lower bound counterpart of Lo's result; that is, solving ($\underline{P}^\sigma$) when only the mean and variance of the underlying asset price at maturity is available. Theorem 2 below, gives a closed-form solution to this problem (which up to our knowledge does not appear in the literature).

Theorem 2 *Let $\sigma = (1, \mu, \gamma\mu^2)$. If $\mu \geq 0$ and $\gamma \geq 1$ then $\underline{p}(\sigma) = (\mu - 1)^+$.*

Proof. See the Appendix. □

The sharp bound on Theorem 2 is consistent with the bound determined by Merton [19, Theorem 1]. Also, from Grundy's closed-form solution of ($\underline{P}^\sigma$), when only the mean of the underlying asset price at maturity is available [8, Proposition 3]; it follows that the variance (second-order) information does not contribute to tighten Grundy's first-order lower semiparametric bound on the payoff of a European call option.

The semiparametric bounds obtained by Lo [18] and Grundy [8], and extensions of these results, are used to set bounds for option prices under the risk-neutral measure pricing theory; to examine the relationship between option prices and the true, as opposed to risk-neutral, distribution of the underlying asset; and to test different model assumptions (see, e.g., [4, 8, 12, 18]).

Figure 1 summarizes the results presented in this section on sharp bounds on the payoff of a European call option when up to second-order moment information on the underlying asset price is given.

2.2 Bounds for up to third-order moment information

We now consider bounds on the expected payoff of a European call option (analogous to those given by Theorems 1 and 2) when up to third-order moment information on the underlying asset price is given. We will represent this information by setting $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$; that is, the mean, variance, and non-central third moment of the asset price are represented by the parameters μ, $(\gamma-1)\mu^2$, and $\beta\mu^3$ respectively. For this purpose, Proposition 2 characterizes the feasibility of $(\overline{P}^\sigma)$ and $(\underline{P}^\sigma)$ in this instance (cf. Proposition 1).

Proposition 2 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. The problems $(\overline{P}^\sigma)$ and $(\underline{P}^\sigma)$ are feasible if and only if one of the following two conditions holds:*

(i) $\gamma \geq 1$, $\beta = \gamma^2$ *and* $\mu \geq 0$.

(ii) $\gamma > 1$, $\beta > \gamma^2$ *and* $\mu > 0$.

Proof. See the Appendix. □

Let us first consider case *(i)* in Proposition 2. For this particular instance, Theorem 3 below gives a closed-form solution to both $(\overline{P}^\sigma)$ and $(\underline{P}^\sigma)$

Theorem 3 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. If $(\overline{P}^\sigma)$ (or $(\underline{P}^\sigma)$) is feasible and $\beta = \gamma^2$ then $\overline{p}(\sigma) = \underline{p}(\sigma) = (\mu - 1/\gamma)^+$.*

Proof. See the Appendix. □

Theorem 3 implies that if the values of the second and third moments of an asset price at maturity satisfy the relation $\beta = \gamma^2$ (following the notation in Proposition 2), then the expected payoff of a European call on this asset is completely determined by the moment information, independent of the exact distribution of the underlying asset price.

Let us now consider case *(ii)* in Proposition 2. For this particular instance, Theorem 4 below gives a closed-form solution to $(\underline{P}^\sigma)$.

Theorem 4 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. If $(\underline{P}^\sigma)$ is feasible and $\beta > \gamma^2$, then*

$$\underline{p}(\sigma) = \begin{cases} \mu - 1 & \text{if } \mu \geq \widetilde{\mu}, \\ \frac{(\gamma\mu-1)^2}{\beta\mu-\gamma} & \text{if } \frac{1}{\gamma} < \mu \leq \widetilde{\mu}, \\ 0 & \text{if } \mu \leq \frac{1}{\gamma}, \end{cases}$$

where

$$\widetilde{\mu} = \frac{2(\gamma-1)}{(\beta-\gamma) - \sqrt{(\beta - 3\gamma + 2)^2 + 4(\gamma-1)^3}}.$$

Proof. This follows from putting toghether Lemmas 11, 12, and 13 in the Appendix.

As Figure 2 shows, for values of $1/\gamma \leq \mu \leq \widetilde{\mu}$, the addition of third-order moment information tightens the sharp lower second-order semiparametric bound

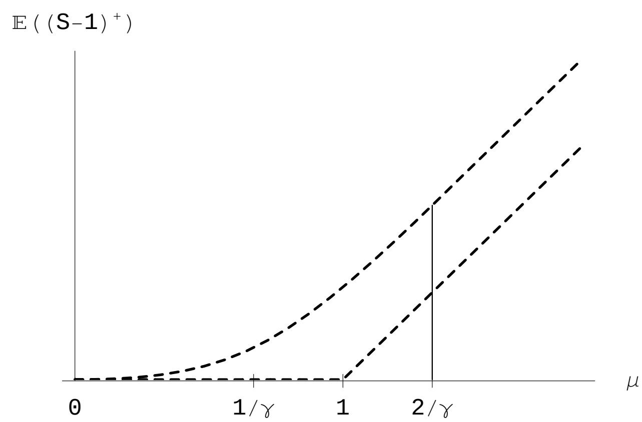

Figure 1: Sharp upper and lower semiparametric bounds for the payoff of a European call option when up to second-order moment information on the underlying asset price is given (plots of $\overline{p}(1, \mu, \gamma\mu^2)$ and $\underline{p}(1, \mu, \gamma\mu^2)$ generated with $\gamma = 1.5$, as a function of μ).

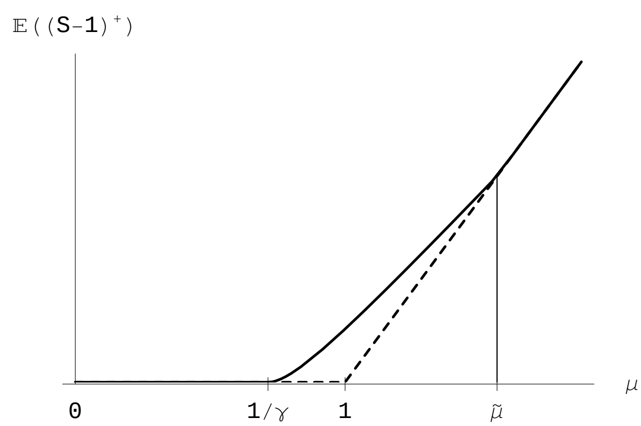

Figure 2: Sharp third-order (solid line) vs. second-order (dashed line) lower semiparametric bounds for the payoff of a European call option (plots of $\underline{p}(1, \mu, \gamma\mu^2)$, $\underline{p}(1, \mu, \gamma\mu^2, \beta\mu^3)$, generated with $\gamma = 1.4$, and $\beta = 2.5$, as a function of μ).

on the payoff of a European call (see Theorem 2). Note that from $\gamma \geq 1$ and $\beta \geq \gamma^2$ (e.g., when $(\underline{P}^\sigma)$ for $n = 3$ is feasible), it follows that $1/\gamma \leq 1 \leq \widetilde{\mu}$. Thus, the tightening occurs in an interval around $\mu = 1$; that is, when the option is *at the money* (recall that the strike price is \$1) and the pricing problem is more interesting.

Now we consider the problem $(\overline{P}^\sigma)$ when case *(ii)* in Proposition 2 holds. In this case, Theorem 5 provides a sharpening of the upper bound given by Lo's Theorem (Theorem 1). The upper bound given by Theorem 5 is exactly $\overline{p}(\sigma)$ for $\mu > 2/\gamma$. In Section 2.4, we will empirically show that for $\mu \leq 2/\gamma$ the upper bound in Theorem 5 gives an accurate approximation of the exact value of $\overline{p}(\sigma)$. To introduce the new upper bound, we need the following definitions:

$$
\begin{aligned}
a(\mu,\gamma,\beta) &= \tfrac{32\mu r_1}{(4r_1+r_2-1)^3}(4\beta\mu^2 - 7\gamma\mu + 3 + r_1(4\gamma\mu - 3) + r_2(1 - r_1 - \gamma\mu)),\\
b(\mu,\gamma) &= \tfrac{1}{2}((\mu - 1) + \sqrt{(\gamma-1)\mu^2 + (\mu-1)^2}),
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
r_1(\mu,\gamma,\beta) &= \sqrt{1 - 2\gamma\mu + \beta\mu^2},\\
r_2(\mu,\gamma,\beta) &= \sqrt{1 + 8r_1(\mu,\gamma,\beta)}.
\end{aligned}
\tag{2}
$$

(Notice that $b(\cdot)$ above comes from Lo's theorem (see Theorem 1).)

