Distributed Consensus Algorithms in Sensor Networks: Quantized Data and Random Link Failures

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Distributed Consensus Algorithms in Sensor Networks: Quantized Data

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Abstract

The paper studies the problem of distributed average consensus in sensor networks with quantized data. We consider two versions of the algorithm: unbounded quantizers and bounded quantizers. To achieve consensus, dither (small noise) is added to the sensor states before quantization. We show by stochastic approximation techniques that consensus is asymptotically achieved to a finite random variable. We then study analytically the tradeoffs between how far away is this limiting random variable from the desired average, the consensus convergence rate, the quantizer parameters, and the network topology. We cast these tradeoff issues as an optimal quantizer design that we solve. A numerical study illustrates the design tradeoffs.

Keywords: Consensus, quantized, stochastic approximation, convergence
I. INTRODUCTION

This paper is concerned with consensus in networks, e.g., a sensor network, when the data exchanges among nodes in the network (sensors, agents) are quantized. Consensus is broadly understood as individuals in a community achieving a consistent view of the World by interchanging information regarding their current state with their neighbors. It has received considerable attention in recent years and arises in numerous applications including: load balancing, [1], alignment, flocking, and multi-agent collaboration, e.g., [2], [3], vehicle formation, [4], gossip algorithms, [5], tracking, data fusion, [6], and distributed inference, [7]. We refer the reader to the recent overviews on consensus, which include [8], [9].

Consensus is a distributed iterative algorithm where the sensor states evolve on the basis of local interactions. Early work on asynchronous distributed algorithms and their convergence include [10]. Reference [3] used spectral graph concepts like graph Laplacian and algebraic connectivity to prove convergence for consensus under several network operating conditions (e.g., delays and switching networks, i.e., time varying). Our own work has been concerned with designing topologies that optimize consensus with respect to the convergence rate, [11], [7]. Topology design is concerned with two issues: 1) the definition of the graph that specifies the neighbors of each sensor—i.e., with whom should each sensor exchange data; and 2) the weights used by the sensors when combining the information received from their neighbors to update their state. Reference [12] considers the problem of weight design, when the topology is specified, in the framework of semi-definite programming. References [13], [14] considered the impact of different topologies on the convergence rate of consensus, in particular, regular, random, and small-world graphs, [15].

Distributed consensus with quantized transmission has been studied recently in [16], [17], [18]. The algorithm in [16] is restricted to integer-valued initial sensor states, where at each iteration the sensors exchange integer-valued data. It is shown there that the sensor states are asymptotically close (in the appropriate sense defined there) to the desired average, but may not reach absolute consensus. In [17], the authors interpret the noise in the consensus algorithm studied in [19] as quantization noise and show by simulation with a small network that the variance of the quantization noise is reduced as the algorithm iterates and the sensors converge to a consensus. Reference [18] studies probabilistic quantized consensus. Each sensor updates its state at each iteration by first quantizing its current state and linearly combining it with the quantized versions of the states of the neighbors. They show that the sensor states reach consensus to a quantized level; only in expectation do they converge to the desired average. If the quantization step size is large this approach will lead to large residual errors. Also, analytical bounds on the mean-squared error are not presented.
In this paper, we consider two variations on consensus with quantized data: 1) Quantized Consensus (QC) where the alphabet of the quantizers at the sensors is countable to account for possibly unbounded initial sensor states; and 2) Quantized Consensus with Finite (QCF) number of bits, i.e., the alphabet is finite. To avoid divergence due to error accumulation, we add a small amount of noise, dither, to the data before quantization and use a sequence of weights that satisfy a persistence condition—their sum diverges, while their square sum is finite. The dither makes QC and QCF randomized distributed iterative consensus algorithms, while the persistent gains enable their convergence, as we show by stochastic approximation techniques. We prove that these quantized consensus algorithms lead to a.s. consensus of the sensor states to a finite random variable. We address in two directions the question of how close can this random variable be made to the desired average of the initial data by tuning certain parameters of the algorithm: we show that, for the QC algorithm, the mean square error can be made arbitrarily small, though penalizing the convergence rate; for the QCF algorithm, we show that asymptotically the quantized states get within a ball of radius $\epsilon$, $\epsilon$-consensus, of the desired average with high probability. We establish analytically several tradeoffs between consensus closeness, consensus convergence rate, quantizer parameters, and network topology, and then illustrate these tradeoffs numerically.

We comment briefly on the organization of the main sections of the paper. Section II summarizes relevant background, including spectral graph theory and average consensus, and presents the dithered quantized consensus problem with the dither satisfying the Schuchman conditions. Sections III and IV consider the convergence of the QC and QCF algorithms. They show a.s. convergence to a random variable, whose m.s.e. is fully characterized. Section IV also studies tradeoffs among different quantizer parameters, e.g., number of bits and quantization step-size, and the network topology to achieve optimal performance under a constraint on the range of the quantizer. Finally, Section V concludes the paper and proposes avenues for future research.

II. CONSENSUS WITH QUANTIZED DATA: PROBLEM STATEMENT

This section presents very briefly preliminaries needed for the analysis of the consensus algorithm with quantized data. The set-up is standard, see the introductory sections of relevant recent papers on consensus, in particular, the companion manuscript [20].

A. Preliminaries: Average Consensus

We consider consensus in the context of spectral graph theory where the sensor network is represented by an undirected, simple, weighted, connected graph $G = (V, E)$. The vertex and edge sets $V$ and $E$,
with cardinalities $|V| = N$ and $|E| = M$, collect the sensors and communication channels or links among sensors in the network. The network topology, i.e., with which sensors does each sensor communicate with, is described by the $N \times N$ discrete Laplacian $L = L^T = D - A \geq 0$. The matrix $A$ is the adjacency matrix of the graph, a $(0,1)$ matrix where $A_{nk} = 1$ signifies that there is a link between sensors $n$ and $k$. The diagonal entries of $A$ are zero. The diagonal matrix $D$ is the degree matrix, whose diagonal $D_{nn} = d_n$ where $d_n$ is the degree of sensor $n$, i.e., the number of links of sensor $n$. The neighbors of a sensor or node $n$, collected in the neighborhood set $\Omega_n$, are those sensors $k$ for which entries $A_{nk} \neq 0$. The Laplacian is positive semidefinite; for a connected network, the algebraic connectivity or the Fiedler value of the network is positive, i.e., the second eigenvalue of the Laplacian $\lambda_2(L) > 0$, where the eigenvalues of $L$ are ordered in increasing order. For detailed treatment of graphs and their spectral theory see, for example, [21], [22], [23].

**Distributed Average Consensus.** Let the sensors in a sensor network measure the data $x_n(0)$, $n = 1, \cdots, N$. We collect these data in the vector $x(0) = [x_1(0) \cdots x_N(0)]^T \in \mathbb{R}^{N \times 1}$. The goal of distributed average consensus is to compute the average $r$ of these initial data

$$
  r &= x_{\text{avg}}(0) = \frac{1}{N} \sum_{n=1}^{N} x_n(0) \\
  &= \frac{1}{N} x(0)^T 1
$$

by local data exchanges among neighboring sensors. In (2), the column vector $1$ has all entries equal to 1.

Consensus is an iterative algorithm where at iteration $i$ each sensor updates its current state $x_n(i)$ by a weighted average of its current state and the states of its neighbors. Standard consensus assumes a fixed connected network topology, i.e., the links stay online permanently, the communication is noiseless, and the data exchanges are analog. Under mild conditions, the states of all sensors reach consensus, converging to the desired average $r$, see [3], [12],

$$
  \lim_{i \to \infty} x(i) = r 1
$$

where $x(i) = [x_1(i) \cdots x_N(i)]^T$ is the state vector that stacks the state of the $N$ sensors at iteration $i$.

We assume here the framework of standard consensus, but consider the data exchanges to be quantized. In [20], we considered consensus when the topologies are random (links fail or become alive at random times) and the links are noisy simultaneously. Subsection III-D will comment briefly on how to extend the analysis to the more general case of quantized consensus with random links and noisy communication.
B. Dithered Quantization: Schuchman Conditions

We write the sensor updating equations for standard consensus with quantized data as

\[ x_n(i + 1) = [1 - \alpha(i)d_n] x_n(i) + \alpha(i) \sum_{l \in \Omega_n} f_{nl,i}[x_l(i)] , \quad 1 \leq n \leq N \] (4)

where: \( \alpha(i) \) is the weight at iteration \( i \); and \( \{f_{nl,i}\}_{1 \leq n,l \leq N, i \geq 0} \) is a sequence of functions (possibly random) modeling the quantization effects. Note that in (4), the weights \( \alpha(i) \) are the same across all links—the equal weights consensus, see [12]—but the weights may change with time. Also, the degree \( d_n \) and the neighborhood \( \Omega_n \) of each sensor \( n, n = 1, \cdots, N \) are not dependent on \( i \) because we are considering a fixed (non random) topology.

