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A Gronwall Inequality for Weakly Lipschitzian Mappings

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1. Introduction

In the study of non–linear oscillators arising in mechanics, there are many examples of systems of ordinary differential equations whose right–hand sides are not Lipschitzian or even one–sided Lipschitzian and, therefore, whose solutions are not covered by classical theorems on uniqueness and continuous dependence. With such examples in mind, OWEN & WANG [1] and WANG [2] have introduced the notion of a weakly Lipschitzian mapping, have established generalizations of classical uniqueness theorems, and have given specific examples from mechanics to which their theorems, but not the classical ones, apply.

Our purpose here is to establish a counterpart for weakly Lipschitzian mappings of the classical Gronwall inequality for Lipschitzian and one–sided Lipschitzian mappings ([3], p. 13). Our inequality implies not only the uniqueness theorems of OWEN & WANG ([1], Theorems 2.1 and 4.1), but also theorems on continuous dependence of solutions upon initial data that apply to all of the examples considered in the studies [1] and [2].

The principal tool that we use here in proving a Gronwall inequality for weakly Lipschitzian mappings is an inequality for concave functions or for convex functions that generalizes inequalities established in previous studies ([1], [4]). The present inequality, here called a "separation inequality", provides a lower bound for a line integral as the underlying curve varies within a specified class of curves, all having the same initial point and all having the same final point. The line integral itself provides an example of an "action with the dissipation property", and the lower bound yields a "lower potential" in the sense of OWEN ([5], Chapter IV; see also [6]). These facts show that the separation inequality has a counterpart in thermodynamics. Moreover, our proof of the separation inequality is considerably less technical than the proofs of its counterparts in the previous articles ([1], [4]).

Although our statements and proofs of the Gronwall inequality and theorem on continuous dependence are given for classical solutions of ordinary differential equations, there are straightforward modifications of our arguments that yield corresponding results for Filippov solutions. The nature of these modifications can be inferred from the discussion in Section 4 of the article [1]. As was noted in that article in the case of "restricted uniqueness" of solutions, the hypotheses that here give "restricted continuous dependence" of Filippov solutions on initial data also give existence of Filippov solutions.

2. A Separation Inequality

In this section we prove a "separation inequality" for concave, non–decreasing or for convex, non–decreasing functions. This inequality generalizes the corresponding inequalities [1, Lemma 2.2], [4, (5.8)] in our earlier studies, and the proof that we give here is substantially less technical than the proofs given in [1] and [4].

Lemma 2.1: Let I be an interval in $\mathbb{R}$, let $G : I \longrightarrow \mathbb{R}$ be a non–decreasing function that is concave or that is convex, and for each $x, y \in I$, put

$$h_G(x,y) = \begin{cases} \displaystyle\int_{\min\{x,y\}}^{\max\{x,y\}} (G(z) - G(\min\{x,y\}))dz & \text{if } G \text{ is concave,} \\ \displaystyle\int_{\min\{x,y\}}^{\max\{x,y\}} (G(\max\{x,y\}) - G(z))dz & \text{if } G \text{ is convex.} \end{cases} \tag{1}$$

Furthermore, for each $t > 0$ and each pair $x(\cdot)$: $[0,t] \longrightarrow I$, $y(\cdot)$: $[0,t] \longrightarrow I$ of absolutely continuous functions, put

$$\mathcal{J}_G(x(\cdot), y(\cdot)) := \int_0^t (G(x(s)) - G(y(s)))\,(\dot{x}(s) - \dot{y}(s))ds. \tag{2}$$

If a) G is concave and $x(\cdot)$ and $y(\cdot)$ are non–decreasing or if b) G is convex and $x(\cdot)$ and $y(\cdot)$ are non–increasing, then there holds

$$\mathcal{J}_G(x(\cdot), y(\cdot)) \geq h_G(x(t), y(t)) - h_G(x(0), y(0)). \tag{3}$$

We note that if $x(0) = y(0)$, the inequality (3) and the definition of h_G in (1) imply

$$\mathcal{J}_G(x(\cdot),\ y(\cdot)) \geq 0, \tag{4}$$

which is the inequality obtained in the articles [1] and [4]; if, in addition, G is increasing, then, by (3) and (1), equality holds in (4) only if $x(t) = y(t)$. Therefore, the integral $\mathcal{J}_G(x(\cdot),\ y(\cdot))$ is positive if the graphs of the two functions $x(\cdot)$ and $y(\cdot)$ coincide initially at the common point $(0, x(0)) = (0,y(0))$ and are separate at time t: $(t,x(t)) \neq (t,y(t))$; the integral vanishes if the graphs never separate. For this reason, we refer to (4) and to its generalization (3) as separation inequalities.

