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Ana Cristina. Barroso
Carnegie Mellon University

Irene Fonseca

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Anisotropic Singular Perturbations - The Vectorial Case

Ana Cristina Barroso
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

and

Irene Fonseca
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

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Abstract. We obtain the $r(L^*(Q)) - \lim$ of the sequence

$$J_\varepsilon(u) = \frac{1}{\varepsilon} E_\varepsilon(u)$$

where $E_\varepsilon$ is the family of anisotropic perturbations

$$E_\varepsilon(u) := \int W(u(x)) \, dx + \varepsilon^2 \int h^*(x, V_u(x)) \, dx$$

of the nonconvex functional of vector-valued functions

$$E_\varepsilon(u) = \int \Delta W(u(x)) \, dx.$$ 

The proof relies on the blow-up argument introduced by Fonseca and Müller [FM1].

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1. Introduction.

In this paper we obtain the $\Gamma(L^1(\Omega))$ - limit of a family of anisotropic singular perturbations of a nonconvex functional in the vector-valued case. The study of this problem was motivated by the analysis of variational problems for phase transitions.

We consider the nonconvex energy

$$E(u) = \int_{\Omega} W(u(x)) \, dx$$  \hspace{1cm} (1.1)

where $\Omega$ is an open, bounded, strongly Lipschitz domain of $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}^P$ and $W$ supports two phases. The problem

\begin{align*}
\text{(P) minimize } E(\cdot) \text{ subject to the constraint } \\
\frac{1}{\text{meas}(\Omega)} \int_{\Omega} u(x) \, dx = m, \text{ where } m = \theta a + (1-\theta)b \text{ for some } \theta \in (0,1) \tag{1.2}
\end{align*}

has infinitely many solutions which are piecewise constant functions of bounded variation, $u = \chi_A a + (1-\chi_A)b$ with $\text{meas}(A) = \theta \text{meas}(\Omega)$. In order to determine a selection criterion for resolving this non-uniqueness one studies the properties of the limits of sequences of minimizers for the quasiconvex perturbed problems

$$E_\varepsilon(u) = \int_{\Omega} [W(u(x)) + \varepsilon^2 h^2(x, \nabla u(x))] \, dx$$  \hspace{1cm} (1.3)

where the relevant notion of convergence in this context is $\Gamma$-convergence as introduced by De Giorgi [DG] (see [At], [DM], [DD] for more recent expositions). Hence we are lead to the problem of identifying the $\Gamma(L^1(\Omega))$ - limit of the rescaled energies

$$J_\varepsilon(u) := \frac{1}{\varepsilon} E_\varepsilon(u).$$

We show that if $W$ satisfies a certain growth condition and attains the minimum value of zero at exactly two points $a$ and $b$ and if $h$ grows at most linearly in the last argument and satisfies some technical continuity conditions (see Section 2) then the $\Gamma(L^1(\Omega))$ - limit of $J_\varepsilon(\cdot)$ is given by

$$J_0(u) = \begin{cases} 
\int_{\Omega^*} K(x, a, b, v(x)) \, dH^{N-1}(x) & \text{if } u \in \{a, b\} \text{ a.e., } u \in BV \\
+\infty & \text{otherwise}
\end{cases}$$

where $v(x)$ is the normal to the interface $\Omega \cap \partial^* \{u=a\}$,

$$K(x, a, b, v(x)) := \inf \left\{ \int_{\Omega^*} \left[ LW(\xi(y)) + \frac{1}{L}(h^*)^2(x, \nabla \xi(y)) \right] \, dy : \xi \in A(a, b, v(x)), L > 0 \right\},$$

$$A(a, b, v) := \{ \xi \in H^1(Q_v ; \mathbb{R}^P) : \xi(y) = a \text{ if } y \cdot v = -1/2, \xi(y) = b \text{ if } y \cdot v = 1/2, \text{ and } \xi \text{ is periodic with period one in the directions of } v_1, \ldots, v_{N-1} \}.$$
$\{v_1, \ldots, v_{N-1}, v\}$ forms an orthonormal basis of $\mathbb{R}^N$, $Q_v$ is the open unit cube centered at the origin with two of its faces normal to $v$ and the recession function $h^{\infty}$ is given by (see [FM2])

$$h^{\infty}(x,A) := \limsup_{t \to +\infty} \frac{h(x,tA)}{t}.$$

We will also show that a sequence of minimizers of (1.3) will single out the solution of (P) for which

$$\int_{\Omega \cap \partial \{u=a\}} K(x,a,b,v(x)) \, dH_{N-1}(x)$$

is a minimum, recovering the Wulff shape as the preferred equilibrium configuration (see [Fo], [FM3], [T1], [T2], [W]).