Quantizer. Each inter-sensor communication channel uses a uniform quantizer with quantization step \( \Delta \). We model the communication channel by introducing the quantizing function, \( q(\cdot) : \mathbb{R} \rightarrow Q, \)

\[ q(y) = k\Delta, \quad (k - \frac{1}{2})\Delta \leq y < (k + \frac{1}{2})\Delta \] (5)

where \( y \in \mathbb{R} \) is the channel input. Writing

\[ q(y) = y + e(y) \] (6)

where \( e(y) \) is the quantization error; we have

\[ -\frac{\Delta}{2} \leq e(y) < \frac{\Delta}{2}, \quad \forall y \] (7)

Conditioned on the input, the quantization error \( e(y) \) is deterministic.

We consider two cases. In the first, quantized consensus (QC), the quantization alphabet

\[ Q = \{k\Delta \mid k \in \mathbb{Z}\} \] (8)

is countably infinite. In the second, quantized consensus with finite (QCF) alphabet, the alphabet is finite. The QCF quantizer model is in Section IV. The QC quantizer alphabet (8) may model the problem where there is no prior knowledge about the range of the initial data measured by the sensors, while QCF assumes that these data are bounded by a fixed constant. The QC results are used to study the convergence of QCF.

We discuss briefly why a naive approach to consensus will fail. If we use directly the quantized state
information, the functions \( f_{nl,i}(\cdot) \) in eqn. (4) are

\[
    f_{nl,i}(x_l(i)) = q(x_l(i)) = x_l(i) + e(x_l(i))
\]

Equations (4) take then the form

\[
    x_n(i + 1) = \left[ (1 - \alpha(i)d_n)x_n(i) + \alpha(i) \sum_{l \in \Omega_n} x_l(i) \right] + \alpha(i) \sum_{l \in \Omega_n} e(x_l(i))
\]

The non-stochastic errors (the most right terms in (11)) lead to error accumulation. If the network topology remains fixed (deterministic topology,) the update in eqn. (11) represents a sequence of iterations that, as observed above, conditioned on the initial state, which then determines the input, are deterministic. If we choose the weights \( \alpha(i) \)'s to decrease to zero very quickly, then (11) may terminate before reaching the consensus set. On the other hand, if the \( \alpha(i) \)'s decay slowly, the quantization errors may accumulate, thus making the states unbounded.

In either case, the naive approach to consensus with quantized data fails to lead to a reasonable solution. This failure is due to the fact that the error terms are not stochastic. To overcome these problems, we introduce in a controlled way noise (dither) to randomize the sensor states prior to quantizing the perturbed stochastic state. We will show that, under appropriate conditions, the resulting quantization errors possess nice statistical properties, leading to the quantized states reaching consensus (in an appropriate sense to be defined below.) Dither places consensus with quantized data in the framework of distributed consensus with noisy communication links; we will apply stochastic approximation arguments to study the limiting behavior; we also used stochastic approximation to show convergence of two versions of consensus with noise that we introduced in [20], the \( A - ND \) and \( A - NC \) algorithms.

**Schuchman conditions.** We assume that the dither we add to randomize the quantization effects satisfies a special condition that we consider now. Let \( \{y(i)\}_{i \geq 0} \) and \( \{\nu(i)\}_{i \geq 0} \) be arbitrary sequences of random variables, and \( q(\cdot) \) be the quantization function (5). When dither is added before quantization, the quantization error sequence, \( \{\varepsilon(i)\}_{i \geq 0} \), is

\[
    \varepsilon(i) = q(y(i) + \nu(i)) - (y(i) + \nu(i))
\]

This corresponds to subtractively dithered systems, see [24], [25].

It can be shown that, if the dither sequence, \( \{\nu(i)\}_{i \geq 0} \), satisfies the Schuchman conditions, [26], then the quantization error sequence, \( \{\varepsilon(i)\}_{i \geq 0} \), in (12) is i.i.d. uniformly distributed on \([-\Delta/2, \Delta/2) \) and
independent of the input sequence \( \{ y(i) \}_{i \geq 0} \) (see [27], [28], [24]). A sufficient condition for \( \{ \nu(i) \} \) to satisfy the Schuchman conditions is for it to be a sequence of i.i.d. random variables uniformly distributed on \([-\Delta/2, \Delta/2]\) and independent of the input sequence \( \{ y(i) \}_{i \geq 0} \). In the sequel, the dither \( \{ \nu(i) \}_{i \geq 0} \) satisfies the Schuchman conditions. Hence, the quantization error sequence, \( \{ \epsilon(i) \} \), is i.i.d. uniformly distributed on \([-\Delta/2, \Delta/2]\) and independent of the input sequence \( \{ y(i) \}_{i \geq 0} \).

C. Dithered Quantized Consensus: Problem Statement

We now return to the problem formulation of consensus with quantized data with dither added. Introducing the sequence, \( \{ \nu_{nl}(i) \}_{i \geq 0, 1 \leq n, l \leq N} \), of i.i.d. random variables, uniformly distributed on \([-\Delta/2, \Delta/2]\), and independent of the input sequence \( \{ y(i) \}_{i \geq 0} \), the state update equation for quantized consensus is:

\[
x_n(i + 1) = (1 - \alpha(i) d_n) x_n(i) + \alpha(i) \sum_{l \in \Omega_n} q [x_l(i) + \nu_{nl}(i)], \quad 1 \leq n \leq N
\] (13)

This equation shows that, before transmitting its state \( x_l(i) \) to the \( n \)-th sensor, the sensor \( l \) adds the dither \( \nu_{nl}(i) \), then the channel between the sensors \( n \) and \( l \) quantizes this corrupted state, and, finally, sensor \( n \) receives this quantized output. Using eqn. (12), the state update is

\[
x_n(i + 1) = (1 - \alpha(i) d_n) x_n(i) + \alpha(i) \sum_{l \in \Omega_n} [x_l(i) + \nu_{nl}(i) + \epsilon_{nl}(i)]
\] (14)

The random variables \( \nu_{nl}(i) \) are independent of the state \( x(j) \), i.e., the states of all sensors at iteration \( j \), for \( j \leq i \). Hence, the collection \( \{ \epsilon_{nl}(i) \} \) consists of i.i.d. random variables uniformly distributed on \([-\Delta/2, \Delta/2]\), and the random variable \( \epsilon_{nl}(i) \) is also independent of the state \( x(j) \), \( j \leq i \).

We rewrite (14) in vector form. Define the random vectors, \( \Upsilon(i) \) and \( \Psi(i) \) \( \in \mathbb{R}^{N \times 1} \) with components

\[
\Upsilon_n(i) = -\sum_{l \in \Omega_n} \nu_{nl}(i)
\] (15)

\[
\Psi_n(i) = -\sum_{l \in \Omega_n} \epsilon_{nl}(i)
\] (16)

The the \( N \) state update equations in (14) become in vector form

\[
x(i + 1) = x(i) - \alpha(i) [Lx(i) + \Upsilon(i) + \Psi(i)]
\] (17)

where \( \Upsilon(i) \) and \( \Psi(i) \) are zero mean vectors, independent of the state \( x(i) \), and have independent components. Also, If \( M \) is the number of edges in the network, eqns. (15) and (16) lead to

\[
\mathbb{E} \left[ \| \Upsilon(i) \|^2 \right] = \mathbb{E} \left[ \| \Psi(i) \|^2 \right] = \frac{M\Delta^2}{6}, \quad i \geq 0
\] (18)
**Persistence condition.** To obtain convergence, we assume that the gains $\alpha(i)$ satisfy the following.

\[
\alpha(i) > 0, \quad \sum_{i \geq 0} \alpha(i) = \infty, \quad \sum_{i \geq 0} \alpha^2(i) < \infty \quad (19)
\]

Condition (19) assures that the gains decay to zero, but not too fast. It is standard in stochastic adaptive signal processing and control; it is also used in consensus with noisy communications in [29], [20].

**Markov property.** Denote the natural filtration of the process $X = \{x(i)\}_{i \geq 0}$ by $\mathcal{F}^X_i$. Because the dither random variables $\nu_{nl}(i)$, $1 \leq n, l \leq N$, are independent of $\mathcal{F}^X_i$ at any time $i \geq 0$, and, correspondingly, the noises $\Upsilon(i)$ and $\Psi(i)$ are independent of $x(i)$, the process $X$ is Markov.

**III. Consensus With Quantized Data: Unbounded Quantized States**

We consider that the dynamic range of the initial sensor data, whose average we wish to compute, is not known. To avoid quantizer saturation, the quantizer output takes values in the countable alphabet (8), and so the channel quantizer has unrestricted dynamic range. This is the quantized consensus (QC) algorithm. Section IV studies the quantized consensus finite-bit (QCF) algorithm, a modification of QC, where the initial sensor data is bounded (the dynamic range is known a priori), enabling the use of channel quantizers that take only a finite number of output values (finite-bit quantizers). As mentioned in Section I, consensus with quantized data has been considered by others, including [16], [17], [18].

We comment briefly on the organization of the remaining of this section. Subsection III-A proves the a.s. convergence of the QC algorithm. We characterize the performance of the QC algorithm and derive expressions for the mean-squared error in Subsection III-B. The tradeoff between m.s.e. and convergence rate is studied in Subsection III-C. Finally, we present generalizations to the approach in Subsection III-D.