Theorem 5 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. If $(\overline{P}^\sigma)$ is feasible and $\beta > \gamma^2$, then*

$$
\begin{aligned}
\overline{p}(\sigma) \leq \quad & \rho(\mu,\gamma,\beta)^+ = \begin{cases} \min\{a(\mu,\gamma,\beta), b(\mu,\gamma)\} & \text{if } \mu \leq \frac{2\gamma}{\beta} \\ b(\mu,\gamma) & \text{if } \frac{2\gamma}{\beta} \leq \mu \leq \frac{2}{\gamma}, \end{cases}\\
\overline{p}(\sigma) = \quad & \mu - \tfrac{1}{\gamma} \qquad \text{if } \mu \geq \tfrac{2}{\gamma},
\end{aligned}
$$

where $a(\cdot), b(\cdot)$ are as in (1).

Proof. See the Appendix. □

As Figure 3 shows, for values of $\mu \leq 2\gamma/\beta$, the addition of third-order moment information tightens the sharp upper second-order semiparametric bound on the payoff of a European call (see Theorem 1).

With the results presented in Theorems 1 through 5, we can now consider how the relationship between the second and third moment of the asset price influences the magnitude by which third-order bounds tighten the second-order semiparametric bounds introduced in Theorems 1 and 2. For this particular analysis we use $\rho(\cdot)^+$ (see Theorem 5) as an upper bound on the option's payoff.

As Figure 4 shows, when the value of $\beta >> \gamma^2$, the magnitude of the tightening due to third-order moment information on the asset price is noticeable only in the lower bound of the option's payoff. As $\beta \approx \gamma^2$, the magnitude of the tightening due to third-order moment information becomes markedly higher for both the lower and upper bounds on the option's payoff. An example of this is shown on Figure 5. In fact, as $\beta \to \gamma^2$, both the third-order upper and lower bounds converge to $(\mu - 1/\gamma)^+$. This is an immediate consequence of Theorem 4 and the following result.

Proposition 3 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. If $(\overline{P}^\sigma)$ is feasible and $\delta := \beta/\gamma^2 - 1 < 1$, then*

$$\overline{p}(\sigma) \le \left(\mu - \frac{1}{\gamma}\right)^+ + 5\delta^{1/3}.$$

Proof. See the Appendix. □

Theorems 4 and 5 also reveal that third-order moment information is not relevant when the mean of the asset is much higher than the strike of the option. This is in agreement with the intuition that option prices should be essentially determined by the mean of the underlying asset price when the mean is bounded away from the strike price.

The discussion in this section shows that third-order moment information on the underlying asset price at maturity can be used to extend the early work of Grundy [8] and Lo [18], by producing tighter bounds on the payoff of a European call option.

2.3 An inventory management application

Although the previous results concern bounds for European call options, the computation of semiparametric bounds is a classical probability problem (cf. Karlin and Studden [11], and Zuluaga and Peña [27]). Hence our results are relevant for a much wider variety of applications. For instance, consider the following classical result of Scarf [25] in inventory management (using the notation introduced here).

Theorem 6 (Scarf [25]) *Let x be the inventory of a single product, c be the product's unit cost, and r the product's unit price. The minimum expected profit over all demands distributions (represented by the random variable S) with given mean $\mu \ge 0$ and variance $(\gamma - 1)\mu^2 \ge 0$ is given by:*

$$\begin{array}{ll} \inf & \mathbb{E}_\pi(r\min\{x, S\} - cx) \\ \text{s.t.} & \mathbb{E}_\pi(1) = 1 \\ & \mathbb{E}_\pi(S) = \mu \\ & \mathbb{E}_\pi(S^2) = \gamma\mu^2 \\ & \pi \quad \textit{a distribution in } \mathrm{I\!R}_+, \end{array} = \begin{cases} x\left(\frac{r}{\gamma} - c\right) & x \le \frac{\gamma\mu}{2}, \\ \frac{r}{2}(\mu + x - \\ \sqrt{\gamma\mu^2 - 2\mu x + x^2}) - cx & x \ge \frac{\gamma\mu}{2}. \end{cases}$$

Scarf [25] uses the result in Theorem 6 to find the level of inventory that maximizes the minimum expected profit; thus, obtaining a robust inventory policy when only second-order moment information on the demand of the product is available. By the simple observation: $\min\{x, S\} = S - (S - x)^+$, it follows that Theorems 3, 4, and 5 can be used to extend Scarf's result by considering third-order moment information on the demand, and upper bounds as well as lower bounds.

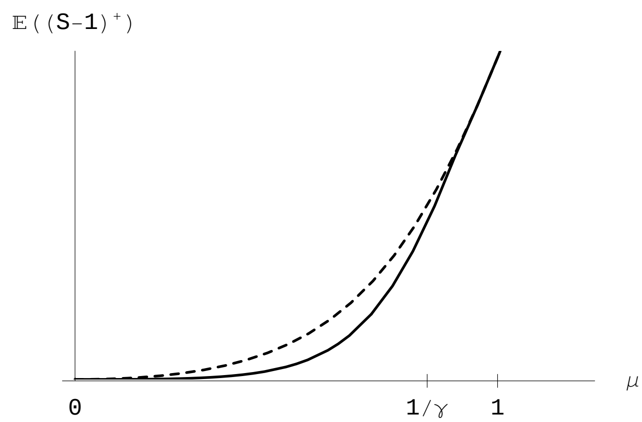

Figure 3: Sharp third-order (solid line) vs. second-order (dashed line) upper semiparametric bounds for the payoff of a European call option (plots of $\overline{p}(1,\mu,\gamma\mu^2)$, $\rho(\mu,\gamma,\beta)^+$, generated with $\gamma = 1.2$, and $\beta = 1.6$, as a function of μ).

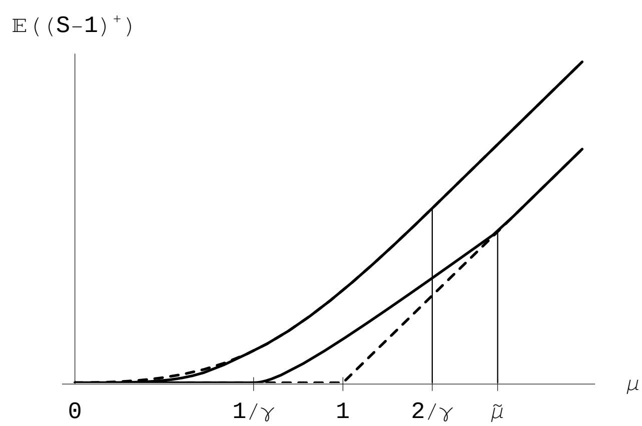

Figure 4: Third-order (solid lines) vs. second-order (dashed lines) semiparametric bounds for the payoff of a European call option when $\beta >> \gamma^2$ (plots of $\overline{p}(1,\mu,\gamma\mu^2)$, $\underline{p}(1,\mu,\gamma\mu^2)$, $\rho(\mu,\gamma,\beta)^+$, $\underline{p}(1,\mu,\gamma\mu^2,\beta\mu^3)$, generated with $\gamma = 1.5$, and $\beta = 3$, as a function of μ).

2.4 Approximation for third-order upper bound

The result in Theorem 5 is not as strong as the one obtained for $(\underline{P}^\sigma)$ in Theorem 4. However, we next introduce a valid lower bound on the value of $\overline{p}(\sigma)$ that will allow us to show empirically that Theorem 5 gives a very accurate approximation for the value of $\overline{p}(\sigma)$ in case *(ii)* of Proposition 2. To introduce this valid lower bound, we need the following definitions:

$$\begin{aligned}
c(\mu,\gamma,\beta) &= \tfrac{1}{2\beta\mu-4\gamma}\left(2-3\gamma\mu+\beta\mu^2+(\gamma\mu-2)r_1(\mu,\gamma,\beta)\right),\\
d(\mu,\gamma,\beta) &= \tfrac{1}{2}\left(\mu-1+\tfrac{1}{r_3(\gamma,\beta)}(2+\beta-3\gamma+(2\gamma^2-\beta-\gamma)\mu)\right),\\
l(\gamma,\beta) &= \tfrac{1}{2(\beta-\gamma^2)}(\beta-\gamma-r_3(\gamma,\beta)),\\
u(\gamma,\beta) &= \tfrac{1}{2\gamma(\gamma-1)}(5\gamma-\beta-4+r_3(\gamma,\beta)),
\end{aligned} \tag{3}$$

where

$$r_3(\gamma,\beta)=\sqrt{\beta^2+\beta(4-6\gamma)+\gamma^2(4\gamma-3)}, \tag{4}$$

and $r_1(\cdot)$ is given in (2).

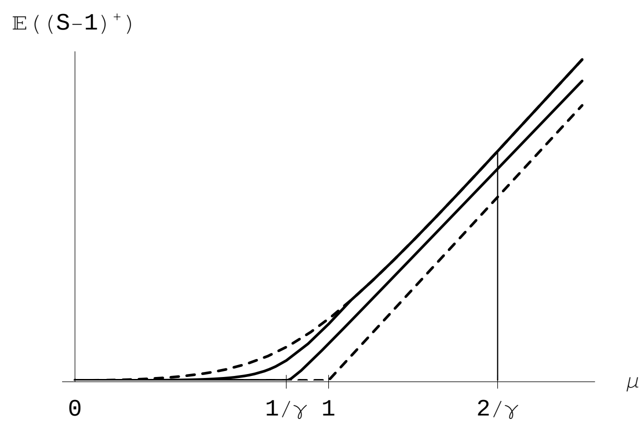

Figure 5: Third-order (solid lines) vs. second-order (dashed lines) semiparametric bounds for the payoff of a European call option when $\beta \approx \gamma^2$ (plots of $\overline{p}(1,\mu,\gamma\mu^2)$, $\underline{p}(1,\mu,\gamma\mu^2)$, $\rho(\mu,\gamma,\beta)^+$, $\underline{p}(1,\mu,\gamma\mu^2,\beta\mu^3)$, generated with $\gamma=1.2$, and $\beta=1.5$, as a function of μ).