Proof of Lemma 2.1: We consider first the case a) G is concave and $x(\cdot)$, $y(\cdot)$ are non–decreasing. By (1) and (2), both $\mathcal{J}_G$ and h_G are symmetric functions, so it suffices to verify (3) when $x(0) \geq y(0)$. Three further possibilities arise

$$y(0) \leq y(t) \leq x(0) \leq x(t), \tag{5}$$

$$y(0) \leq x(0) \leq y(t) \leq x(t), \tag{6}$$

$$y(0) \leq x(0) \leq x(t) \leq y(t). \tag{7}$$

For the case (5), we consider the curvilinear triangle

$$\mathcal{A} := \{(x,y) \mid x(0) < x < x(s),\ y(s) < y < y(t) \quad \text{for some } s \in [0,t]\} \tag{8}$$

and the trapezoid

$$\mathcal{T} := \{(x,y) \mid y(0) < x \le y(t), \quad y(0) < y < x\}$$

$$\cup \{(x,y) \mid y(t) < x \le x(0), \quad y(0) < y < y(t)\}. \tag{9}$$

The open region $\mathcal{A} \cup \mathcal{T}$ is bounded by the curve $C: \ s \longmapsto (x(s),\ y(s))$, parameterized on $[0,t]$, the horizontal segment H_L moving to the left from $(x(0),\ y(0))$ to $(y(0),\ y(0))$, the slanted segment S from $(y(0),\ y(0))$ to $(y(t),\ y(t))$, and the horizontal segment H_R moving to the right from $(y(t),\ y(t))$ to $(x(t),\ y(t))$. Writing $\mathcal{J}_G$ in (2) as a line integral

$$\mathcal{J}_G(x(\cdot),\ y(\cdot)) = \int_C (G(x) - G(y)dx + (G(y) - G(x))dy = \int_C \vec{F} \cdot d\vec{r}, \tag{10}$$

we can apply Green's Theorem to write

$$\int_C \vec{F} \cdot d\vec{r} - \int_{H_L} \vec{F} \cdot d\vec{r} - \int_S \vec{F} \cdot d\vec{r} - \int_{H_R} \vec{F} \cdot d\vec{r}$$

$$= \iint_{\mathcal{A} \cup \mathcal{T}} (G'(y) - G'(x))dxdy. \tag{11}$$

(Because G is non–decreasing, concave, and, therefore, absolutely continuous and because $x(\cdot)$ and $y(\cdot)$ are non–decreasing and absolutely continuous, it is easy to use appropriate forms of Fubini's Theorem and the Fundamental Theorem of Calculus to establish Green's Theorem in the present context.) Because G is concave and $y \le x$ for all $(x,y) \in \mathcal{A} \cup \mathcal{T}$, the double integral in (11) is non–negative, and evaluation of the line integrals along H_L, S and H_R yields in view of (10), (11), and (1):

$$\mathcal{J}_G(x(\cdot), y(\cdot)) \geq \int_{x(0)}^{y(0)} (G(x) - G(y(0))dx$$

$$+ \int_{y(t)}^{x(t)} (G(x) - G(y(t))dx$$

$$= -h_G(x(0), y(0)) + h_G(x(t), y(t)),$$

which is the separation inequality (3) when $(x(0), y(0))$ and $(x(t), y(t))$ satisfy (5).