**A. QC Algorithm: Convergence**

We start with the definition of the consensus subspace $C$ given as

\[
C = \{x \in \mathbb{R}^{N \times 1} \mid x = a1, \ a \in \mathbb{R}\} \quad (20)
\]

We show that (17), under the model in Subsection II-C, converges a.s. to a finite point in $C$.

Define the component-wise average as

\[
x_{avg}(i) = \frac{1}{N}1^T x(i) \quad (21)
\]

We prove the a.s. convergence of the QC algorithm in two stages. Theorem 2 proves that the state vector sequence $\{x(i)\}_{i \geq 0}$ converges a.s. to the consensus subspace $C$. Theorem 4 then completes the
proof by showing that the sequence of component-wise averages, \( \{x_{\text{avg}}(i)\}_{i \geq 0} \) converges a.s. to a finite random variable \( \theta \). The proofs of these Theorems need a basic result on convergence of Markov processes.

**Stochastic approximation: Convergence of Markov processes.** We state a slightly modified form, suitable to our needs, of a result from [30]. We start by introducing notation, following [30], see also [20].

Let \( \mathbf{X} = \{\mathbf{x}(i)\}_{i \geq 0} \) be Markov in \( \mathbb{R}^{N \times 1} \). The generating operator \( \mathcal{L} \) is

\[
\mathcal{L}V(i, \mathbf{x}) = \mathbb{E}\left[V(i+1, \mathbf{x}(i+1)) \mid \mathbf{x}(i) = \mathbf{x}\right] - V(i, \mathbf{x}) \text{ a.s.}
\]  

(22)

for functions \( V(i, \mathbf{x}) \), \( i \geq 0 \), \( \mathbf{x} \in \mathbb{R}^{N \times 1} \), provided the conditional expectation exists. We say that \( V(i, \mathbf{x}) \in D \mathcal{L} \) in a domain \( A \), if \( \mathcal{L}V(i, \mathbf{x}) \) is finite for all \( (i, \mathbf{x}) \in A \).

Let the Euclidean metric be \( \rho(\cdot) \). Define the \( \epsilon \)-neighborhood of \( B \subset \mathbb{R}^{N \times 1} \) and its complementary set

\[
U_\epsilon(B) = \left\{ \mathbf{x} \mid \inf_{y \in B} \rho(x, y) < \epsilon \right\}
\]

(23)

\[
V_\epsilon(B) = \mathbb{R}^{N \times 1} \backslash U_\epsilon(B)
\]

(24)

**Theorem 1 (Convergence of Markov Processes)** Let: \( \mathbf{X} \) be a Markov process with generating operator \( \mathcal{L} \); \( V(i, \mathbf{x}) \in D \mathcal{L} \) a non-negative function in the domain \( i \geq 0 \), \( \mathbf{x} \in \mathbb{R}^{N \times 1} \), and \( B \subset \mathbb{R}^{N \times 1} \). Assume:

1) **Potential function:**

\[
\inf_{i \geq 0, \mathbf{x} \in V_\epsilon(B)} V(i, \mathbf{x}) > 0, \quad \forall \epsilon > 0
\]

(25)

\[
V(i, \mathbf{x}) \equiv 0, \quad \mathbf{x} \in B
\]

(26)

\[
\lim_{\mathbf{x} \to B} \sup_{i \geq 0} V(i, \mathbf{x}) = 0
\]

(27)

2) **Generating operator:**

\[
\mathcal{L}V(i, \mathbf{x}) \leq g(i)(1 + V(i, \mathbf{x})) - \alpha(i)\varphi(i, \mathbf{x})
\]

(28)

where \( \varphi(i, \mathbf{x}) \), \( i \geq 0 \), \( \mathbf{x} \in \mathbb{R}^{N \times 1} \) is a non-negative function such that

\[
\inf_{i, \mathbf{x} \in V_\epsilon(B)} \varphi(i, \mathbf{x}) > 0, \quad \forall \epsilon > 0
\]

(29)

\[
\alpha(i) > 0, \quad \sum_{i \geq 0} \alpha(i) = \infty
\]

(30)

\[
g(i) > 0, \quad \sum_{i \geq 0} g(i) < \infty
\]

(31)

Then, the Markov process \( \mathbf{X} = \{\mathbf{x}(i)\}_{i \geq 0} \) with arbitrary initial distribution converges a.s. to \( B \) as \( i \to \infty \)

\[
\mathbb{P}\left(\lim_{i \to \infty} \rho(\mathbf{x}(i), B) = 0\right) = 1
\]

(32)

**Proof:** For proof, see [20].
**Theorem 2 (a.s. convergence to consensus subspace)** Consider the quantized distributed averaging algorithm given in eqns. (17). Then, for arbitrary initial condition, \( x(0) \), we have

\[
P \left[ \lim_{i \to \infty} \rho(x(i), C) = 0 \right] = 1 \tag{33}
\]

**Proof:** The key idea of the proof is to show that the quantized iterations satisfy the assumptions of Theorem 1. Define the potential function, \( V(i, x) \), for the Markov process \( X \) as

\[
V(i, x) = x^T L x \tag{34}
\]

Then, using the properties of the graph Laplacian \( L \) and the continuity of \( V(i, x) \),

\[
V(i, x) \equiv 0, \ x \in C \text{ and } \lim_{x \to C} \sup_{i \geq 0} V(i, x) = 0 \tag{35}
\]

We now recall a standard result, see also [20].

**Lemma 3** Given \( x \in \mathbb{R}^{N \times 1} \) and the orthogonal decomposition \( x = x_C + x_{C^\perp} \), then \( \rho(x, C) = \|x_{C^\perp}\| \).

Using Lemma 3, it successively follows

\[
\begin{align*}
x & \in V_i(C) : \quad \|x_{C^\perp}\| \geq \epsilon \tag{36} \\
x & \in V_i(C) : \quad V(i, x) = x^T L x \geq \lambda_2(L) \|x_{C^\perp}\|^2 \\
 & \quad \geq \lambda_2(L) \epsilon^2
\end{align*}
\]

Then, we get

\[
\inf_{i \geq 0, x \in V_i(C)} V(i, x) \geq \lambda_2(L) \epsilon^2 > 0 \tag{38}
\]

since \( \lambda_2(L) > 0 \) (connected topology.) This shows, together with (35), that \( V(i, x) \) satisfies (25)–(27).
Now consider $\mathcal{L}V(i, x)$. We have

\[
\mathcal{L}V(i, x) = \mathbb{E}\left[ (x(i) - \alpha(i)Lx(i) - \alpha(i)\Psi(i))T L (x(i) - \alpha(i)Lx(i) - \alpha(i)\Psi(i)) \right] - \alpha(i)\Psi(i) | x(i) = x - x^T L x
\]

\[
= -2\alpha(i)x^T L^2 x + 2\alpha^2(i)x^T L^3 x + 2\alpha^2(i)\mathbb{E} [\Psi^T(i) L \Psi(i) | x(i) = x] + \alpha^2(i)\mathbb{E} [\Psi^T(i) L \Psi(i)]
\]

\[
\leq -2\alpha(i)x^T L^2 x + 2\alpha^2(i)x^T L^3 x + 2\alpha^2(i) (\mathbb{E} [||\Psi(i)||^2])^{1/2} (\mathbb{E} [||L \Psi(i)||^2])^{1/2} + \alpha^2(i)\mathbb{E} [\Psi^T(i) L \Psi(i)]
\]

\[
\leq -2\alpha(i)x^T L^2 x + 2\alpha^2(i)\lambda^3_N(L) ||x_{C+}||^2 + 2\alpha^2(i)\lambda_N(L) (\mathbb{E} [||\Psi(i)||^2])^{1/2} (\mathbb{E} [||\Psi(i)||^2])^{1/2} + \alpha^2(i)\lambda_N(L) \mathbb{E} [||\Psi(i)||^2]
\]

(39)

We now use the fact that $x^T L x \geq \lambda_2(L) ||x_{C+}||^2$ and eqn. (18) to get

\[
\mathcal{L}V(i, x) \leq -2\alpha(i)x^T L^2 x + 2\alpha^2(i)\lambda^3_N(L) ||x_{C+}||^2 + \frac{\alpha^2(i)M\Delta^2 \lambda_N(L)}{3} + \frac{\alpha^2(i)M\Delta^2 \lambda_N(L)}{6}
\]

\[
\leq -2\alpha(i)x^T L^2 x + \frac{\alpha^2(i)\lambda^3_N(L)}{\lambda_2(L)} x^T L x + \frac{2\alpha^2(i)M\Delta^2 \lambda_N(L)}{3}
\]

\[
\leq -\alpha(i)\varphi(i, x) + g(i) [1 + V(i, x)]
\]

(40)

where

\[
\varphi(i, x) = 2x^T L^2 x, \quad g(i) = \alpha^2(i) \max \left( \frac{\lambda^3_N(L)}{\lambda_2(L)}, \frac{2M\Delta^2 \lambda_N(L)}{3} \right)
\]

(41)

Clearly, $\mathcal{L}V(i, x)$ and $\varphi(i, x), g(i)$ satisfy the remaining assumptions (28)–(31) of Theorem 1; hence,

\[
\mathbb{P} \left[ \lim_{i \to \infty} \rho(x(i), C) = 0 \right] = 1
\]

(42)

The convergence proof for QC will now be completed in the next Theorem.