Proposition 4 *Let* $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$ *and*

$$\rho(\mu,\gamma,\beta)^- = \begin{cases} c(\mu,\gamma,\beta) & 0 \le \mu \le l(\gamma,\beta), \\ \max\{c(\mu,\gamma,\beta), d(\mu,\gamma,\beta)\} & l(\gamma,\beta) \le \mu \le \frac{2\gamma-2}{\beta-\gamma}, \\ d(\mu,\gamma,\beta) & \frac{2\gamma-2}{\beta-\gamma} \le \mu \le u(\gamma,\beta), \\ \mu - \frac{1}{\gamma} & \mu \ge u(\gamma,\beta), \end{cases}$$

where $c(\cdot), d(\cdot), l(\cdot), u(\cdot)$ *are as in (3). If* $(\overline{P}^\sigma)$ *is feasible and* $\beta > \gamma^2$*, then* $\overline{p}(\sigma) \ge \rho(\mu,\gamma,\beta)^-$*.*

Proof. See the Appendix. □

As Figure 6 shows, using Proposition 4 we can now empirically show that $\rho(\mu,\gamma,\beta)^+$ and $\rho(\mu,\gamma,\beta)^-$ (see Theorem 5 and Proposition 4) give a very accurate interval for the value of $\overline{p}(1,\mu,\gamma\mu,\beta\mu^3)$.

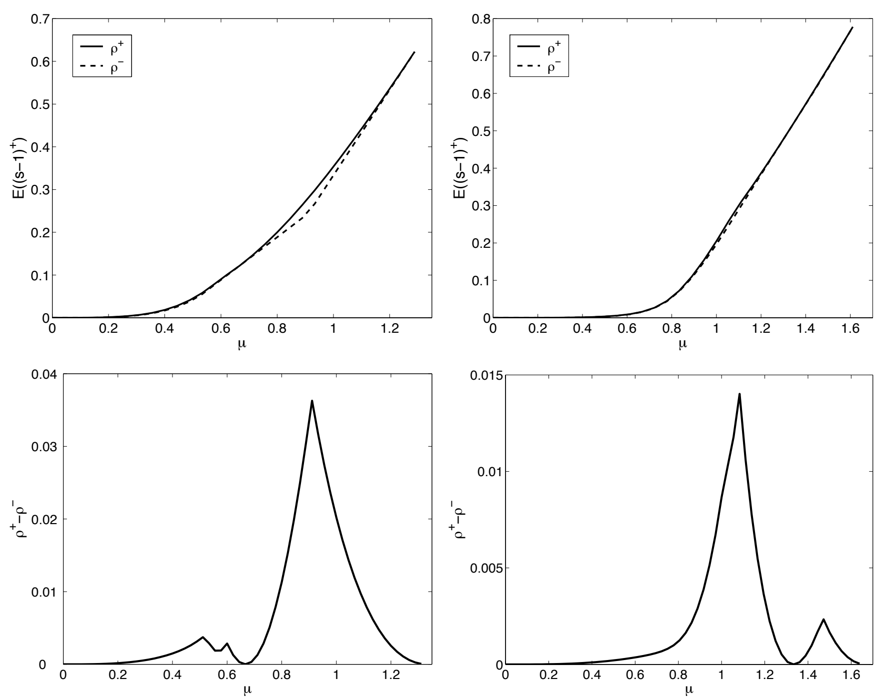

Figure 6: Valid upper and lower bounds for $\overline{p}(1,\mu,\gamma\mu^2,\beta\mu^3)$ for $0 \le \mu \le 2/\gamma$. Top: $\rho(\mu,\gamma,\beta)^+$ and $\rho(\mu,\gamma,\beta)^-$ for $\gamma = 1.5$, $\beta = 3$ (left), and $\gamma = 1.2$, $\beta = 1.5$ (right), as a function of μ. Bottom: Absolute error $\rho(\mu,\gamma,\beta)^+ - \rho(\mu,\gamma,\beta)^-$ for $\gamma = 1.5$, $\beta = 3$ (left), and $\gamma = 1.2$, $\beta = 1.5$ (right), as a function of μ.

3 Bounds on the risk of a European call payoff

In this section, we consider the problem of finding sharp bounds on the risk associated to the expected payoff of a European call option. This, given only up to second-order moment (i.e., mean and variance) information on the underlying asset price at maturity (without making any other assumption on the distribution of the asset price). To measure the risk, we consider the variance of the option's payoff. As a first step towards finding such bounds, we start by considering this problem when information on the expected payoff of the European call is also given.

3.1 Risk bounds given option's payoff expectation

Finding the sharp upper, and the sharp lower bound for the above described problem, when information on the expected payoff of the option is also given, can be (respectively) formulated as the following optimization problems:

$$
(\overline{Q}) \qquad \begin{array}{rll} \overline{q}(\mu,\gamma,\widehat{\mu}) = & \sup & \mathbb{V}\mathrm{ar}_\pi((S-1)^+) \\ & \text{s.t.} & \mathbb{E}_\pi(1) = 1 \\ & & \mathbb{E}_\pi(S) = \mu \\ & & \mathbb{E}_\pi(S^2) = \gamma\mu^2 \\ & & \mathbb{E}_\pi((S-1)^+) = \widehat{\mu} \\ & & \pi \text{ a distribution on } \mathbb{R}_+, \end{array} \tag{5}
$$

and

$$
(\underline{Q}) \qquad \begin{array}{rll} \underline{q}(\mu,\gamma,\widehat{\mu}) = & \inf & \mathbb{V}\mathrm{ar}_\pi((S-1)^+) \\ & \text{s.t.} & \mathbb{E}_\pi(1) = 1 \\ & & \mathbb{E}_\pi(S) = \mu \\ & & \mathbb{E}_\pi(S^2) = \gamma\mu^2 \\ & & \mathbb{E}_\pi((S-1)^+) = \widehat{\mu} \\ & & \pi \text{ a distribution on } \mathbb{R}_+. \end{array} \tag{6}
$$

Here, analogous to the semiparametric bound problems considered in Section 2, the random variable S represents the price of the underlying asset at maturity. Also, the formulations assume, without loss of generality, that the strike price of the call option is \$1. Thus, the problem $(\overline{Q})$ (resp. $(\underline{Q})$) maximizes (minimizes) the variance of a European call option's payoff ($(S-1)^+ := \max\{0, S-1\}$) over all probability distributions π in $\mathbb{R}_+$ whose moments satisfy the given information on the mean (μ) and variance ($(\gamma-1)\mu^2$) of the underlying asset price at maturity, and the given expected option's payoff ($\widehat{\mu}$).

In what follows, the aim is to provide closed-form solutions to $(\overline{Q})$ and $(\underline{Q})$. For this purpose, we begin by addressing the feasibility of $(\underline{Q})$ and $(\overline{Q})$.

Proposition 5 *The problems $(\underline{Q})$ and $(\overline{Q})$ are feasible if and only if*

$$\mu \geq 0, \gamma \geq 1,$$

and

$$(\mu-1)^+ \le \widehat{\mu} \le \begin{cases} \mu - \frac{1}{\gamma} & \text{if } \mu \ge \frac{2}{\gamma}, \\ \frac{1}{2}((\mu-1)+\sqrt{(\gamma-1)\mu^2+(\mu-1)^2}) & \text{if } \mu \le \frac{2}{\gamma}. \end{cases}$$

Proof. The first condition is the same as that of Proposition 1. The second condition follows from Lo's upper semiparametric bound (see Theorem 1), and its corresponding lower semiparametric bound given in Theorem 2. □

First, we present the closed-form solution of $(\underline{Q})$.

Theorem 7 *If $(\underline{Q})$ is feasible then*

$$\underline{q}(\mu,\gamma,\widehat{\mu}) = \begin{cases} \mu(\gamma\mu-1)-\widehat{\mu}-\widehat{\mu}^2 & \text{if } \widehat{\mu} \le \mu-\frac{1}{\gamma}, \\ \frac{1}{2}(2\widehat{\mu}(\mu-1)+\mu((\gamma-1)\mu-r(\mu,\gamma,\widehat{\mu}))-\widehat{\mu}^2 & \text{if } \widehat{\mu} \ge \mu-\frac{1}{\gamma}, \end{cases}$$

where $r(\mu,\gamma,\widehat{\mu}) = \sqrt{(\gamma-1)((\gamma-1)\mu^2+4\widehat{\mu}(\mu-1)-4\widehat{\mu}^2)}$.