As remarked by Gurtin [G2], the assumption that $W$ has two potential wells of equal depth involves no loss of generality; indeed, because of the constraint (1.2) we can always add an affine function of $u$ to the integrand in (1.1) without changing the solution set of (P).

In the isotropic scalar case, i.e. if $u : \Omega \to \mathbb{R}$ and $h = I_1\|\cdot\|$, the $\Gamma(L^1(\Omega))$ limit of $J_\varepsilon(\cdot)$, $J_0(\cdot)$, was studied by Gurtin [G1], [G2] and Modica [Mo] who showed that

$$J_0(u) := \inf_{\{u_n\}} \left\{ \liminf_{n \to +\infty} \int_{\Omega} f(x,u_n(x),\nabla u_n(x)) \, dx : u_n \in W^{1,1}(\Omega;\mathbb{R}), u_n \to u \text{ in } L^1 \right\}$$

where $f(x,u,A) = 2\sqrt{W(u)} \, h(A)$. This result was generalized by Owen and Sternberg [OS] to anisotropic functions $h$ with linear growth for which $h^2$ is convex. The isotropic vector valued case, i.e. if $u : \Omega \to \mathbb{R}^p (p>1)$ and $h = I_1\|\cdot\|$, was studied by Kohn and Sternberg [KS], by Sternberg [S] and by Fonseca and Tartar [FT] who obtained the representation

$$J_0(u) := \left\{ \begin{array}{ll} K \text{ Per}_\Omega(\{u=a\}) & \text{if } u(x) \in \{a,b\} \text{ a.e.} \\ +\infty & \text{otherwise} \end{array} \right.$$ 

where

$$K = 2 \inf \left\{ 1 \int_{-1}^1 \sqrt{W(s)} |g'(s)| \, ds : g \text{ is piecewise } C^1, g(-1) = a, g(1) = b \right\}.$$

The paper is organized as follows; in Section 2 we mention some results on functions of bounded variation and sets of finite perimeter and state the theorem characterizing the $\Gamma$ - limit of the functionals $J_\varepsilon$ (see Theorem 2.9). In Section 3 we obtain a lower bound for the $\Gamma$ - limit and in Section 4 we conclude the proof of Theorem 2.9 by constructing sequences $u_n \in H^1(\Omega;\mathbb{R}^p)$ and $\varepsilon_n \to 0^+$ such that

$$\lim_{n \to +\infty} J_{\varepsilon_n}(u_n) = J_0(u).$$

The results of Sections 3 and 4 rely on a lemma (cf. Lemma 3.2) which allows us to modify a sequence near the boundary without increasing its total energy. In Section 5 we show that the $\Gamma$ - limit of a sequence of minimizers of (1.3) selects the solution of (P) which minimizes the integral over the interface of the surface energy density.

In what follows $\Omega \subset \mathbb{R}^N$ is an open, bounded, strongly Lipschitz domain, $p$, $N \geq 1$, \( \{e_1, \ldots, e_N\} \) is the standard orthonormal basis of $\mathbb{R}^N$ and $M^{pN}$ is the vector space of all $p \times N$ real matrices. If $A \in M^{pN}$ let $\|A\| := (\text{tr}(A^T A))^{1/2}$.

Given $v \in S^{N-1} := \{ x \in \mathbb{R}^N : \|x\| = 1 \}$ we denote by $Q_v$ the open unit cube centered at the origin with two of its faces normal to $v$, i.e. if $\{v_1, \ldots, v_{N-1}, v\}$ is an orthonormal basis of $\mathbb{R}^N$ then

$$Q_v := \{ x \in \mathbb{R}^N : \|x \cdot v\| < \frac{1}{2}, \|x \cdot v\| < \frac{1}{2}, \text{ i} = 1, \ldots, N-1 \}.$$ 