**Theorem 4 (Consensus to finite random variable)** Consider (17), with arbitrary initial condition $x(0) \in \mathbb{R}^{N \times 1}$ and the state sequence $\{x(i)\}_{i \geq 0}$. Then, there exists a finite random variable $\theta$ such that

\[
\mathbb{P} \left[ \lim_{i \to \infty} x(i) = \theta 1 \right] = 1
\]

(43)
Proof: Define the filtration \( \{\mathcal{F}_i\}_{i \geq 0} \) as
\[
\mathcal{F}_i = \sigma \left\{ x(0), \{\Upsilon(j)\}_{0 \leq j < i}, \{\Psi(j)\}_{0 \leq j < i} \right\}
\] (44)

We will now show that the sequence \( \{x_{\text{avg}}(i)\}_{i \geq 0} \) is an \( L_2 \)-bounded martingale w.r.t. \( \{\mathcal{F}_i\}_{i \geq 0} \). In fact,
\[
x_{\text{avg}}(i + 1) = x_{\text{avg}}(i) - \alpha(i) \Upsilon(i) - \alpha(i) \Psi(i)
\] (45)

where \( \Upsilon(i) \) and \( \Psi(i) \) are the component-wise averages given by
\[
\Upsilon(i) = \frac{1}{N} 1^T \Upsilon(i), \quad \Psi(i) = \frac{1}{N} 1^T \Psi(i)
\] (46)

Then,
\[
\mathbb{E} \left[ x_{\text{avg}}(i + 1) \mid \mathcal{F}_i \right] = x_{\text{avg}}(i) - \alpha(i) \mathbb{E} \left[ \Upsilon(i) \mid \mathcal{F}_i \right] - \alpha(i) \mathbb{E} \left[ \Psi(i) \mid \mathcal{F}_i \right]
\] (47)
\[
= x_{\text{avg}}(i) - \alpha(i) \mathbb{E} \left[ \Upsilon(i) \right] - \alpha(i) \mathbb{E} \left[ \Psi(i) \right]
\]
\[
= x_{\text{avg}}(i)
\]

where the last step follows from the fact that \( \Upsilon(i) \) is independent of \( \mathcal{F}_i \), and
\[
\mathbb{E} \left[ \Psi(i) \mid \mathcal{F}_i \right] = \mathbb{E} \left[ \Psi(i) \mid x(i) \right]
\] (48)
\[
= 0
\]
because \( \Psi(i) \) is independent of \( x(i) \) as argued in Section II-B.

Thus, the sequence \( \{x_{\text{avg}}(i)\}_{i \geq 0} \) is a martingale. For proving \( L_2 \) boundedness, note
\[
\mathbb{E} \left[ x_{\text{avg}}^2(i + 1) \right] = \mathbb{E} \left[ x_{\text{avg}}(i) - \alpha(i) \Upsilon(i) - \alpha(i) \Psi(i) \right]^2
\] (49)
\[
= \mathbb{E} \left[ x_{\text{avg}}^2(i) \right] + \alpha^2(i) \mathbb{E} \left[ \Upsilon^2(i) \right] + \alpha^2(i) \mathbb{E} \left[ \Psi^2(i) \right] + 2 \alpha^2(i) \mathbb{E} \left[ \Upsilon(i) \Psi(i) \right]
\]
\[
\leq \mathbb{E} \left[ x_{\text{avg}}^2(i) \right] + \alpha^2(i) \mathbb{E} \left[ \Upsilon^2(i) \right] + \alpha^2(i) \mathbb{E} \left[ \Psi^2(i) \right] + 2 \alpha^2(i) \left( \mathbb{E} \left[ \Upsilon^2(i) \right] \right)^{1/2} \left( \mathbb{E} \left[ \Psi^2(i) \right] \right)^{1/2}
\]

Again, it can be shown by using the independence properties and (18) that
\[
\mathbb{E} \left[ \Upsilon^2(i) \right] = \mathbb{E} \left[ \Psi^2(i) \right] = \frac{M \Delta^2}{6N^2}
\] (50)
where $M$ is the number of edges in the network. It then follows from eqn. (49) that

$$
\mathbb{E}[x_{\text{avg}}^2(i+1)] \leq \mathbb{E}[x_{\text{avg}}^2(i)] + \frac{2\alpha^2(i)M\Delta^2}{3N^2}
$$

Finally, the recursion leads to

$$
\mathbb{E}[x_{\text{avg}}^2(i)] \leq x_{\text{avg}}^2(0) + \frac{2M\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j)
$$

Thus $\{x_{\text{avg}}(i)\}_{i \geq 0}$ is an $L_2$-bounded martingale; hence, it converges a.s. and in $L_2$ to a finite random variable $\theta$ ([31]). In other words,

$$
P\left[ \lim_{i \to \infty} x_{\text{avg}}(i) = \theta \right] = 1
$$

Again, Theorem 2 implies that as $i \to \infty$ we have $x(i) \to x_{\text{avg}}(i)$ a.s. This and (53) prove the Theorem.

\section*{B. QC Algorithm: Mean-Squared Error}

Theorem 4 shows that the sensors reach consensus asymptotically and in fact converge a.s. to a finite random variable $\theta$. Viewing $\theta$ as an estimate of the initial average $r$ (see eqn. (1)), we characterize its desirable statistical properties in the following Lemma.

\textbf{Lemma 5}  Let $\theta$ be as given in Theorem 4 and $r$, the initial average, as given in eqn. (1). Define

$$
\zeta = \mathbb{E}[\theta - r]^2
$$

to be the m.s.e. Then, we have:

1) Unbiasedness: \hspace{1cm} $\mathbb{E}[\theta] = r$

2) M.S.E. Bound: \hspace{1cm} $\zeta \leq \frac{2M\Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j)$

\textit{Proof:} The proof is omitted, see [20].

Lemma 5 shows that, for given $\eta$ and $\Delta$, $\zeta$ can be made arbitrarily small by properly scaling the weight sequence, $\{\alpha(i)\}_{i \geq 0}$. We formalize this by introducing some notation here, which will be used in the sequel. Given an arbitrary weight sequence, $\{\alpha(i)\}_{i \geq 0}$, which satisfies the persistence condition (19), define the scaled weight sequence, $\{\alpha_s(i)\}_{i \geq 0}$, as

$$
\alpha_s(i) = s\alpha(i), \forall i \geq 0
$$
where, $s > 0$, is a constant scaling factor. Clearly, such a scaled weight sequence satisfies the persistence condition (19), and the m.s.e. $\zeta_s$ obtained by using this scaled weight sequence is given by

$$\zeta_s \leq \frac{2M\Delta^2s^2}{3N^2} \sum_{j \geq 0} \alpha^2(j)$$

showing that, by proper scaling of the weight sequence, the m.s.e. can be made arbitrarily small.

However, reducing the m.s.e. by scaling the weights in this way will reduce the convergence rate of the algorithm and, this tradeoff is considered in the next subsection.

C. QC Algorithm: Convergence Rate

The QC algorithm falls under the framework of stochastic approximation algorithms and, hence, a detailed convergence rate analysis can be done through the ODE method (see, for example, [32]). We do not pursue it in this paper; rather, we present a simpler convergence rate analysis, involving the mean state vector sequence only. From the asymptotic unbiasedness of $\theta$,

$$\lim_{i \to \infty} E[x(i)] = r1$$

Our objective is to determine the rate at which the sequence $\{E[x(i)]\}_{i \geq 0}$ converges to $r1$.

**Lemma 6** Without loss of generality, make the assumption

$$\alpha(i) \leq \frac{2}{\lambda_2(L) + \lambda_N(L)}, \forall i$$

(We note that this holds eventually, as the $\alpha(i)$ decrease to zero.) Then,

$$\|E[x(i)] - r1\| \leq \left(e^{-\lambda(L)(\sum_{0 \leq j \leq -1} \alpha(j))}\right) \|E[x(0)] - r1\|$$

**Proof:** We note that the mean state propagates as

$$E[x(i+1)] = (I - \alpha(i)L)E[x(i)], \forall i$$

The proof then follows from [20] and is omitted.

It follows from Lemma 6 that the rate at which the sequence $\{E[x(i)]\}_{i \geq 0}$ converges to $r1$ is closely related to the rate at which the weight sequence, $\alpha(i)$, sums to infinity. On the other hand, to achieve a small bound $\zeta$ on the m.s.e, see lemma 54 in Subsection III-B, we need to make the weights small, which reduces the convergence rate of the algorithm.
Note that this tradeoff is established between the convergence rate of the mean state vectors and the m.s.e. of the limiting consensus variable $\theta$. But, in general, even for more appropriate measures of the convergence rate, we expect that, intuitively, the same tradeoff will be exhibited, in the sense that the rate of convergence will be closely related to the rate at which the weight sequence, $\alpha(i)$, sums to infinity.