Next, suppose that $(x(0), y(0))$ and $(x(t), y(t))$ satisfy (6) and, with $\mathcal{A}$ given by (8), put

$$\mathcal{A}_1 := \{(x,y) \in \mathcal{A} \mid y \leq x\} \tag{12}$$

$$\mathcal{A}_2 := \{(x,y) \in \mathcal{A} \mid x < y\} \tag{13}$$

$$\mathcal{A}_3 := \{(x,y) \in (x(0), x(t)) \times (y(0), y(t)) \mid x < y\} \tag{14}$$

$$\mathcal{U} := \{(x,y) \mid y(0) < x < x(0), \quad y(0) < y < x\}. \tag{15}$$

Moreover, let V be the vertical path from $(x(0), y(0))$ to $(x(0), y(t))$, let H be the horizontal path from $(x(0), y(t))$ to $(x(t), y(t))$, and let C, H_L, S, and H_R be as above.

We apply Green's Theorem twice to obtain the relations

$$\int_C \vec{F}\cdot d\vec{r} - \int_V \vec{F}\cdot d\vec{r} - \int_H \vec{F}\cdot d\vec{r} = \iint_{\mathscr{A}} (G'(y) - G'(x))dxdy$$

$$\int_V \vec{F}\cdot d\vec{r} + \int_H \vec{F}\cdot d\vec{r} - \int_{H_R} \vec{F}\cdot d\vec{r} - \int_S \vec{F}\cdot d\vec{r} - \int_{H_L} \vec{F}\cdot d\vec{r}$$

$$= \iint_{\mathscr{U}} (G'(y) - G'(x)dxdy$$

$$- \iint_{\mathscr{A}_3} (G'(y) - G'(x))dxdy,$$

which yield upon addition and use of $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$

$$\int_C \vec{F}\cdot d\vec{r} - \int_{H_R} \vec{F}\cdot d\vec{r} - \int_S \vec{F}\cdot d\vec{r} - \int_{H_L} \vec{F}\cdot d\vec{r} =$$

$$\iint_{\mathscr{A}_1\cup\mathscr{A}_2} (G'(y) - G'(x))dxdy + \iint_{\mathscr{U}} (G'(y) - G'(x))dxdy$$

$$- \iint_{\mathscr{A}_3} (G'(y) - G'(x))dxdy. \qquad (16)$$

Because G is concave and $\mathscr{A}_2 \subset \mathscr{A}_3$, we have

$$\iint_{\mathscr{A}_1 \cup \mathscr{U}} (G'(y) - G'(x))dxdy \geq 0$$

$$\iint_{\mathscr{A}_3} (G'(y) - G'(x))dxdy \leq \iint_{\mathscr{A}_2} (G'(y) - G'(x))dxdy \leq 0,$$

and (16) then tells us that

$$\int_C \vec{F} \cdot d\vec{r} - \int_{H_R} \vec{F} \cdot d\vec{r} - \int_S \vec{F} \cdot d\vec{r} - \int_{H_L} \vec{F} \cdot d\vec{r} \geq 0,$$

which, as in the case when (5) holds, is the separation inequality (3).

In the remaining situation for $(x(0), y(0))$ and $(x(t), y(t))$ when (7) holds, the argument for the case when (6) holds also can be applied, provided only that one replaces H_R by the vertical segment V^* from $(x(t), x(t))$ to $(x(t), y(t))$ and S by the slanted segment S^* from $(y(0), y(0))$ to $(x(t), x(t))$.

The verification of (3) when b) G is convex and $x(\cdot)$ and $y(\cdot)$ are non–increasing follows from the case a) and from the facts:

(i) for all $s \in [0,t]$

$$(G(x(s)) - G(y(s)))(\dot{x}(s) - \dot{y}(s)) =$$

$$(-G(-(-y(s))) - [-G(-(-x(s)))])(-\dot{y}(s) - (-\dot{x}(s))),$$

(ii) $-x(\cdot)$ and $-y(\cdot)$ are non–decreasing when $x(\cdot)$ and $y(\cdot)$ are non–increasing,

(iii) $z \longmapsto -G(-z)$ is non–decreasing and concave whenever G is non–decreasing and convex, and

(iv) $h_{z \longmapsto -G(-z)}(-x, -y) = h_G(x, y)$. ■

We note that the separation inequality (3) is not sharp, but that the arguments presented here can be used to obtain a sharp version of (3). Because the form of the sharp version is rather complicated and is not useful in subsequent sections, we do not record it here.