**Definition 2.1.** ([DG]) $J_0(.)$ is the $\mathcal{H}^{\text{lip}}(\Omega)$ - limit of the sequence $J_{\varepsilon}(.)$ if and only if

i) given any $u \in L^1(\Omega; \mathbb{R}^P)$ and any sequence $u_\varepsilon$ such that $u_\varepsilon \rightharpoonup u$ in $L^1(\Omega; \mathbb{R}^P)$,

$$J_0(u) \leq \liminf_{\varepsilon \to 0^+} J_{\varepsilon}(u_\varepsilon);$$

ii) given any $u \in L^1(\Omega; \mathbb{R}^P)$ there exists a sequence $u_\varepsilon \rightharpoonup u$ in $L^1(\Omega; \mathbb{R}^P)$ such that

$$J_0(u) = \lim_{\varepsilon \to 0^+} J_{\varepsilon}(u_\varepsilon).$$

We recall briefly some facts on functions of bounded variation and sets of finite perimeter which will be of later use in this paper. For more details we refer the reader to Evans and Gariepy [EG], Federer [F], Giusti [G] and Ziemer [Z].

**Definition 2.2.** A function $u \in L^1(\Omega; \mathbb{R}^P)$ is said to be of bounded variation, $u \in BV(\Omega; \mathbb{R}^P)$, if for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, N\}$ there exists a Radon measure $\mu_{ij}$ such that

$$\int_{\Omega} u_i(x) \frac{\partial \phi}{\partial x_j}(x) \, dx = - \int_{\Omega} \phi(x) \, d\mu_{ij}$$

for every $\phi \in C_0^1(\Omega)$. The distributional derivative $Du$ is the matrix-valued measure with components $\mu_{ij}$.

**Definition 2.3.** A set $A \subset \Omega$ is said to be of finite perimeter in $\Omega$ if $\chi_A \in BV(\Omega)$, where $\chi_A$ denotes the characteristic function of $A$. The perimeter of $A$ in $\Omega$ is defined by

$$\text{Per}_\Omega(A) := \sup \{ \int_A \text{div} \phi(x) \, dx : \phi \in C_0^1(\Omega; \mathbb{R}^N), \|\phi\|_{L^\infty} \leq 1 \}. \quad (2.1)$$
The approximate upper and lower limit of each component \( u^i \), for all \( i \in \{1, ..., p\} \), are given by

\[
\begin{align*}
u^*_i(x) &:= \inf \{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \varepsilon N \{ u^i > t \cap B(x, \varepsilon) \} = 0 \} \\
\end{align*}
\]

and

\[
\begin{align*}
u^T(x) &:= \sup \{ t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \varepsilon N \{ u^i < t \cap B(x, \varepsilon) \} = 0 \}
\end{align*}
\]

where \( B(x, \varepsilon) \) is the open ball centered at \( x \) and with radius \( \varepsilon \). The set \( X(u) \) is called the singular set of \( u \) or jump set and is defined as

\[
K_u = \{ x \in \Omega : u^T(x) < u^f(x) \}.
\]

It is well known that \( \varepsilon(u) \) is N-1 rectifiable, i.e.

\[
\mathcal{H}^1(X(u)) = \mathcal{H}^1(K_u) \cap \mathcal{H}^1(E)
\]

where \( \mathcal{H}^1(E) = 0 \) and \( K_u \) is a compact subset of a \( C^1 \) hypersurface.

**Theorem 2.4.** If \( u \in BV(\mathbb{R}^2; [0,1]) \) then for \( \mathcal{H}^1 \) a.e. \( x \in X(u) \) there exists a unit vector \( v(x) \in S^{N-1} \), normal to \( \varepsilon(u) \) at \( x \), and there exist vectors \( u^-(x), u^+(x) \in \mathbb{R}^p \) such that

\[
\begin{align*}
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon N} \int_{\{ y \in B(x, \varepsilon) : (y-x).v(x) > 0 \}} |u(y) - u^+(x)|^{N/(N-1)} dy &= 0, \\
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon N} \int_{\{ y \in B(x, \varepsilon) : (y-x).v(x) < 0 \}} |u(y) - u^-(x)|^{N/(N-1)} dy &= 0.
\end{align*}
\]

We note that it may happen that \( u^*_i(x) \neq (u^+(x))_i \).