**D. QC Algorithm: Generalizations**

The QC algorithm can be extended to handle more complex situations of imperfect communication. For instance, we may have random link failures in addition to quantization effects. In that case, the potential function $V(\cdot)$ needs to be redefined, but, the remaining of the analysis will still hold.

**IV. Consensus with Quantized Data: Bounded Initial Sensor State**

In this section, we consider consensus with quantized data when the initial sensor states are bounded, and this bound is known *a priori*. In this case, we show that finite bit quantizers (quantizers, whose outputs take only a finite number of values) will suffice. The algorithm QCF that we now consider is a modification of the QC algorithm of Section III. The good performance of the QCF algorithm relies mainly on the fact that, if the initial sensor states are bounded, then the state sequence, $\{x(i)\}_{i \geq 0}$ generated by the QC algorithm remains uniformly bounded with high probability. In that case, channel quantizers with only a finite dynamic range will perform very well with high probability.

We next briefly state the QCF problem in Subsection IV-A. Then, Subsection IV-B shows that with high probability the sample paths generated by the QC algorithm are uniformly bounded, when the initial sensor states are bounded. Subsection IV-C proves that QCF achieves asymptotic consensus. Finally, Subsections IV-D and IV-E analyze its statistical properties, performance, and tradeoffs.

**A. QCF Algorithm: Statement**

The QCF algorithm modifies the QC algorithm by restricting the alphabet to be finite. It assumes that the initial sensor state $x(0)$, whose average we wish to compute, is known to be bounded. Of course, even if the initial state is bounded, the states of QC can become unbounded. The good performance of QCF is a consequence of the fact that, as our analysis will show, the states $\{x(i)\}_{i \geq 0}$ generated by the QC algorithm when started with a bounded initial state $x(0)$ remain uniformly bounded with high probability.

The following are the assumptions underlying QCF.
1) Bounded initial state. The QCF initial state $\tilde{x}(0)$ is bounded to the set $B$ known a priori

$$B = \{ y \in \mathbb{R}^{N \times 1} \mid |y_n| \leq b < +\infty \}$$

for some $b > 0$.

2) Uniform quantizers and finite alphabet. Each inter-sensor communication channel in the network uses a uniform $\lceil \log_2(2p + 1) \rceil$ bit quantizer with step-size $\Delta$, where $p > 0$ is an integer. In other words, the quantizer output takes only $2p + 1$ values, and the quantization alphabet is given by

$$\tilde{Q} = \{ l\Delta \mid l = 0, \pm 1, \ldots, \pm p \}$$

Clearly, such a quantizer will not saturate if the input falls in the range $[(-p - 1/2)\Delta, (p + 1/2)\Delta)$; if the input goes out of that range, the quantizer saturates.

3) Uniform i.i.d. dither. Like with QC, the $\{\nu_{nl}(i)\}_{i \geq 0, 1 \leq n, l \leq N}$ are a sequence of i.i.d. random variables uniformly distributed on $[-\Delta/(2a/2), \Delta/2)$.

Given this setup, we present the distributed QCF algorithm, assuming that the sensor network is connected.

The state sequence, $\{\tilde{x}(i)\}_{i \geq 0}$ is given by the following Algorithm.

**Algorithm 1: QCF**

**Initialize**

$\tilde{x}_n(0) = x_n(0), \forall n$;

$i = 0$;

**begin**

**while** $\sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |(\tilde{x}_l(i) + \nu_{nl}(i))| < (p + 1/2)\Delta$ **do**

$$\tilde{x}_n(i + 1) = (1 - \alpha(i)d_n)\tilde{x}_n(i) + \alpha(i)\sum_{l \in \Omega_n} q(\tilde{x}_l(i) + \nu_{nl}(i)), \forall n;$$

$$i = i + 1;$$

**end**

Stop the algorithm and reset all the sensor states to zero

The last step of the algorithm can be distributed, since the network is connected.

**B. High Probability Bounds on Uniform Boundedness of Sample Paths of QC**

We show that the state vector sequence, $\{x(i)\}_{i \geq 0}$, generated by the QC algorithm is uniformly bounded with high probability. The proof is lengthy and uses mainly maximal inequalities for submartingale and supermartingale sequences.
Recall that the state vector at any time $i$ can be decomposed orthogonally as

$$x(i) = x_{\text{avg}}(i)1 + x_{C^\perp}(i)$$

where the consensus subspace, $C$, is given in eqn. (20). We provide probability bounds on the sequences $\{x_{\text{avg}}(i)\}_{i \geq 0}$ and $\{x_{C^\perp}(i)\}_{i \geq 0}$ and then use an union bound to get the final result.

Before proceeding, we need another slight modification of a result from [30] that we prove in [20].

**Lemma 7** Assume that the conditions of Theorem 1 hold for the Markov process, $X = \{x(i)\}_{i \geq 0}$. Define the function $W(i, x)$, $i \geq 0$, $x \in \mathbb{R}^{N \times 1}$, as

$$W(i, x) = (1 + V(i, x)) \prod_{j \geq i} [1 + g(j)]$$

Then, the process $\{W(i, x(i))\}_{i \geq 0}$ is a non-negative supermartingale with respect to the natural filtration $\mathcal{F}^X$ of the Markov process $X$.

The next Lemma bounds the sequence $\{x_{C^\perp}(i)\}_{i \geq 0}$.

**Lemma 8** Let $\{x(i)\}_{i \geq 0}$ be the state vector sequence generated by the QC algorithm, with an initial state $x(0) \in \mathbb{R}^{N \times 1}$. Consider the orthogonal decomposition:

$$x(i) = x_{\text{avg}}(i)1 + x_{C^\perp}(i), \forall i$$

Then, for any $a > 0$, we have

$$\mathbb{P} \left[ \sup_{j \geq 0} \|x_{C^\perp}(i)\|^2 > a \right] \leq \frac{(1 + x(0)^T L x(0)) \prod_{j \geq 0} (1 + g(j))}{1 + a \lambda_2(L)}$$

where

$$g(j) = \alpha^2(j) \max \left( \frac{\lambda^3_N(L)}{\lambda_2(L)}, \frac{2M \Delta^2 N(L)}{3} \right), \forall j \geq 0$$

**Proof:** For any $a > 0$, we have

$$\|x_{C^\perp}(i)\|^2 > a \implies x(i)^T L x(i) \geq a \lambda_2(L), \forall i$$

Define the potential function $V(i, x)$ as in Theorem 2 and eqn. (34) and the $W(i, x)$ in (64) in Lemma 7. It then follows from eqn. (68 that

$$\|x_{C^\perp}(i)\|^2 > a \implies W(i, x(i)) = 1 + a \lambda_2(L), \forall i$$
By Lemma 7, the process \((W(i, x(i)), \mathcal{F}_i^X)\) is a non-negative supermartingale. A maximal inequality for non-negative supermartingales then says that, for any \(b > 0\) (see [33]),

\[
P \left[ \max_{0 \leq j \leq i} W(j, x(j)) \geq b \right] \leq \frac{\mathbb{E}[W(0, x(0))]}{b}, \quad \forall i
\]  

(70)

Also, we note that

\[
\left\{ \sup_{j \geq 0} W(j, x(j)) > b \right\} \iff \bigcup_{i \geq 0} \left\{ \max_{0 \leq j \leq i} W(j, x(j)) > b \right\}
\]  

(71)

Since \(\{\max_{0 \leq j \leq i} W(j, x(j)) > a\}\) is a non-decreasing sequence of sets in \(i\), it follows from the continuity of probability measures

\[
P \left[ \sup_{j \geq 0} W(j, x(j)) > b \right] = \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} W(j, x(j)) > b \right]
\]  

(72)

Combining eqns. (70,72), we get

\[
P \left[ \sup_{j \geq 0} W(j, x(j)) > b \right] = \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} W(j, x(j)) > b \right] \leq \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} W(j, x(j)) \geq b \right] \leq \frac{\mathbb{E}[W(0, x(0))]}{b}
\]  

(73)

It follows from eqn. (69) using similar arguments that

\[
P \left[ \sup_{j \geq 0} \|x_{C^\perp(i)}\|^2 > a \right] = \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} \|x_{C^\perp(i)}\|^2 > a \right] \leq \lim_{i \to \infty} P \left[ \max_{0 \leq j \leq i} W(j, x(j)) > 1 + a\lambda_2(L) \right] = P \left[ \sup_{j \geq 0} W(j, x(j)) > 1 + a\lambda_2(L) \right]
\]  

(74)

Finally, from eqns. (73,74) and using the fact that

\[
W(0, x(0)) = (1 + x(0)^TLx(0)) \prod_{j \geq 0} (1 + g(j))
\]
is deterministic, we have
\[
P \left[ \sup_{j \geq 0} \| x_{C^+} (i) \|^2 > a \right] \leq P \left[ \sup_{j \geq 0} W(j, x(j)) > 1 + a \lambda_2 (L) \right] (75)
\]
\[
\leq \frac{\left( 1 + x(0)^T L x(0) \right) \prod_{j \geq 0} (1 + g(j))}{1 + a \lambda_2 (L)}
\]

Next, we provide high probability bounds on the uniform boundedness of \( \{ x_{\text{avg}} (i) \}_{i \geq 0} \).