3. A Gronwall Inequality

Let $n \in \mathbb{N}$, $D \subset \mathbb{R}^{n+1}$, $f : D \longrightarrow \mathbb{R}^n$. and $T > 0$ be given, and consider the ordinary differential equation

$$\dot{x}(t) = f(t, x(t)), \quad t \in [0,T]. \tag{17}$$

We say that $x : [0, T] \longrightarrow \mathbb{R}^n$ is a classical solution of (17) if x is absolutely continuous, $(t, x(t)) \in D$ for all $t \in [0, T]$, and x satisfies (17) for almost every $t \in [0,T]$. In this section, we use the separation inequality (3) to prove a Gronwall inequality for classical solutions of (17) when f is <u>weakly Lipschitzian</u> on D in the following sense [1, Section 2]: there exist $m \in \{0, 1, ..., n\}$ and, for each $j \in \{m + 1, ..., n\}$, an increasing mapping $G_j : I_j \longrightarrow \mathbb{R}$, with I_j an interval in $\mathbb{R}$, satisfying

(WL1) for each $j \in \{m + 1, ..., n\}$, $f_j \geq 0$ and G_j is concave, or $f_j \leq 0$ and G_j is convex;

(WL2) there exists a locally integrable function $L : [0, \infty) \longrightarrow [0, \infty)$ such that for every $(t, x), (t, y)$ in D, with x_j and y_j in I_j for all $j \in \{m + 1, ..., n\}$, there holds

$$(Px - Py) \cdot (Pf(t, x) - Pf(t, y)) + \sum_{j=m+1}^{n} (G_j(x_j) - G_j(y_j)) (f_j(t, x) - f_j(t, y)) \leq L(t) \, \| Px - Py \|^2; \tag{18}$$

here, for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we put

$$Px = P(x_1, ..., x_n) = (x_1, x_2, ..., x_m, 0, ..., 0), \tag{19}$$

and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

We note that if f is monotone, Lipschitzian, or one–sided Lipschitzian, then f is weakly Lipschitzian, and the examples given in the paper [1] provide many functions that are weakly Lipschitzian but not monotone, Lipschitzian, or one–sided Lipschitzian. Although a given function f can be weakly Lipschitzian on D for many choices of the intervals I_j, in practice one typically chooses the intervals to be maximal.

We assume now that f is weakly Lipschitzian and consider two classical solutions $x : [0, T] \longrightarrow \mathbb{R}^n$, $y : [0, T] \longrightarrow \mathbb{R}^n$ of (17) that satisfy for each $j \in \{m + 1, ..., n\}$ and for all $t \in [0, T]$:

$$x_j(t) \in I_j \tag{20a}$$

and

$$y_j(t) \in I_j. \tag{20b}$$

By (17), (18) and (20), we conclude that for almost every $s \in [0, T]$,

$$(Px(s) - Py(s)) \cdot (P\dot{x}(s) - P\dot{y}(s))$$

$$+ \sum_{j=m+1}^{n} (G_j(x_j(s)) - G_j(y_j(s)) (\dot{x}_j(s) - \dot{y}_j(s))$$

$$\leq L(s) \| Px(s) - Py(s) \|^2. \tag{21}$$

For each $t \in [0, T]$, we integrate both members of (21) from 0 to t and use the separation inequality (3) to obtain

$$\frac{1}{2} \| Px(t) - Py(t) \|^2 - \frac{1}{2} \| Px(0) - Py(0) \|^2$$

$$+ \sum_{j=m+1}^{n} (h_j(x_j(t), y_j(t)) - h_j(x_j(0), y_j(0)))$$

$$\leq \int_0^t L(s) \| Px(s) - Py(s) \|^2 ds. \tag{22}$$

In this relation, we have put for each $j \in \{m + 1, ..., n\}$ and $\xi, \eta \in I_j$.