If \( u_n \in BV(\Omega; [0,1]) \) converges to \( u \) in \( L^1(\Omega; [0,1]) \) then

\[
\text{Id}_u(Q) \leq \liminf_{n \to \infty} \text{Id}_{u_n}(Q)
\]

where \( \text{Id}_u \) denotes the total variation measure of \( Du \). If \( u \in BV(\Omega; [0,1]) \) then \( Du \) may be represented as

\[
Du = Vu dx + (u^+ - u^-) \otimes v d\mathcal{H}^N![I(u) + C(u)]
\]

where \( Vu \) is the density of the absolutely continuous part of \( Du \) with respect to the \( N \) dimensional Lebesgue measure \( \mathcal{L}^N \) and \( H_{N,i} \) is the \( N \)-dimensional Hausdorff measure. The three measures in (2.3) are mutually singular, if \( H_{N,i}(B) < +\infty \) then \( IC(u)l = 0 \) and there exists a Bonel set \( E \) such that \( J\mathcal{L}^N(E) = 0 \) and \( IC(u)l(B) = IC(u)l(B \cap E) \) for all Borel sets \( BC D^N \). The following version of the Besicovitch Differentiation Theorem was proven by Ambrosio and Dal Maso, [ADM] Proposition 2.2.
Theorem 2.5. If $\lambda$ and $\mu$ are Radon measures in $\Omega$, $\mu \geq 0$, then there exists a Borel set $E$ \( \subset \Omega \) such that $\mu(E) = 0$ and for every $x \in (\text{supp } \mu) \setminus E$

\[
\frac{d\lambda}{d\mu}(x) := \lim_{\varepsilon \to 0^+} \frac{\lambda(x+\varepsilon C)}{\mu(x+\varepsilon C)}
\]

exists and is finite whenever $C$ is a bounded, convex, open set containing the origin.

Theorem 2.6. Let $A$ be a subset of $\Omega$ such that $\text{Per}_\Omega(A) < +\infty$. There exists a sequence of polyhedral sets \{\(A_k\)\} (i.e. $A_k$ are bounded, strongly Lipschitz domains with $\partial A_k = H_1 \cup H_2 \cup \ldots \cup H_p$ where each $H_i$ is a closed subset of a hyperplane $\{x \in \mathbb{R}^N : x \cdot v_i = \alpha_i\}$) satisfying the following properties:

i) $\mathcal{L}_N[((A_k \cap \Omega) \setminus A) \cup (A \setminus (A_k \cap \Omega))] \to 0$ as $k \to +\infty$;

ii) $\text{Per}_\Omega(A_k) \to \text{Per}_\Omega(A)$ as $k \to +\infty$;

iii) $H_{N-1}(\partial A_k \cap \partial \Omega) = 0$;

iv) $\mathcal{L}_N(A_k) = \mathcal{L}_N(A)$.

It can be shown that (see [FM2], Lemma 2.6) if $\text{Per}_\Omega(A) < +\infty$ then for $H_{N-1}$ a.e. $x \in \Omega \cap \partial^* A$

\[
\lim_{\delta \to 0^+} \frac{1}{\delta^{N-1}} H_{N-1}((\Omega \cap \partial^* A) \cap (x+\delta Qv(x))) = 1. \tag{2.4}
\]

Let $W : \mathbb{R}^p \to [0, +\infty)$ and $h : \Omega \times \mathcal{M}^{p*} \to [0, +\infty)$ be continuous functions satisfying the following hypotheses:

(H1) $W(u) = 0$ if and only if $u \in \{a,b\}$;

(H2) there exist constants $c_1, c > 0$ such that

\[
c_1 \|u\|^q - c \leq W(u) \leq c (1 + \|u\|^q)
\]

for all $u \in \mathbb{R}^p$ and for some $q \geq 2$;

(H3) there exist constants $C_1, C > 0$ such that

\[
C_1 \|A\| - C \leq h(x, A) \leq C (1 + \|A\|)
\]

for all $x \in \Omega$ and for all $A \in \mathcal{M}^{p*}$.

Let $h^\infty : \Omega \times \mathcal{M}^{p*} \to [0, +\infty)$ be the recession function, i.e.

\[
h^\infty(x, A) := \limsup_{t \to +\infty} \frac{h(x, tA)}{t}.
\]

In addition to (H1)-(H3) we will also need the following hypotheses:

(H4) there exist $0 < m < 2, C, L > 0$ such that

\[
\| h^\infty \|^2(x, A) - \frac{h^2(x, tA)}{t^2} \leq C \frac{\|A\|^{2-m}}{t^m}
\]

for all $(x, A) \in \Omega \times \mathcal{M}^{p*}$ and for all $t > 0$ such that $t \|A\| > L$;

(H5) for all $x_0 \in \Omega$ and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that
\[ | h^2(x_0, A) - h^2(x, A) | \leq \varepsilon C (1 + \| A \|^2) \]

whenever \( |x - x_0| < \delta \).