**Lemma 9** Let \( \{ x_{\text{avg}} (i) \}_{i \geq 0} \) be the average sequence generated by the QC algorithm, with an initial state \( x(0) \in \mathbb{R}^{N \times 1} \). Then, for any \( a > 0 \),
\[
P \left[ \sup_{j \geq 0} | x_{\text{avg}} (j) | > a \right] \leq \frac{\left[ x_{\text{avg}}^2 (0) + \frac{2M \Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2 (j) \right]^{1/2}}{a} (76)
\]

**Proof:** It was shown in Theorem 4 that the sequence \( \{ x_{\text{avg}} (i) \}_{i \geq 0} \) is a martingale. It then follows that the sequence, \( \{ | x_{\text{avg}} (i) | \}_{i \geq 0} \), is a non-negative submartingale (see [31]).

The submartingale inequality then states that
\[
P \left[ \max_{0 \leq j \leq i} | x_{\text{avg}} (j) | \geq a \right] \leq \frac{\mathbb{E} \left[ | x_{\text{avg}} (i) | \right]}{a}, \forall i \quad (77)
\]

Clearly, from the continuity of probability measures,
\[
P \left[ \sup_{j \geq 0} | x_{\text{avg}} (j) | > a \right] = \lim_{i \to \infty} P \left[ \max_{0 \leq j \geq i} | x_{\text{avg}} (j) | > a \right] (78)
\]

Thus, we have
\[
P \left[ \sup_{j \geq 0} | x_{\text{avg}} (j) | > a \right] \leq \lim_{i \to \infty} \frac{\mathbb{E} \left[ | x_{\text{avg}} (i) | \right]}{a} (79)
\]

(the limit on the right exists because \( x_{\text{avg}} (i) \) converges in \( \mathcal{L}_1 \).)

Also, we have from eqn. (52), for all \( i \),
\[
\mathbb{E} \left[ | x_{\text{avg}} (i) | \right] \leq \left[ \mathbb{E} \left[ | x_{\text{avg}} (i) |^2 \right] \right]^{1/2} (80)
\]
\[
\leq \left[ x_{\text{avg}}^2 (0) + \frac{2M \Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2 (j) \right]^{1/2}
\]
Combining eqns. (79,80), we have

\[ p \left[ \sup_{j \geq 0} |x_{av}(j)| > a \right] \leq \frac{\left[ x_{av}^2(0) + 2M \Delta^2 \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} \]  

(81)

**Theorem 10**  Let \( \{x(i)\}_{i \geq 0} \) be the state vector sequence generated by the QC algorithm, with an initial state \( x(0) \in \mathbb{R}^{N \times 1} \). Then, for any \( a > 0 \),

\[ p \left[ \sup_{j \geq 0} \|x(j)\| > a \right] \leq \frac{\left[ 2N x_{av}^2(0) + \frac{4M \Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} + \frac{(1 + x(0)^T L x(0)) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{a^2}{2} \lambda_2(L)} \]  

(82)

where

\[ g(j) = \alpha^2(j) \max \left( \frac{\lambda_3^3(L)}{\lambda_2(L)}, \frac{2M \Delta^2 \lambda_N(L)}{3} \right), \forall j \geq 0 \]  

(83)

**Proof:** Since, \( \|x(j)\|^2 = N x_{av}^2(i) + \|x_{C+}\|^2(j) \), we have

\[ p \left[ \sup_{j \geq 0} \|x(j)\|^2 > a \right] \leq p \left[ \sup_{j \geq 0} N |x_{av}(j)|^2 > \frac{a}{2} \right] + p \left[ \sup_{j \geq 0} \|x_{C+}(i)\|^2 > \frac{a}{2} \right] \]  

(84)

\[ = p \left[ \sup_{j \geq 0} |x_{av}(j)| > \left( \frac{a}{2N} \right)^{1/2} \right] + p \left[ \sup_{j \geq 0} \|x_{C+}(i)\|^2 > \frac{a}{2} \right] \]

We thus have from Lemmas 8 and 9,

\[ p \left[ \sup_{j \geq 0} \|x(j)\|^2 > a \right] \leq \frac{\left[ x_{av}^2(0) + \frac{2M \Delta^2}{3N^2} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} + \frac{(1 + x(0)^T L x(0)) \prod_{j \geq 0} (1 + g(j))}{1 + \left( \frac{a^2}{2} \right) \lambda_2(L)} \]  

(85)

We now state as a Corollary the result on the boundedness of the sensor states, which will be used in analyzing the performance of the QCF algorithm.

**Corollary 11** Assume that the initial sensor state, \( x(0) \in B \), where \( B \) is given in eqn. (61). Then, if \( \{x(i)\}_{i \geq 0} \) is the state sequence generated by the QC algorithm starting from the initial state, \( x(0) \), we have, for any \( a > 0 \),

\[ p \left[ \sup_{1 \leq n \leq N, j \geq 0} |x_n(j)| > a \right] \leq \frac{\left[ 2N b^2 + \frac{4M \Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a} + \frac{(1 + N \lambda_N(L) b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \left( \frac{a^2}{2} \right) \lambda_2(L)} \]  

(86)
where
\[ g(j) = \alpha^2(j) \max \left( \frac{\lambda_N^2(L)}{\lambda_2(L)} , \frac{2M\Delta^2\lambda_N(L)}{3} \right), \forall j \geq 0 \]  

(87)

**Proof:** We note that, for \( x(0) \in B \),
\[ x_{avg}^2(0) \leq b^2, \quad x(0)^TLx(0) \leq N\lambda_N(L)b^2 \]

(88)

From Theorem 10, we then get,
\[
P \left[ \sup_{1 \leq n \leq N; j \geq 0} |x_n(j)| > a \right] \leq P \left[ \sup_{j \geq 0} \|x(j)\| > a \right]
\]
\[
\leq \frac{\left[ 2N x_{avg}^2(0) + \frac{4M\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a}
\]
\[
+ \frac{(1 + x(0)^TLx(0)) \prod_{j \geq 0} (1 + g(j))}{1 + a^2 \lambda_2(L)}
\]
\[
\leq \frac{\left[ 2Nb^2 + \frac{4M\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{a}
\]
\[
+ \frac{(1 + N\lambda_N(L)b^2) \prod_{j \geq 0} (1 + g(j))}{1 + a^2 \lambda_2(L)}
\]

(89) (90)

\[ \square \]

C. Algorithm QCF: Asymptotic Consensus

We show that the QCF algorithm, given in Subsection IV-A, converges a.s. to a finite random variable and the sensors reach consensus asymptotically.

**Theorem 12 (QCF: a.s. asymptotic consensus)** Let \( \{\bar{x}(i)\}_{i \geq 0} \) be the state vector sequence generated by the QCF algorithm, starting from an initial state \( \bar{x}(0) = x(0) \in B \). Then, the sensors reach consensus asymptotically a.s. In other words, there exists an a.s. finite random variable \( \bar{\theta} \) such that
\[
P \left[ \lim_{i \to \infty} \bar{x}(i) = \bar{\theta} \mathbf{1} \right] = 1
\]

(91)

**Proof:** For the proof, consider the sequence \( \{x(i)\}_{i \geq 0} \) generated by the QC algorithm, with the same initial state \( x(0) \). Let \( \theta \) be the a.s. finite random variable (see eqn. 42) such that
\[
P \left[ \lim_{i \to \infty} x(i) = \theta \mathbf{1} \right] = 1
\]

(92)
It is then clear that
\[
\tilde{\theta} = \begin{cases} 
\theta & \text{on } \{\sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |x_l(i) + \nu_{nl}(i)| < (p + \frac{1}{2})\Delta \} \\
0 & \text{otherwise}
\end{cases}
\]  
(93)

In other words, we have
\[
\tilde{\theta} = \theta \mathbb{I}\left(\sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |x_l(i) + \nu_{nl}(i)| < (p + \frac{1}{2})\Delta \right)
\]  
(94)

where \(\mathbb{I}(\cdot)\) is the indicator function. Since \(\{\sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |x_l(i) + \nu_{nl}(i)| < (p + 1/2)\Delta \}\) is a measurable set, it follows that \(\tilde{\theta}\) is a random variable.

\[\square\]

D. QCF: \(\epsilon\)-Consensus

Recall the QCF algorithm in Subsection IV-A and the assumptions 1)–3). A key step is that, if we run the QC algorithm, using finite bit quantizers with finite alphabet \(\tilde{Q}\) as given in eqn. (62), the only way for an error to occur is for one of the quantizers to saturate. This was, in fact, the main intuition behind the design of the QCF algorithm.