Proof. This follows from putting together Lemmas 14 and 15 in the Appendix. □

Now, we present the closed-form solution of $(\overline{Q})$.

Theorem 8 *If $(\overline{Q})$ is feasible then*

$$\overline{q}(\mu,\gamma,\widehat{\mu}) = \frac{1}{2}(2\widehat{\mu}(\mu-1)+\mu((\gamma-1)\mu-r(\mu,\gamma,\widehat{\mu})))-\widehat{\mu}^2,$$

where $r(\mu,\gamma,\widehat{\mu}) = -\sqrt{(\gamma-1)((\gamma-1)\mu^2+4\widehat{\mu}(\mu-1)-4\widehat{\mu}^2)})$.

Proof. See the Appendix. □

With Theorems 7 and 8, we can compute semiparametric bounds on the risk associated to a given expectation of the payoff of a European call option. In Figure 7 we compute these bounds for two particular instances of the parameters of the problem. Analogous to the results presented in Section 2, these bounds can be used to set bounds on the risk associated to a European call under the risk-neutral measure pricing theory, and to test the validity of different model assumptions. In the following section, we will introduce results that will allow us to compute similar risk bounds when no information on the expectation of a European call option is given.

3.2 Risk bounds without option's payoff expectation

Now, we are ready to reconsider the problem described at the beginning of the section; namely, finding sharp bounds on the risk (i.e., variance) associated to the expected payoff of a European call option. This, given only up to second-order moment (i.e., mean and variance) information on the underlying asset price at maturity. Finding the sharp upper, and the sharp lower bound for this problem, can be (respectively) formulated as the following optimization problems:

$$(\overline{R}) \qquad \begin{array}{rll} \overline{r}(\mu,\gamma) = & \sup & \mathbb{V}\mathrm{ar}_\pi((S-1)^+) \\ & \text{s.t.} & \mathbb{E}_\pi(1) = 1 \\ & & \mathbb{E}_\pi(S) = \mu \\ & & \mathbb{E}_\pi(S^2) = \gamma\mu^2 \\ & & \pi \text{ a distribution on } \mathbb{R}_+, \end{array}$$

and

$$(\underline{R}) \qquad \begin{array}{rll} \underline{r}(\mu,\gamma) = & \inf & \mathbb{V}\mathrm{ar}_\pi((S-1)^+) \\ & \text{s.t.} & \mathbb{E}_\pi(1) = 1 \\ & & \mathbb{E}_\pi(S) = \mu \\ & & \mathbb{E}_\pi(S^2) = \gamma\mu^2 \\ & & \pi \text{ a distribution on } \mathbb{R}_+. \end{array}$$

From the problems introduced in Section 3.1 (eq. (5) and (6)) and Proposition 5, we can readily obtain closed-form solutions for $(\overline{R})$ and $(\underline{R})$.

Theorem 9 *Problem $(\overline{R})$ is feasible if and only if $\mu \geq 0$ and $\gamma \geq 1$. Furthermore, if $(\overline{R})$ is feasible, then $\overline{r}(\mu,\gamma) = (\gamma-1)\mu^2$.*

Proof. The feasibility conditions of $(\overline{R})$ are straightforward (see Proposition 5). Let $\mu \geq 0$ and $\gamma \geq 1$. From (5) and Proposition 5, it follows that $\overline{r}(\mu,\gamma) = \sup\{\overline{q}(\mu,\gamma,\widehat{\mu}) : (\mu-1)^+ \leq \widehat{\mu} \leq \widehat{\mu}^+\}$, where $\widehat{\mu}^+ = \mu - 1/\gamma$ if $\mu \geq 2/\gamma$ and $\widehat{\mu}^+ = (1/2)((\mu-1) + \sqrt{(\gamma-1)\mu^2 + (\mu-1)^2})$ if $\mu \leq 2/\gamma$. Since $\overline{q}(\mu,\gamma,\widehat{\mu})$ is decreasing with $\widehat{\mu}$, it follows that $\overline{r}(\mu,\gamma) = \overline{q}(\mu,\gamma,(\mu-1)^+) = (\gamma-1)\mu^2$. $\square$

Theorem 10 *Problem $(\underline{R})$ is feasible if and only if $\mu \geq 0$ and $\gamma \geq 1$. Furthermore, if $(\underline{R})$ is feasible, then $\underline{r}(\mu,\gamma) = (\gamma-1)((\gamma\mu-1)^+)^2/\gamma^2$.*

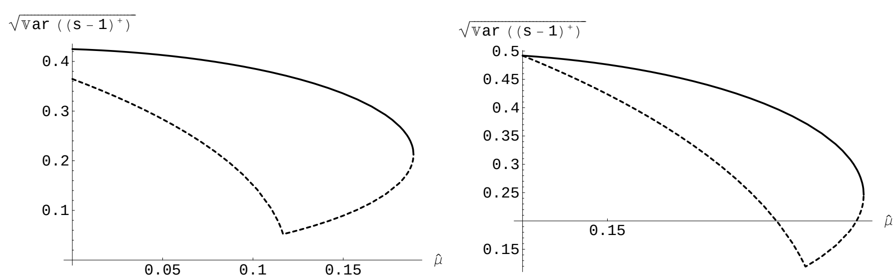

Figure 7: Upper and lower semiparametric bounds for the risk associated to a given expectation on the payoff of a European call option. Left: $\underline{q}(\mu,\gamma,\widehat{\mu})$ (dashed) and $\overline{q}(\mu,\gamma,\widehat{\mu})$ (solid) for $\gamma = 1.2, \mu = 0.95$, as a function of $\widehat{\mu}$. Right: $\underline{q}(\mu,\gamma,\widehat{\mu})$ (dashed) and $\overline{q}(\mu,\gamma,\widehat{\mu})$ (solid) for $\gamma = 1.2, \mu = 1.1$, as a function of $\widehat{\mu}$.

Proof. It follows analogous to the proof of Theorem 9.

In particular, Theorem 9 shows that the risk associated to the payoff of a European call option is not higher than the variance of the underlying asset price at maturity, when information about the asset's mean and variance is available.

3.3 A supply chain management application

As in Section 2.3, here we present an application of the results presented in this section to a so-called supply chain game. Consider a scenario where two retailers compete on product availability (cf. Parlar [20], Lippman and McCardle [17], and Avsar and Baykal-Gursoy [1]). Denote by

$$\begin{array}{rcl} p_i & = & \text{sale price per unit for retailer } i \\ c_i & = & \text{purchase cost price per unit for retailer } i \\ q_i & = & \text{order quantity for retailer } i \\ D_i & = & \text{demand for retailer } i \\ X_i & = & \text{effective demand for retailer } i, \end{array}$$

for $i = 1, 2$. Then

$$X_i = D_i + (D_j - q_j)^+,$$

where $(D_j - q_j)^+$ is the leftover from retailer $j \neq i$. The payoff function is therefore:

$$\begin{aligned} G_i(q_1, q_2) &= \mathbb{E}(p_i \min\{X_i, q_i\}) - c_i q_i \\ &= \mathbb{E}(p_i \min\{D_i + (D_j - q_j)^+, q_i\}) - c_i q_i, \end{aligned}$$

for $i, j = 1, 2, i \neq j$. Each retailer wants to maximize its own profit G_i. Suppose D_i has a distribution with mean $\mathbb{E}(D_i) = \mu_i$ and variance $\mathbb{V}\text{ar}(D_i) = (\gamma_i - 1)\mu_i^2$, for $i = 1, 2$. In this setting, it is interesting (as in Scarf's result, presented in Theorem 6) to compute sharp lower bounds on the retailer's profit, over all demand distributions with the given mean and variance infomation. One way to address this problem using the results outlined here, is the following. Let us focus on retailer 1. Assuming that D_1 and D_2 are independent, it follows from the available mean and variance information on D_1 and D_2 that

$$\mathbb{E}(X_1) = \mu_1 + \widehat{\mu}_2,$$

and (recalling (5) and (6)),

$$(\gamma_1 - 1)\mu_1^2 + \underline{q}\left(\frac{\mu_2}{q_2}, \gamma_2, \frac{\widehat{\mu}_2}{q_2}\right) \leq \mathbb{V}\text{ar}(X_1) \leq (\gamma_1 - 1)\mu_1^2 + \overline{q}\left(\frac{\mu_2}{q_2}, \gamma_2, \frac{\widehat{\mu}_2}{q_2}\right),$$

for some $(\mu_2 - q_2)^+ \leq \widehat{\mu}_2 \leq \widehat{\mu}_2^+$, where

$$\widehat{\mu}_2^+ = \begin{cases} \mu_2 - \frac{q_2}{\gamma_2} & \text{if } \mu_2 > \frac{2q_2}{\gamma_2}, \\ \frac{1}{2}((\mu_2 - q_2) + \sqrt{(\gamma_2 - 1)\mu_2^2 + (\mu_2 - q_2)^2}) & \text{if } \mu \leq \frac{2q_2}{\gamma_2}. \end{cases}$$