$$h_j(\xi,\eta) := h_{G_j}(\xi, \eta). \tag{23}$$

Hence, we can write

$$\frac{1}{2} \| Px(t) - Py(t) \|^2 + \sum_{j=m+1}^{n} h_j(x_j(t), y_j(t))$$

$$\leq \frac{1}{2} \| Px(0) - Py(0) \|^2 + \sum_{j=m+1}^{n} h_j(x_j(0), y_j(0))$$

$$+ \int_0^t L(s) \| Px(s) - Py(s) \|^2 ds, \tag{24}$$

and, noting that

$$2h_j(\xi, \eta) \geq 0 \quad \text{for all} \quad \xi, \eta \in I_j \text{ and } j \in \{m+1, \ldots, n\}, \tag{25}$$

we conclude that

$$\Psi(t) \leq \Psi(0) + \int_0^t 2L(s)\, \Psi(s) ds, \tag{26}$$

where we have put for each $t \in [0, T]$

$$\Psi(t) := \| Px(t) - Py(t) \|^2 + 2 \sum_{j=m+1}^{n} h_j(x_j(t), y_j(t)). \tag{27}$$

A standard argument applied to (26) yields the <u>Gronwall inequality</u>:

$$\Psi(t) \leq \Psi(0) \exp\left(\int_0^t 2L(s) ds \right), \quad \text{for all } t \in [0, T], \tag{28}$$

<u>with</u> $\Psi(t)$ <u>given by</u> (27) <u>and with</u> $x : [0, T] \longrightarrow \mathbb{R}^n$ <u>and</u> $y : [0, T] \longrightarrow \mathbb{R}^n$ <u>two classical solutions of</u> (17) <u>that satisfy</u> (20).

We note that (28) remains valid when relation (18) in (WL2) is replaced by the weaker condition

$$(Px - Py) \cdot (Pf(t,x) - Pf(t,y))$$

$$+ \sum_{j=m+1}^{n} (G_j(x_j) - G_j(y_j))\,(f_j(t,x) - f_j(t,y))$$

$$\leq L(t)\,(\| Px - Py \|^2 + 2 \sum_{j=m+1}^{n} h_j(x_j, y_j)). \tag{18$'$}$$

4. A Theorem on Restricted Continuous Dependence

When f is one–sided Lipschitzian on D, then we may put $m = n$ in (WL1) and in (WL2); the set $\{m + 1, ..., n\}$ is then empty, the sums from $m + 1$ to n in (27) and (28) are zero, and the Gronwall inequality (28) becomes

$$\| x(t) - y(t) \|^2 \leq \| x(0) - y(0) \|^2 \exp\left(\int_0^t 2L(s)ds\right), \tag{29}$$

which is the classical Gronwall inequality for one–sided Lipschitzian mappings. Because $m = n$ in the case of one–sided Lipschitzian functions, the condition that x and y satisfy (20) places no additional restriction on the functions x and y beyond the requirement that x and y are classical solutions of (17). Therefore, (29) implies the following classical theorem on continuous dependence of solutions: if f is one–sided Lipschitzian then for each classical solution $x : [0, T] \longrightarrow \mathbb{R}^n$ of (17) and each sequence $p \longmapsto y^p : [0, T] \longrightarrow \mathbb{R}^n$ of classical solutions of (17), if $\lim_{p\to\infty} y^p(0) = x(0)$, then the sequence $p \longmapsto y^p$ converges uniformly to x on $[0, T]$.

We now shall prove a theorem on "restricted continuous dependence of solutions "when f is only weakly Lipschitzian.

Theorem 4.1: Let $f : D \longrightarrow \mathbb{R}^n$ be weakly Lipschitzian on D, let $x : [0, T] \longrightarrow \mathbb{R}^n$ be a classical solution of (17) that satisfies (20a), and let $p \longmapsto y^p : [0, T] \longrightarrow \mathbb{R}^n$ be a sequence of classical solutions (17) satisfying (20b) for each $p \in \mathbb{N}$ as well as the condition

$$\lim_{p\to\infty} y^p(0) = x(0). \tag{30}$$

It follows that for every $t \in [0, T]$,

$$\lim_{p\to\infty} y^p(t) = x(t). \tag{31}$$

If, in addition, there is a positive number A such that, for all $j \in \{m+1, ..., n\}$ and all $\xi, \eta \in I_j$ with $\xi \neq \eta$, there holds

$$\frac{G_j(\xi) - G_j(\eta)}{\xi - \eta} \geq A, \tag{32}$$

then the convergence in (31) is uniform on $[0, T]$.