Theorem 12 shows that the QCF sensor states asymptotically reach consensus, converging a.s. to a finite random variable \(\tilde{\theta}\). The next series of results address the question of how close is this consensus to the desired average \(r\) in (1). These results express how this consensus to the desired average depends on the QCF design: 1) the quantizer parameters (like the number of levels \(2p + 1\) or the quantization step \(\Delta\)); 2) the network topology (given by the Laplacian \(L\) of its graph \(G\)); and 3) the gains \(\alpha\). The next Lemma shows that \(\tilde{\theta}\), with appropriate design of the quantizer, is arbitrarily close to \(r\) with high probability. Theorem 15 shows that QCF achieves \(\epsilon\)-consensus (see below), i.e., for arbitrary \(\epsilon > 0\), the QCF quantizers can be designed so that the QCF states get within an \(\epsilon\)-ball of \(r\) with the desired probability. Proposition 16 considers several tradeoffs between the probability of achieving consensus and the quantizer parameters and network topology.

Definition 13 (\(\epsilon\)-consensus and probability of \(\epsilon\)-consensus) QCF achieves \(\epsilon\)-consensus iff for every \(0 < \delta < 1\) and \(\epsilon > 0\), there exist quantizer parameters such that the probability of \(\epsilon\)-consensus \(T(G, b, \alpha, \epsilon, p, \Delta)\)

\[
T(G, b, \alpha, \epsilon, p, \Delta) = \mathbb{P} \left[ \lim_{i \to \infty} \sup_{1 \leq n \leq N} |\tilde{x}_n(i) - r| < \epsilon \right] > 1 - \delta
\]  
(95)

Lemma 14 Let \(\tilde{\theta}\) be defined as in Lemma 12, with the initial state \(\tilde{x}(0) = x(0) \in B\). The desired average,
\( r \), is given in (1). Then, for any \( \epsilon > 0 \), we have
\[
\mathbb{P}\left[ |\tilde{\theta} - r| \geq \epsilon \right] \leq \frac{2M\Delta^2}{3N^2\epsilon^2} \sum_{j \geq 0} \alpha^2(j) + \frac{\left[ 2Nb^2 + \frac{4M\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{p\Delta} \\
+ \frac{(1 + N\lambda_N(L)b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{p^2\Delta^2}{2} \lambda_2(L)} \tag{96}
\]

where
\[
g(j) = \alpha^2(j) \max \left( \frac{\lambda^3_N(L)}{\lambda_2(L)}, \frac{2M\Delta^2\lambda_N(L)}{3} \right), \quad \forall j \geq 0 \tag{98}
\]

**Proof:** For the proof, consider the sequence \( \{x(i)\}_{i \geq 0} \) generated by the QC algorithm, with the same initial state \( x(0) \). Let \( \theta \) be the a.s. finite random variable (see eqn. 42) such that
\[
\mathbb{P}\left[ \lim_{i \to \infty} x(i) = \theta 1 \right] = 1 \tag{99}
\]
We note that
\[
\mathbb{P}\left[ |\tilde{\theta} - r| \geq \epsilon \right] = \mathbb{P}\left[ (|\tilde{\theta} - r| \geq \epsilon) \cap (\tilde{\theta} = \theta) \right] + \mathbb{P}\left[ (|\tilde{\theta} - r| \geq \epsilon) \cap (\tilde{\theta} \neq \theta) \right]
= \mathbb{P}\left[ (|\theta - r| \geq \epsilon) \cap (\tilde{\theta} = \theta) \right] + \mathbb{P}\left[ (|\tilde{\theta} - r| \geq \epsilon) \cap (\tilde{\theta} \neq \theta) \right]
\leq \mathbb{P}\left[ |\theta - r| \geq \epsilon \right] + \mathbb{P}\left[ \tilde{\theta} \neq \theta \right] \tag{100}
\]

From Chebyshev’s inequality, we have
\[
\mathbb{P}\left[ |\theta - r| \geq \epsilon \right] \leq \mathbb{E}\left[ |\theta - r|^2 \right] \leq \frac{2M\Delta^2}{3N^2\epsilon^2} \sum_{j \geq 0} \alpha^2(j) \tag{101}
\]
Next, we bound \( \mathbb{P}\left[ \tilde{\theta} \neq \theta \right] \). To this end, we note that
\[
\sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |x_l(i) + \nu_{nl}(i)| \leq \sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |x_l(i)| + \sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |\nu_{nl}(i)|
\leq \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| + \sup_{i \geq 0} \sup_{1 \leq n \leq N} \sup_{l \in \Omega_n} |\nu_{nl}(i)|
\leq \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| + \frac{\Delta}{2} \tag{102}
\]
Then, for any $\delta > 0$,
\[
\mathbb{P} \left[ \bar{\theta} \neq \theta \right] = \mathbb{P} \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_t(i) + \nu_m(i)| \geq \left( p + \frac{1}{2} \right) \Delta \right] \\
\leq \mathbb{P} \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| + \frac{\Delta}{2} \geq \left( p + \frac{1}{2} \right) \Delta \right] \\
= \mathbb{P} \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| \geq p\Delta \right] \\
\leq \mathbb{P} \left[ \sup_{i \geq 0} \sup_{1 \leq n \leq N} |x_n(i)| > p\Delta - \delta \right] \\
\leq \frac{\left[ 2Nb^2 + \frac{4M\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{p\Delta - \delta} + \frac{(1 + N\lambda_N(L)b^2) \prod_{j \geq 0} (1 + g(j))}{1 + (p\Delta - \delta)^2 \lambda_2(L)} \tag{103}
\]
where, in the last step, we use eqn. (89.) Since the above holds for arbitrary $\delta > 0$, we have

\[
\mathbb{P} \left[ \bar{\theta} \neq \theta \right] \leq \lim_{\delta \downarrow 0} \frac{\left[ 2Nb^2 + \frac{4M\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2}}{p\Delta} + \frac{(1 + N\lambda_N(L)b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{p\Delta^2}{2} \lambda_2(L)} \tag{104}
\]
Combining eqns. (100,101,104), we get the result.

We now state the main result of this Section, which provides a performance guarantee for QCF.

**Theorem 15 (QCF: $\epsilon$-consensus)** QCF attains $\epsilon$-consensus: for any $\epsilon > 0$, the probability of $\epsilon$-consensus $T(G, b, \alpha, \epsilon, p, \Delta)$ is bounded below

\[
\mathbb{P} \left[ \lim_{i \to \infty} \sup_{1 \leq n \leq N} |\bar{x}_n(i) - r| < \epsilon \right] \geq 1 - \frac{2M\Delta^2}{3N^2\epsilon^2} \sum_{j \geq 0} \alpha^2(j) - \frac{2Nb^2 + \frac{4M\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j)}{p\Delta} \\
- \frac{(1 + N\lambda_N(L)b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{p\Delta^2}{2} \lambda_2(L)} \tag{105}
\]
where

\[
g(j) = \alpha^2(j) \max \left( \frac{\lambda_N^3(L)}{\lambda_2(L)}, \frac{2M\Delta^2\lambda_N(L)}{3} \right), \quad \forall j \geq 0 \tag{106}
\]

**Proof:** It follows from Theorem 12 that

\[
\lim_{i \to \infty} \bar{x}_n(i) = \bar{\theta} \text{ a.s., } \forall 1 \leq n \leq N \tag{107}
\]
The rest of the proof follows from Lemma 14.

The lower bound on $T(\cdot)$, given by (105), is uniform, in the sense that it is applicable for all initial states $\mathbf{x}(0) \in \mathcal{B}$. Recall the scaled weight sequence $\alpha_s$, given by eqn. (55). We introduce the zero-rate probability of $\epsilon$-consensus, $T^z(G,b,\epsilon,p,\Delta)$ by

$$T^z(G,b,\epsilon,p,\Delta) = \lim_{\Delta \to 0} T(G,b,\alpha_s,\epsilon,p,\Delta)$$

(108)

The next proposition studies the dependence of the $\epsilon$-consensus probability $T(\cdot)$ and of the zero-rate probability $T^z(\cdot)$ on the network and algorithm parameters.

**Proposition 16 (QCF: Tradeoffs)**

1) **Limiting quantizer.** For fixed $G,b,\alpha,\epsilon$, we have

$$\lim_{\Delta \to 0, \ p\Delta \to \infty} T(G,b,\alpha,\epsilon,p,\Delta) = 1$$

(109)

Since, this holds for arbitrary $\epsilon > 0$, we note that, as $\Delta \to 0, \ p\Delta \to \infty$,

$$\mathbb{P}\left[ \lim_{i \to \infty} \bar{x}(i) = r \mathbf{1} \right] = \lim_{\epsilon \to 0} \mathbb{P}\left[ \lim_{i \to \infty} \sup_{1 \leq n \leq N} |\bar{x}_n(i) - r| < \epsilon \right]$$

$$= \lim_{\epsilon \to 0} \left[ \lim_{\Delta \to 0, \ p\Delta \to \infty} T(G,b,\alpha,\epsilon,p,\Delta) \right]$$

$$= 1$$

(110)

In other words, the QCF algorithm leads to a.s. consensus to the desired average $r$, as $\Delta \to 0, \ p\Delta \to \infty$.