This follows from Theorems 1, 2, 7 and 8. Thus, to get a sharp bound on the minimum profit of retailer 1 with the moment information and assumptions discussed above, one could solve the problem:

$$\inf\{\underline{G}_1(\widehat{\mu}_2) : (\mu_2 - q_2)^+ \leq \widehat{\mu}_2 \leq \widehat{\mu}_2^+\},$$

where

$$\begin{array}{rll}
\underline{G}_1(\widehat{\mu}_2) = & \inf & \mathbb{E}_\pi(p_1 \min\{X_1, q_1\} - c_1 q_1) \\
& & \\
& \text{s.t.} & \mathbb{E}_\pi(1) = 1 \\
& & \mathbb{E}_\pi(X_1) = \mu_1 + \widehat{\mu}_2 \\
& & \mathbb{E}_\pi(X_1^2) \geq \gamma_1 \mu_1^2 + \underline{q}\left(\frac{\mu_2}{q_2}, \gamma_2, \frac{\widehat{\mu}_2}{q_2}\right) + 2\mu_1\widehat{\mu}_2 + \widehat{\mu}_2^2 \\
& & \mathbb{E}_\pi(X_1^2) \leq \gamma_1 \mu_1^2 + \overline{q}\left(\frac{\mu_2}{q_2}, \gamma_2, \frac{\widehat{\mu}_2}{q_2}\right) + 2\mu_1\widehat{\mu}_2 + \widehat{\mu}_2^2 \\
& & \pi \quad \text{a distribution in } \mathbb{R}_+.
\end{array}$$

We can also solve the maximization problem to get a sharp bound $\overline{G}_1(\hat{\mu}_2)$ on the maximum profit of retailer 1.

These two problems are slight variations of the one addressed by Scarf [25] (see Theorem 6). It thus follows that for the minimization problem the second constraint on $\mathbb{E}_\pi(X_1^2)$ is binding and the exact value of $\overline{G}_1(\hat{\mu}_2)$ follows from Theorem 6. Likewise, for the maximization problem the first constraint on $\mathbb{E}_\pi(X_1^2)$ is binding and the exact value of $\underline{G}_1(\hat{\mu}_2)$ follows from Theorem 2.

4 Appendix

Proof of Theorem 2. It is immediate that $\underline{p}(\sigma) \geq (\mu-1)^+$. To show the equality, let k be a positive integer and consider the following distribution

$$\pi_k(s) = \begin{cases} \frac{1}{k(1+k^2)} & \text{if } s = \mu\left(1 + k\sqrt{k(\gamma-1)}\right), \\ 1 - \frac{1}{k} & \text{if } s = \mu, \\ \frac{k}{1+k^2} & \text{if } s = \mu\left(1 - \frac{1}{k}\sqrt{k(\gamma-1)}\right). \end{cases}$$

Observe that π_k is a feasible distribution for $(\underline{P}^\sigma)$ as long as k is sufficiently large. If $\mu > 1$ then taking k such that $\mu(1 - \frac{1}{k}\sqrt{k(\gamma-1)}) > 1$, we have $\mathbb{E}_{\pi_k}((S-1)^+) = \mathbb{E}_{\pi_k}(S-1) = \mu - 1$. Hence $\underline{p}(\sigma) \leq \mu - 1$ if $\mu > 1$. On the other hand, if $\mu \leq 1$ then taking k such that $\mu(1 - \frac{1}{k}\sqrt{k(\gamma-1)}) \geq 0$, we have $\mathbb{E}_{\pi_k}((S-1)^+) = (\mu(1+k\sqrt{k(\gamma-1)})-1)^+(\frac{1}{k(1+k^2)})$. Taking the limit as $k \to \infty$, we get $\lim_{k\to\infty} \mathbb{E}_{\pi_k}((S-1)^+) = 0$. Hence in either case $\underline{p}(\sigma) \leq (\mu-1)^+$, which completes the proof. □

Now we introduce the following notation.

Definition 1 *Let $\mathcal{M}_{n+1}$ be the set of vectors $\sigma \in \mathbb{R}^{n+1}$ for which there exists a distribution π in $\mathbb{R}_+$ such that $\sigma_i = \mathbb{E}_\pi(S^i)$ for $i = 0, 1, \ldots, n$.*

Proof of Proposition 2. Evidently, $(\underline{P}^\sigma)$ and $(\overline{P}^\sigma)$ are feasible if and only if $\sigma \in \mathcal{M}_4$. The case $\mu = 0$ is trivial as $\sigma \in \mathcal{M}_4 \iff \sigma = (1, 0, 0, 0)$. Hence, let us assume $\mu > 0$. It is known that for any $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3) \in \mathcal{M}_4$ the following two matrices are positive semidefinite (see Theorem V.10.1 in [11], p. 173) : $R_2 = \begin{bmatrix} 1 & \mu \\ \mu & \gamma\mu^2 \end{bmatrix}$, $R_3 = \begin{bmatrix} \mu & \gamma\mu^2 \\ \gamma\mu^2 & \beta\mu^3 \end{bmatrix}$. The matrices R_2 and R_3 are positive semidefinite if and only if $\gamma \geq 1$, $\beta \geq \gamma^2$ and $\mu \geq 0$. To finish the *only if* direction, just note that $\gamma = 1$ implies $\beta = \gamma^2$.

For the *if* direction, it is known that if R_2 and R_3 are positive definite; i.e., if $\gamma > 1$, $\beta > \gamma^2$ and $\mu > 0$, then $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3) \in \mathcal{M}_4$ (see Theorem V.10.1 in [11], p. 173). Finally, for the case $\beta = \gamma^2$, $\gamma \geq 1$, and $\mu > 0$ the following distribution shows that $\sigma = (1, \mu, \gamma\mu^2, \gamma^2\mu^3) \in \mathcal{M}_4$:

$$\pi(s) = \begin{cases} \frac{1}{\gamma} & \text{if } s = \gamma\mu, \\ 1 - \frac{1}{\gamma} & \text{if } s = 0. \end{cases} \tag{7}$$

$\square$

In order to prove some of the results that follow, we need to introduce the dual problems corresponding to $(\overline{P}^\sigma)$ and $(\underline{P}^\sigma)$ (see, e.g., Bertsimas and Popescu [3]). The dual problem corresponding to problem $(\overline{P}^\sigma)$ can be written as:

$$(\overline{D}^\sigma) \qquad \begin{array}{rll} \overline{d}(\sigma) = & \inf & \displaystyle\sum_{i=0}^n \sigma_i y_i \\ & \text{s.t.} & \displaystyle\sum_{i=0}^n y_i s^i \geq (s-1)^+ \quad \text{for all } s \in \mathrm{I\!R}_+. \end{array}$$

It is easy to see that weak duality holds between $(\overline{P}^\sigma)$ and $(\overline{D}^\sigma)$:

$$\overline{d}(\sigma) \geq \overline{p}(\sigma). \tag{8}$$

The dual problem corresponding to problem $(\underline{P}^\sigma)$ can be written as:

$$(\underline{D}^\sigma) \qquad \begin{array}{rll} \underline{d}(\sigma) = & \sup & \displaystyle\sum_{i=0}^n \sigma_i y_i \\ & \text{s.t.} & \displaystyle\sum_{i=0}^n y_i s^i \leq (s-1)^+ \quad \text{for all } s \in \mathrm{I\!R}_+. \end{array}$$

It is easy to see that weak duality holds between $(\underline{P}^\sigma)$ and $(\underline{D}^\sigma)$:

$$\underline{d}(\sigma) \leq \underline{p}(\sigma). \tag{9}$$

Furthermore, for both the upper and lower bound problems, strong duality holds under a suitable Slater condition (see Theorem XII.2.1 in Karlin and Studden (1966), p. 472). In particular, we will rely on the following fact:

Remark 1 *If $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3) \in \mathcal{M}_4$ with $\mu > 0$ and $\beta > \gamma^2$, then $\overline{d}(\sigma) = \overline{p}(\sigma)$ and $\underline{d}(\sigma) = \underline{p}(\sigma)$.*

We also rely on the following straightforward property of cubic polynomials.

Proposition 6 *Let $p(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3$ where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Assume $p(s)$ has real roots $r_1 \le r_2 \le r_3$. If $a_3 > 0$ then $p(s) \ge 0$ for $s \in [r_1, r_2] \cup [r_3, \infty)$. If $a_3 < 0$ then $p(s) \le 0$ for $s \in [r_1, r_2] \cup [r_3, \infty)$.*

We repeatedly use Proposition 6 to verify that a given vector (y_0, y_1, y_2, y_3) is feasible for $(\overline{D}^{\sigma})$ or for $(\underline{D}^{\sigma})$ when $n = 3$. In all cases this is done as follows:

(i) Compute (sometimes with the help of a symbolic computation package such as `Maple` or `Mathematica`) the roots of the polynomials $p_1(s) = y_0 + y_1 s + y_2 s^2 + y_3 s^3$ and $p_2(s) = y_0 + y_1 s + y_2 s^2 + y_3 s^3 - (s-1)$.