Proof: If we put for each $t \in [0, T]$ and $p \in \mathbb{N}$

$$\Psi^p(t) := \| Px(t) - Py^p(t) \|^2 + 2 \sum_{j=m+1}^{n} h_j(x_j(t), y_j^p(t)), \tag{33}$$

then (28) applies with y replaced by y^p, and we can write

$$\Psi^p(t) \leq \Psi^p(0) \exp \left(\int_0^t 2L(s)ds\right) \tag{34}$$

for all $t \in [0, T]$ and $p \in \mathbb{N}$. Therefore, for each $t \in [0, T]$ we conclude from (30), (33), and (34) that $\lim_{p\to\infty} \Psi^p(t) = 0$ and, because every term on the right–hand side of (33) is non–negative,

$$\lim_{p\to\infty} \| Px(t) - Py^p(t) \|^2 = 0 \tag{35}$$

and, for each $j \in \{m+1, ..., n\}$,

$$\lim_{p \to \infty} h_j(x_j(t), y_j^p(t)) = 0. \tag{36}$$

Thus, in order to verify (31), it suffices to show for each $j \in \{m+1, ..., n\}$ that (36) implies

$$\lim_{p \to \infty} y_j^p(t) = x_j(t). \tag{37}$$

Suppose, on the contrary, that (37) fails to hold for some $j \in \{m+1, ..., n\}$. Specifically, we suppose that there is a subsequence $p' \longmapsto y_j^{p'}(t)$ of $p \longmapsto y_j^p(t)$ and $\delta > 0$ such that for every p' there holds

$$y_j^{p'}(t) \geq x_j(t) + \delta. \tag{38}$$

Suppose in addition that G_j is concave. Relations (23) and (1) then yield for all p'

$$h_j(x_j(t), y_j^{p'}(t)) = \int_{x_j(t)}^{y_j^{p'}(t)} (G_j(y) - G_j(x_j(t))dy \geq \int_{x_j(t)}^{x_j(t)+\delta} (G_j(y) - G_j(x_j(t))dy > 0 \tag{39}$$

which contradicts (36). A contradiction can be obtained in a similar manner when (38) is replaced by

$$y_j^{p'}(t) \leq x_j(t) - \delta.$$

or when G_j is convex. Thus, (37) holds for each $j \in \{m+1, ..., n\}$ and (31) is verified.

To verify that the convergence in (31) is uniform when (32) holds, we let $j \in \{m+1, \ldots, n\}$ be given, assume G_j is concave, and note that for all $t \in [0, T]$ and $p \in \mathbb{N}$ we have

$$h_j(x_j(t), y_j^p(t)) = \int_{x_j(t)}^{y_j^p(t)} (G_j(y) - G_j(x_j(t))dy$$

$$\geq \int_{x_j(t)}^{y_j^p(t)} A\,(y - x_j(t))dy = \frac{A}{2}(y_j^p(t) - x_j(t))^2 \qquad (40)$$

if $y_j^p(t) \geq x_j(t)$, and we have

$$h_j(x_j(t), y_j^p(t)) = \int_{y_j^p(t)}^{x_j(t)} (G_j(x) - G_j(y^p(t))dx$$

$$\geq \int_{y_j^p(t)}^{x_j(t)} A(x - y_j^p(t))dx = \frac{A}{2}(x_j(t) - y_j^p(t))^2 \qquad (41)$$

if $x_j(t) \geq y_j^P(t)$. (The proof when G_j is convex is similar.) Relations (33), (34), (40) and (41) then imply for every $t \in [0, T]$ and $p \in \mathbb{N}$

$$\| Px(t) - Py^p(t) \|^2 + A \sum_{j=m+1}^{n} | x_j(t) - y_j^p(t) |^2 \leq \Psi^p(0) \exp \left(\int_0^t 2L(s)ds\right). \qquad (42)$$

Relations (23), (30), (42), and the fact that A is positive and L is non–negative then yield the desired conclusion. ■

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