2) **zero-rate $\epsilon$-consensus probability.** Then, for fixed $G,b,\epsilon,p,\Delta$, we have

$$T^z(G,b,\epsilon,p,\Delta) \geq 1 - \frac{(2Nb^2)^{1/2}}{p\Delta} \cdot \frac{1 + N\lambda N(L)b^2}{1 + \frac{M^2\Delta^2}{2}\lambda_2(L)}$$

(111)

3) **Optimum quantization step-size $\Delta$.** For fixed $G,b,\epsilon,p$, the optimum quantization step-size $\Delta$, which maximizes the probability of $\epsilon$-consensus, $T(G,b,\alpha,\epsilon,p,\Delta)$, is given by

$$\Delta^*(G,b,\alpha,\epsilon,p) = \arg\inf_{\Delta \geq 0} \left[ \frac{2M\Delta^2}{3N^2\epsilon^2} \sum_{j \geq 0} \alpha^2(j) \right]^{1/2} \frac{4M\Delta^2}{3N} \sum_{j \geq 0} \alpha^2(j) \frac{1}{p\Delta}$$

$$+ \frac{(1 + N\lambda N(L)b^2) \prod_{j \geq 0} (1 + g(j))}{1 + \frac{M^2\Delta^2}{2}\lambda_2(L)}$$

(112)

where

$$g(j) = \alpha^2(j) \max \left( \frac{\lambda^3 N(L)}{\lambda_2(L)}, \frac{2M\Delta^2\lambda N(L)}{3} \right), \forall j \geq 0$$

(113)
Proof: For item 2), we note that, as $s \to 0$,

$$\sum_{j \geq 0} \alpha^2_s(j) \to 0, \quad \prod_{j \geq 0} (1 + g_s(j)) \to 1$$

The rest follows by simple inspection of eqn. (105).

We comment on Proposition 16. Item 1) shows that the algorithm QCF is consistent, in the sense that we can achieve arbitrarily good performance by decreasing the step-size $\Delta$ and the number of quantization levels, $2p + 1$, appropriately. Indeed, decreasing the step-size increases the precision of the quantized output and increasing $p$ increases the dynamic range of the quantizer. However, the fact that $\Delta \to 0$ but $p\Delta \to \infty$ implies that the rate of growth of the number of levels $2p + 1$ should be higher than the rate of decay of $\Delta$, guaranteeing that in the limit we have asymptotic consensus with probability one.

For interpreting item 2), we recall the m.s.e. versus convergence rate tradeoff for the QC algorithm, studied in Subsection III-B. There, we considered a quantizer with a countably infinite number of output levels (as opposed to the finite number of output levels in the QCF) and observed that the m.s.e. can be made arbitrarily small by rescaling the weight sequence. By Chebyshev’s inequality, this would imply, that, for arbitrary $\epsilon > 0$, the probability of $\epsilon$-consensus, i.e., that we get within an $\epsilon$-ball of the desired average, can be made as close to 1 as we want. However, this occurs at a cost of the convergence rate, which decreases as the scaling factor $s$ decreases. Thus, for the QC algorithm, in the limiting case, as $s \to 0$, the probability of $\epsilon$-consensus (for arbitrary $\epsilon > 0$) goes to 1; we call “limiting probability” the zero-rate probability of $\epsilon$-consensus, justifying the m.s.e. vs convergence rate tradeoff.\footnote{Note that, for both the algorithms, QC and QCF, we can take the scaling factor, $s$, arbitrarily close to 0, but not zero, so that, these limiting performance values are not achievable, but we may get arbitrarily close to them.} Item 2) shows, that, similar to the QC algorithm, the QCF algorithm exhibits a tradeoff between probability of $\epsilon$-consensus vs. the convergence rate, in the sense that, by scaling (decreasing $s$), the probability of $\epsilon$-consensus can be increased. However, contrary to the QC case, scaling will not lead to probability of $\epsilon$-consensus arbitrarily close to 1, and, in fact, the zero-rate probability of $\epsilon$-consensus is strictly less than one, as given by eqn. (111). In other words, by scaling, we can make $T(G, b, \alpha_s, \epsilon, p, \Delta)$ as high as $T^z(G, b, \epsilon, p, \Delta)$, but no higher.

We now interpret the lower bound on the zero-rate probability of $\epsilon$-consensus, $T^z(G, b, \epsilon, p, \Delta)$, and show that the network topology plays an important role in this context. We note, that, for a fixed number, $N$, of sensor nodes, the only way the topology enters into the expression of the lower bound is through
the third term on the R.H.S. Then, assuming that,
\[ N\lambda_N(L)b^2 \gg 1, \quad \frac{p^2\Delta^2}{2}\lambda_2(L) \gg 1 \]
we may use the approximation
\[ \frac{1 + N\lambda_N(L)b^2}{1 + \frac{p^2\Delta^2}{2}\lambda_2(L)} \approx \left( \frac{2Nb^2}{p^2\Delta^2} \right) \frac{\lambda_N(L)}{\lambda_2(L)} \]  
(114)
Thus, for a fixed number, \( N \), of sensor nodes, topologies with smaller \( \lambda_N(L)/\lambda_2(L) \), will lead to higher zero-rate probability of \( \epsilon \)-consensus and, hence, are preferable. We note that, in this context, for fixed \( N \), the class of non-bipartite Ramanujan graphs give the smallest \( \lambda_N(L)/\lambda_2(L) \) ratio, given a constraint on the number, \( M \), of network edges (see [7].)

Item 3) shows that, for given graph topology \( G \), initial sensor data, \( b \), the link weight sequence \( \alpha \), tolerance \( \epsilon \), and the number of levels in the quantizer \( p \), the step-size \( \Delta \) plays a significant role in determining the performance. This gives insight into the design of quantizers to achieve optimal performance, given a constraint on the number of quantization levels, or, equivalently, given a bit budget on the communication.

In the next Subsection, we present some numerical studies on the QCF algorithm, which demonstrate practical implications of the results just discussed.

E. QCF: Numerical Studies

We present a set of numerical studies on the quantizer step-size optimization problem, considered in Item 3) of Proposition 16. We consider a sensor network of \( N = 230 \) nodes, with communication topology given by an LPS-II Ramanujan graph (see [7]), of degree 6.\(^2\) We fix \( \epsilon \) at .05, and take the initial sensor data bound, \( b \), to be 30. We numerically solve the step-size optimization problem given in (112) for varying number of levels, \( 2p + 1 \). Specifically, we consider two instances of the optimization problem: In the first instance, we consider the weight sequence, \( \alpha(i) = .01/(i + 1) \), \( (s = .01) \), and numerically solve the optimization problem for varying number of levels. In the second instance, we repeat the same experiment, with the weight sequence, \( \alpha(i) = .001/(i + 1) \), \( (s = .001) \). As in eqn. (112), \( \Delta^*(G,b,\alpha_s,\epsilon,p) \) denotes the optimal step-size. Also, let \( T^*(G,b,\alpha_s,\epsilon,p) \) be the corresponding optimum probability of \( \epsilon \)-consensus. Fig. 1 on the left plots \( T^*(G,b,\alpha_s,\epsilon,p) \) for varying \( 2p + 1 \) on the vertical axis, while on the horizontal axis, we plot the corresponding quantizer bit-rate \( BR = \log_2(2p + 1) \). The two plots correspond to two different scalings, namely, \( s = .01 \) and \( s = .001 \) respectively. The result is

\(^2\)This is a 6-regular graph, i.e., all the nodes have degree 6.
in strict agreement with Item 2) of Proposition 16, and shows that, as the scaling factor decreases, the probability of $\epsilon$-consensus increases, till it reaches the zero-rate probability of $\epsilon$-consensus.

Fig. 1 on the right plots $\Delta^*(G, b, \alpha_s, \epsilon, p)$ for varying $2p+1$ on the vertical axis, while on the horizontal axis, we plot the corresponding quantizer bit-rate $BR = \log_2(2p + 1)$. The two plots correspond to two different scalings, namely, $s = .01$ and $s = .001$ respectively. The results are again in strict agreement to Proposition 16 and further show that optimizing the step-size is an important quantizer design problem, because the optimal step-size value is sensitive to the number of quantization levels, $2p + 1$.

![Figure 1](image1.png)

![Figure 1](image2.png)

Fig. 1. Left: $T^*(G, b, \alpha_s, \epsilon, p)$ vs. $2p + 1$ ($BR = \log_2(2p + 1)$.) Right: $\Delta^*(G, b, \alpha_s, \epsilon, p)$ vs. $2p + 1$ ($BR = \log_2(2p + 1)$.)

V. CONCLUSION

The paper considers distributed average consensus with quantized information exchange. We address two versions of the problem: when the quantizers’ alphabet is unbounded, the QC algorithm; and when the quantizers alphabet is bounded, the QCF algorithm. To achieve consensus, we add dither to the sensor states before quantization. We demonstrate that, asymptotically, with high probability, the sensor states achieve consensus to a random variable whose mean is the desired average. We show that the variance of this random variable can be made small by tuning parameters of the algorithm (rate of decay of the gains), the network topology, and quantizers parameters. We consider several metrics of performance and show that with high probability the sensors states achieve $\epsilon$-consensus, i.e., they stay within a ball of radius $\epsilon$ of the true desired average. Given a finite bit-budget, we cast the quantizer design as an optimization problem. Analytical expressions and a numerical study illustrate this design problem and several interesting tradeoffs among design parameters.
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