(ii) Verify, via Proposition 6, that $p_1(s)$ and $p_2(s)$ satisfy the relevant inequalities. That is, for $(\overline{D}^{\sigma})$ we verify that $p_1(s) \ge 0$ for $s \in [0,1]$ and $p_2(s) \ge 0$ for $s \in [1, \infty)$, whereas for $(\underline{D}^{\sigma})$ we verify that $p_1(s) \le 0$ for $s \in [0,1]$ and $p_2(s) \le 0$ for $s \in [1, \infty)$.

To avoid redundant exposition, when we write

"By Proposition 6 the vector (y_0, y_1, y_2, y_3) is feasible for $(\overline{D}^{\sigma})$"

it is to be understood that the above steps (i) and (ii) are performed.

Proof of Theorem 3. By Proposition 2 it suffices to consider the case $\mu > 0$ and $\gamma \ge 1$. Consider the distribution $\pi(s)$ defined by (7). Notice that π is feasible for both $(\underline{P}^{\sigma})$ and $(\overline{P}^{\sigma})$. Thus

$$\underline{p}(\sigma) \le \mathbb{E}_{\pi}(S-1)^+ = \left(\mu - \frac{1}{\gamma}\right)^+ \le \overline{p}(\sigma). \tag{10}$$

Now we show that the equalities actually hold. We consider two cases:

Case 1: $\gamma\mu = 1$. In this case it can be shown that the moment constraints imply that π given by (7) is the only feasible distribution for $(\underline{P}^{\sigma})$ or $(\overline{P}^{\sigma})$. Then, it follows that $\underline{p}(\sigma) = \overline{p}(\sigma) = (\mu - 1/\gamma)^+$.

Case 2: $\gamma\mu \ne 1$. First, consider the vector $y = (y_0, y_1, y_2, y_3)$ defined as follows

$$y = \begin{cases} (0,0,0,0)^T & \text{for } \mu < \frac{1}{\gamma}, \\ (\frac{1}{(\gamma\mu)})^2 (0, -2\gamma\mu, 2\gamma\mu + 1, -1)^T & \text{for } \mu > \frac{1}{\gamma}. \end{cases}$$

By Proposition 6 the vector y is feasible for $(\underline{D}^{\sigma})$. Furthermore, for this y we have $\sum_{i=0}^{3} \sigma_i y_i = (\mu - \frac{1}{\gamma})^+$. Thus $\underline{d}(\sigma) \ge (\mu - 1/\gamma)^+$ which together with (9) and (10) gives $\underline{d}(\sigma) = \underline{p}(\sigma) = (\mu - 1/\gamma)^+$. On the other hand, consider the vector $y = (y_0, y_1, y_2, y_3)$ defined as follows

$$y = \begin{cases} \frac{(\gamma\mu)^2}{(1-\gamma\mu)(3-\gamma\mu)} (0, 1, -\frac{2}{\gamma\mu}, (\frac{1}{\gamma\mu})^2)^T & \text{for } \mu < \frac{1}{\gamma}, \\ (0, a, \frac{1}{\gamma\mu^2}(2\mu(1-a) - \frac{3}{\gamma}), \frac{1}{\gamma^2\mu^3}(-\mu(1-a) + \frac{2}{\gamma}))^T & \text{for } \mu > \frac{1}{\gamma}, \end{cases}$$

where $a \geq (\gamma\mu - \frac{3}{2})^2/(\gamma\mu(\gamma\mu - 1))$. By Proposition 6 the vector y is feasible for $(\overline{D}^{\sigma})$. Furthermore, for this y we have $\sum_{i=0}^{3} \sigma_i y_i = (\mu - 1/\gamma)^+$. Thus $\overline{d}(\sigma) \leq (\mu - 1/\gamma)^+$ which together with (8) and (10) gives $\overline{d}(\sigma) = \overline{p}(\sigma) = (\mu - 1/\gamma)^+$.□

The proofs of some of the results that we present next, rely on the following simple but crucial observations: Given $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3) \in \mathcal{M}_4$, consider the moment vectors

$$\sigma' = (1, \mu, \gamma\mu^2) \in \mathcal{M}_3 \ \text{ and } \ \sigma'' = (1, \mu, \gamma\mu^2, \gamma^2\mu^3) \in \mathcal{M}_4.$$

Because $(\overline{P}^{\sigma})$ is more constrained than $(\overline{P}^{\sigma'})$, it follows that

$$\overline{p}(\sigma') \geq \overline{p}(\sigma). \tag{11}$$

Likewise,

$$\underline{p}(\sigma') \leq \underline{p}(\sigma). \tag{12}$$

Notice also that $(\overline{D}^{\sigma})$ and $(\overline{D}^{\sigma''})$ have the same feasible set. Furthermore, any point (y_0, y_1, y_2, y_3) in this feasible set must satisfy $y_3 \geq 0$. Thus, since $\beta \geq \gamma^2$,

$$\overline{d}(\sigma) \geq \overline{d}(\sigma''). \tag{13}$$

Likewise,

$$\underline{d}(\sigma) \leq \underline{d}(\sigma''). \tag{14}$$

Lemma 11 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. If $(\underline{P}^{\sigma})$ is feasible and $\beta > \gamma^2$, then $\underline{p}(\sigma) = 0$ for $\mu \leq 1/\gamma$.*

Proof. From Proposition 2 it follows that $\mu > 0$. Since $\beta > \gamma^2$, strong duality holds for $(\underline{P}^{\sigma})$, and $(\underline{D}^{\sigma})$; i.e., $\underline{d}(\sigma) = \underline{p}(\sigma)$ (see Remark 1). By Theorem 3 and weak duality (9), $\underline{d}(\sigma'') \leq \underline{p}(\sigma'') = (\mu - 1/\gamma)^+$. Since $\underline{p}(\sigma) \geq 0$, it then follows from (14) that $\underline{p}(\sigma) = 0$ for $\mu \leq 1/\gamma$. □

Lemma 12 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. If $(\underline{P}^{\sigma})$ is feasible and $\beta > \gamma^2$, then $\underline{p}(\sigma) = \mu - 1$ for $\mu \geq \widetilde{\mu}$, where $\widetilde{\mu}$ is as defined in Theorem 4.*

Proof. From Proposition 2 it follows that $\mu > 0$, and $\beta > \gamma^2$. Observe that the solution $(y_o, y_1, y_2, y_3) = (-1, 1, 0, 0)$ is feasible for $(\underline{D}^{\sigma})$. Hence by weak duality (9):

$$\underline{p}(\sigma) \geq \underline{d}(\sigma) \geq \sum_{i=0}^{3} \sigma_i y_i = \mu - 1. \tag{15}$$

Now consider the following distribution

$$\pi(s) = \begin{cases} p & \text{if } s = \mu + b, \\ 1 - p & \text{if } s = \mu - a, \end{cases}$$

where $a = \widetilde{\mu} - 1$, $b = (\widetilde{\mu} - 1) + \frac{\beta - 3\gamma + 2}{\gamma - 1}\mu$ and $p = \frac{(\gamma-1)^2\mu^2}{((\beta - 3\gamma+2)\mu + 2a(\gamma-1))b}$. The distribution π is feasible for $(\underline{P}^\sigma)$. For $\mu \geq \widetilde{\mu}$ we have $\mu - a \geq 1$, so

$$\underline{p}(\sigma) \leq \mathbb{E}_{\pi(s)}(S-1)^+ = \mu - 1. \tag{16}$$

Putting together (15) and (16), we get $\underline{p}(\sigma) = \mu - 1$ for $\mu \geq \widetilde{\mu}$. □

Lemma 13 *Let $\sigma = (1, \mu, \gamma\mu^2, \beta\mu^3)$. If $(\underline{P}^\sigma)$ is feasible and $\beta > \gamma^2$, then $\underline{p}(\sigma) = \frac{(\gamma\mu-1)^2}{\beta\mu - \gamma}$ for $1/\gamma \leq \mu \leq \widetilde{\mu}$, where $\widetilde{\mu}$ is as defined in Theorem 4.*

Proof. Consider the vector $y = (y_0, y_1, y_2, y_3)$ given by

$$(y_0, y_1, y_2, y_3) = \left(0, \frac{2(\gamma\mu - 1)}{(\gamma - \beta\mu)\mu}, \frac{(\gamma\mu - 1)(2\beta\mu^2 - \gamma\mu - 1)}{(\gamma - \beta\mu)^2\mu^2}, -\frac{(\gamma\mu-1)^2}{(\gamma - \beta\mu)^2\mu^2}\right).$$

By Proposition 6 the vector y is feasible for $(\underline{D}^\sigma)$ for $1/\gamma < \mu \leq \tilde{\mu}$. Furthermore, the objective value of the vector (y_0, y_1, y_2, y_3) is equal to $(\gamma\mu - 1)^2/(\beta\mu - \gamma) \leq \underline{d}(\sigma)$. Thus, from weak duality it follows that $(\gamma\mu - 1)^2/\beta\mu - \gamma \leq \underline{p}(\sigma)$.

For the reverse inequality consider the following probability distribution:

$$\pi(s) = \begin{cases} \frac{\gamma^3\mu^3 - 3\gamma^2\mu^2 + 3\gamma\mu - 1}{\beta^2\mu^3 - 3\beta\gamma\mu^2 + (2\gamma^2 + \beta)\mu - \gamma} & \text{if } s = \frac{(\beta\mu - \gamma)\mu}{\gamma\mu - 1}, \\ \frac{(\beta - \gamma^2)\mu^3}{\beta\mu^2 - 2\gamma\mu + 1} & \text{if } s = 1, \\ \frac{(\gamma^2 - \beta)\mu^2 + (\beta - \gamma)\mu - \gamma + 1}{\beta\mu - \gamma} & \text{if } s = 0. \end{cases}$$

If $(\underline{P}^\sigma)$ is feasible, $\beta > \gamma^2$, and $1/\gamma < \mu \leq \widetilde{\mu}$, then $\pi(s)$ is feasible for $(\underline{P}^\sigma)$ and its objective value is equal to $(\gamma\mu - 1)^2/(\beta\mu - \gamma)$. It thus follows that $\underline{p}(\sigma) \leq (\gamma\mu - 1)^2/(\beta\mu - \gamma)$. □

Proof of Theorem 5. By Theorem 1 and Theorem 3, $\overline{p}(\sigma') = \overline{p}(\sigma'') = \mu - 1/\gamma$ for $\mu \geq 2/\gamma$. Furthermore, since $\beta > \gamma^2$ and $\mu \geq 2/\gamma > 0$ then strong duality holds for $(\overline{P}^\sigma)$, and $(\overline{D}^\sigma)$; i.e., $\overline{d}(\sigma) = \overline{p}(\sigma)$ (see Remark 1). Also by weak duality (8), $\overline{d}(\sigma'') \geq \overline{p}(\sigma'')$. From (11) and (13) it then follows that $\overline{p}(\sigma) = \mu - 1/\gamma$ for $\mu \geq 2/\gamma$.

Consider the vector $y = (y_0, y_1, y_2, y_3)$ given by:

$$\begin{aligned} y_0 &= 0 \\ y_1 &= \tfrac{s_1^3 - 6s_1s_2 + s_1^2 s_2 + 4s_1s_2^2}{(s_1 - s_2)^3} \\ y_2 &= \tfrac{3s_1 - 2s_1^2 + 3s_2 - 2s_1s_2 - 2s_2^2}{(s_1 - s_2)^3} \\ y_3 &= \tfrac{-2 + s_1 + s_2}{(s_1 - s_2)^3}, \end{aligned}$$

where $s_1 = 1 - r_1(\mu, \gamma, \beta)$ and $s_2 = \frac{1}{4}(3 + r_2(\mu, \gamma, \beta))$ (see (2)). By Proposition 6 the vector y is feasible for $(\overline{D}^\sigma)$ if $(\overline{P}^\sigma)$ is feasible and $\mu \leq 2\gamma/\beta$. Furthermore, the objective value of the vector (y_0, y_1, y_2, y_3) is equal to $a(\mu, \gamma, \beta) \geq \overline{d}(\sigma)$. The bound $b(\mu,\gamma,\beta) \geq \overline{d}(\sigma)$ follows from Theorem 1. Hence from weak duality it follows that $\overline{p}(\sigma) \leq \overline{d}(\sigma) \min\{a(\mu,\gamma,\beta), b(\mu,\gamma,\beta)\}$. $\square$

Proof of Proposition 3. Consider two separate cases:

Case 1: $|\gamma\mu - 1| \leq \delta^{2/3}$. Let S be any non-negative random variable whose first three moments are $(\mu, \gamma\mu^2, \beta\mu^3)$. The non-negativity of S and Tchebycheff's inequality yield

$$\mathbb{E}((S-1)^+)^2 \leq \mathbb{E}(((S-1)^+)^2) \leq \mathbb{E}(S(S-1)^2) = \beta\mu^3 - 2\gamma\mu^2 + \mu.$$

But $|\gamma\mu - 1| \leq \delta^{2/3}$ and $\frac{\beta}{\gamma^2} = 1 + \delta$ imply

$$\beta\mu^3 - 2\gamma\mu^2 + \mu \leq \mu\left(\frac{\beta(1+\delta)^2}{\gamma^2} - 1 + 2\delta\right) \leq \mu(5\delta^{2/3} + 3\delta^{4/3} + \delta^2).$$

By Proposition 2, $\gamma \geq 1$ so $\mu \leq 1 + \delta^{2/3}$. Hence

$$\mathbb{E}((S-1)^+) \leq (\beta\mu^3 - 2\gamma\mu + \mu)^{1/2} \leq 5\delta^{1/3}.$$

Since this holds for any non-negative random variable S whose first three moments are $(\mu, \gamma\mu^2, \beta\mu^3)$, we get $\overline{p}(\sigma) \leq 5\delta^{1/3} \leq \left(\mu - \frac{1}{\gamma}\right)^+ + 5\delta^{1/3}$.

Case 2: $|\gamma\mu - 1| > \delta^{2/3}$. Consider the vector $y = (y_0, y_1, y_2, y_3)$ defined as follows:

$$y = \begin{cases} \frac{(\gamma\mu)^2}{(1-\gamma\mu)(3-\gamma\mu)}(0, 1, -\frac{2}{\gamma\mu}, (\frac{1}{\gamma\mu})^2) & \text{for } \gamma\mu < 1 - \delta^{2/3}, \\ (0, a, \frac{1}{\gamma\mu^2}(2\mu(1-a) - \frac{3}{\gamma}), \frac{1}{\gamma^2\mu^3}(-\mu(1-a) + \frac{2}{\gamma})) & \text{for } \gamma\mu > 1 + \delta^{2/3}, \end{cases}$$

where $a = \frac{(\gamma\mu - \frac{3}{2})^2}{\gamma\mu(\gamma\mu - 1)}$. By Proposition 6 the vector y is feasible for $(\overline{D}^\sigma)$. Using weak duality (8) and $\overline{d}(\sigma) \leq \sum_{i=0}^3 \sigma_i y_i$, we get

$$\overline{p}(\sigma) \leq \begin{cases} \frac{(\gamma\mu)^2}{(1-\gamma\mu)(3-\gamma\mu)}\mu(\frac{\beta}{\gamma^2} - 1) & \text{for } \gamma\mu < 1 - \delta^{2/3}, \\ (\mu(2-a) - \frac{3}{\gamma})) + \frac{\beta}{\gamma^2}(-\mu(1-a) + \frac{2}{\gamma}) & \text{for } \gamma\mu > 1 + \delta^{2/3}. \end{cases}$$

From these expressions and Theorem 5 it follows that $\overline{p}(\sigma) \leq \left(\mu - \frac{1}{\gamma}\right)^+ + 5\delta^{1/3}$.
$\square$

Proof of Proposition 4. Consider the following probability distribution:

$$\pi(s) = \begin{cases} \frac{1}{2}(1 - \frac{2+\beta-3\gamma}{r_3(\gamma,\beta)}) & \text{if } s = \frac{(\beta-\gamma+r_3(\gamma,\beta))\mu}{2(\gamma-1)}, \\ \frac{1}{2}(1 + \frac{2+\beta-3\gamma}{r_3(\gamma,\beta)}) & \text{if } s = \frac{(\beta-\gamma-r_3(\gamma,\beta))\mu}{2(\gamma-1)}, \end{cases}$$

where $r_3(\cdot)$ is given by (4). If $(\overline{P}^\sigma)$ is feasible and $l(\gamma,\beta) \leq \mu \leq u(\gamma,\beta)$ (see (3)), then $\pi(s)$ is feasible for $(\overline{P}^\sigma)$, and its corresponding objective value is equal to $d(\mu,\gamma,\beta) \leq \overline{p}(\sigma)$.

Consider the following probability distribution:

$$\pi'(s) = \begin{cases} \frac{1}{2(2\gamma-\beta\mu)}\left(2-\gamma\mu-\frac{2-3\gamma\mu+\beta\mu^2}{r_1(\gamma,\beta)}\right) & \text{if } s = 1 + r_1(\mu,\gamma,\beta), \\ \frac{1}{2(2\gamma-\beta\mu)}\left(2-\gamma\mu+\frac{2-3\gamma\mu+\beta\mu^2}{r_1(\gamma,\beta)}\right) & \text{if } s = 1 - r_1(\mu,\gamma,\beta), \\ \frac{2(\gamma-1)+(\gamma-\beta)\mu}{2\gamma-\beta\mu} & \text{if } s = 0 \end{cases}$$

For $0 \le \mu \le \frac{2\gamma-2}{\beta-\gamma}$, $\pi'(s)$ is feasible for $(\overline{P}^{\sigma})$, and its corresponding objective value is equal to $c(\mu,\gamma,\beta) \le \overline{p}(\sigma)$. Finally for $\mu \ge u(\gamma,\beta) \ge 1/\gamma$ Remark 1, (13), and Theorem 3 yield $\overline{p}(\sigma) = \overline{d}(\sigma) \ge \overline{d}(\sigma'') = (\mu - 1/\gamma)^+ = \mu - 1/\gamma$. $\square$

To prove the following lemmas we need to introduce the dual problems corresponding to $(\underline{Q})$, and $(\overline{Q})$. The dual problem corresponding to $(\underline{Q})$ can be written as:

$$(\underline{Q}^*) \qquad \begin{array}{rll} \underline{q}(\mu,\gamma,\widehat{\mu})^* = & \sup & y_o + y_1\mu + y_2\gamma\mu^2 + \widehat{y}_1\widehat{\mu} - \widehat{\mu}^2 \\ & \text{s.t.} & y_o + y_1 s + y_2 s^2 + \widehat{y}_1 (s-1)^+ \le ((s-1)^+)^2 \\ & & \forall s \in \mathbb{R}_+ \end{array}$$

Clearly, weak duality holds between $(\underline{Q})$ and $(\underline{Q}^*)$; that is:

$$\underline{q}(\mu,\gamma,\widehat{\mu}) \ge \underline{q}(\mu,\gamma,\widehat{\mu})^* \tag{17}$$

The dual problem corresponding to $(\overline{Q})$ can be written as:

$$(\overline{Q}^*) \qquad \begin{array}{rll} \overline{q}(\mu,\gamma,\widehat{\mu})^* = & \inf & y_o + y_1\mu + y_2\gamma\mu^2 + \widehat{y}_1\widehat{\mu} - \widehat{\mu}^2 \\ & \text{s.t.} & y_o + y_1 s + y_2 s^2 + \widehat{y}_1 (s-1)^+ \ge ((s-1)^+)^2 \\ & & \forall\ s \in \mathbb{R}_+ \end{array}$$

Clearly, weak duality holds between $(\overline{Q})$ and $(\overline{Q}^*)$; that is:

$$\overline{q}(\mu,\gamma,\widehat{\mu}) \le \overline{q}(\mu,\gamma,\widehat{\mu})^* \tag{18}$$

Lemma 14 *If $(\underline{Q})$ is feasible and $\widehat{\mu} \le \mu - \frac{1}{\gamma}$ then $\underline{q}(\mu,\gamma,\widehat{\mu}) = \mu(\gamma\mu-1) - \widehat{\mu} - \widehat{\mu}^2$.*

Proof. Consider the solution given by $y_o = 0$, $y_1 = -1$, $y_2 = 1$, and $\widehat{y}_1 = -1$. This solution is feasible for $(\underline{Q}^*)$, and its corresponding objective value is equal to $\mu(\gamma\mu-1) - \widehat{\mu} - \widehat{\mu}^2 \le \underline{q}(\mu,\gamma,\widehat{\mu})^*$. Thus from weak duality (17) it follows that $\mu(\gamma\mu-1) - \widehat{\mu} - \widehat{\mu}^2 \le \underline{q}(\mu,\gamma,\widehat{\mu})$. For the reverse inequality consider the following probability distribution:

$$\pi(s) = \begin{cases} \frac{\widehat{\mu}^2}{\mu(\gamma\mu-1)-\widehat{\mu}} & \text{if } s = \frac{\mu(\gamma\mu-1)}{\widehat{\mu}}, \\ \frac{\mu^2(\gamma(\mu-\widehat{\mu})-1)}{\mu(\gamma\mu-1)-\widehat{\mu}} & \text{if } s = 1, \\ 1-\mu+\widehat{\mu} & \text{if } s = 0. \end{cases}$$

If $(\underline{Q})$ is feasible and $\widehat{\mu} \leq \mu - 1/\gamma$, then $\pi(s)$ is feasible for $(\underline{Q})$ and its corresponding objective value is equal to $\mu(\gamma\mu - 1) - \widehat{\mu} - \widehat{\mu}^2 \geq \underline{q}(\mu, \gamma, \widehat{\mu})$. $\square$

Lemma 15 *Let $r(\mu, \gamma, \widehat{\mu}) = \sqrt{(\gamma-1)((\gamma-1)\mu^2 + 4\widehat{\mu}(\mu-1) - 4\widehat{\mu}^2)}$. If $(\underline{Q})$ is feasible and $\widehat{\mu} \geq \mu - \frac{1}{\gamma}$, then $\underline{q}(\mu, \gamma, \widehat{\mu}) = \frac{1}{2}\left(2\widehat{\mu}(\mu-1) + \mu\left((\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu})\right)\right) - \widehat{\mu}^2$.*

Proof. Consider the solution given by

$$
\begin{aligned}
y_o &= -\tfrac{1}{2}\left((2\widehat{\mu} - \mu)\mu + \tfrac{\mu}{r(\mu,\gamma,\widehat{\mu})}\left(2(\gamma-2)\widehat{\mu}^2 + (\gamma-1)\mu^2 - 2\widehat{\mu}(1 + (\gamma-2)\mu)\right)\right)\\
y_1 &= \widehat{\mu} - \mu - \tfrac{1}{r(\mu,\gamma,\widehat{\mu})}\left(2\widehat{\mu}^2 - (\gamma-1)\mu^2 + \widehat{\mu}(2 + (\gamma-3)\mu)\right)\\
y_2 &= -\tfrac{1}{2}\left(\tfrac{1}{\mu r(\mu,\gamma,\widehat{\mu})}\left((\gamma-1)\mu^2 + 2\widehat{\mu}(\mu-1) - 2\widehat{\mu}^2\right) - 1\right)\\
\widehat{y}_1 &= (\mu - 1) - \tfrac{1}{r(\mu,\gamma,\widehat{\mu})}\left((\gamma-1)\mu(\mu - 1 - 2\widehat{\mu})\right)
\end{aligned}
\tag{19}
$$

Under the hypothesis conditions, this solution is feasible for $(\underline{Q}^*)$, and its corresponding objective value is equal to $\frac{1}{2}\left(2\widehat{\mu}(\mu-1) + \mu\left((\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu})\right)\right) - \widehat{\mu}^2 \leq \underline{q}(\mu,\gamma,\widehat{\mu})^*$. Thus from weak duality (17) it follows that $\frac{1}{2}\left(2\widehat{\mu}(\mu-1) + \mu\left((\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu})\right)\right) - \widehat{\mu}^2 \leq \underline{q}(\mu,\gamma,\widehat{\mu})$. For the reverse inequality let

$$
\begin{aligned}
s_1 &= \tfrac{\mu}{2\widehat{\mu}}\left(2\widehat{\mu} + (\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu})\right),\\
p_1 &= \tfrac{1}{2(1-2\mu+\gamma\mu^2)}\left(\mu\left(2\widehat{\mu} + (\gamma-1)\mu + r(\mu,\gamma,\widehat{\mu})\right) - 2\widehat{\mu}\right),
\end{aligned}
\tag{20}
$$

and consider the following probability distribution:

$$
\pi(s) = \begin{cases} p_1 & \text{if } s = s_1,\\[2ex] 1 - p_1 & \text{if } s = \frac{\mu(1-\gamma\mu) + \widehat{\mu}s_1}{1+\widehat{\mu}-\mu}. \end{cases}
\tag{21}
$$

If $(\underline{Q})$ is feasible and $\widehat{\mu} \geq \mu - 1/\gamma$, then $\pi(s)$ is feasible for $(\underline{Q})$, and its corresponding objective value is equal to $\frac{1}{2}(2\widehat{\mu}(\mu-1) + \mu((\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu}))) - \widehat{\mu}^2 \geq \underline{q}(\mu,\gamma,\widehat{\mu})$. $\square$

Proof of Theorem 8. Consider the solution given by (19) with $r(\mu,\gamma,\widehat{\mu})$ as in the statement of Theorem 8. This solution is feasible for $(\overline{Q}^*)$, and its corresponding objective value is equal to $\frac{1}{2}\left(2\widehat{\mu}(\mu-1) + \mu\left((\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu})\right)\right) - \widehat{\mu}^2 \geq \overline{q}(\mu,\gamma,\widehat{\mu})^*$. Thus from weak duality (18) it follows that $\frac{1}{2}\left(2\widehat{\mu}(\mu-1) + \mu\left((\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu})\right)\right) - \widehat{\mu}^2 \geq \overline{q}(\mu,\gamma,\widehat{\mu})$. For the reverse inequality consider the probability distribution given by (20), and (21) where $r(\mu,\gamma,\widehat{\mu})$ is again as in the statement of Theorem 8. If $(\overline{Q})$ is feasible, then this probability distribution is feasible for $(\overline{Q})$, and its corresponding objective value is equal to $\frac{1}{2}\left(2\widehat{\mu}(\mu-1) + \mu\left((\gamma-1)\mu - r(\mu,\gamma,\widehat{\mu})\right)\right) - \widehat{\mu}^2 \leq \overline{q}(\mu,\gamma,\widehat{\mu})$. $\square$

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