It is an easy consequence of the definition of recession function that

**Lemma 2.7.** Under the hypotheses \((H3)\) and \((H5)\) the following hold:

i) \( C, \| A \| \leq h^\infty(x, A) \leq C \| A \|, \) for every \((x, A) \in \Omega \times M_{p_1}^{p_2} \);

ii) For all \( x_0 \in \Omega \) and for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ | (h^\infty)^2(x_0, A) - (h^\infty)^2(x, A) | \leq \varepsilon C \| A \|^2 \]

whenever \( |x - x_0| < \delta \).

Let \((a, b, v) \in \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1}\), let \( \{v_1, \ldots, v_{N-1}, v\} \) form an orthonormal basis of \( \mathbb{R}^N \) and define the class of admissible functions

\[ \mathcal{A}(a, b, v) := \{ \xi \in H^1(Q_v; \mathbb{R}^p) : \xi(y) = a \text{ if } y \cdot v = -1/2, \xi(y) = b \text{ if } y \cdot v = 1/2, \text{ and } \xi \text{ is periodic with period one in the directions of } v_1, \ldots, v_{N-1} \}, \]

where the boundary values of \( \xi \) are understood in the sense of traces. A function \( \xi \) is said to be *periodic with period one in the direction of* \( v \), if

\[ \xi(y) = \xi(y + kv_i) \]

for all \( k \in \mathbb{Z}, y \in Q_v \).

Our surface energy density \( K : \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1} \rightarrow [0, +\infty) \) is defined by

\[ K(x, a, b, v) := \inf \left\{ \int_{Q_v} [LW(\xi(y)) + \frac{1}{L}(h^\infty)^2(x, \nabla \xi(y))] \, dy : \xi \in \mathcal{A}(a, b, v), L > 0 \right\}. \]

We examine some continuity properties of \( K \). In what follows \( C \) denotes a generic constant.

**Proposition 2.8.** Under the hypotheses \((H2), (H3)\) and \((H5)\) we have:

i) \( 0 \leq K(x, a, b, v) \leq C (1 + \| A \|^q + \| b \|^q + \| b - a \|^2) \) for all \((x, a, b, v) \in \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1} \);

ii) For all \( x_0 \in \Omega \) and for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |x - x_0| < \delta \) implies

\[ |K(x, a, b, v) - K(x_0, a, b, v)| < \varepsilon C (1 + \| A \|^q + \| b \|^q + \| b - a \|^2). \]

**Proof.** We follow here the proof of Fonseca and Rybka [FR].

i) Fix \((x, a, b, v) \in \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1} \) and let

\[ \xi(y) = (b - a) (y \cdot v) + \frac{a + b}{2}. \]

Clearly \( \xi \in \mathcal{A}(a, b, v) \) so, by \((H2)\) and Lemma 2.7 i),

\[ K(x, a, b, v) = K(x_0, a, b, v) = \int_{Q_v} [LW(\xi(y)) + \frac{1}{L}(h^\infty)^2(x, \nabla \xi(y))] \, dy \]

for all \( x \in \Omega \times \mathbb{R}^p \times \mathbb{R}^p \times S^{N-1} \).
0 \leq K(x, a, b, v) \leq \int_{Q} [LW(\xi(y)) + \frac{1}{L}(h^\omega)^2(x, \nabla \xi(y))] \, dy \leq \\

\leq \int_{Q} \left[LC(1 + \|\xi(y)\|^q) + \frac{C}{L} \|\nabla \xi(y)\|^2 \right] \, dy \leq \text{const.} \left(1 + \|a\|^q + \|b\|^q + \|\omega\|_{-2}^2\right),

\text{since } \|\xi(y)\| \leq \frac{1}{2}\|\omega\|_{-1} + \frac{1}{2}\|\omega\|_{-2} + \|\omega\|_{-2} \leq \|a\| + \|b\|.

ii) Fix \( x_0 \in \Omega \) and \( \varepsilon > 0 \). By Lemma 2.7 ii) choose \( \delta > 0 \) such that \( |x - x_0| < \delta \) implies

\begin{equation}
(\omega)^2(x_0, A) - (\omega)^2(x, A) \leq \varepsilon \|\omega\|_{-2}.
\end{equation}

For all \( n \in \mathbb{N} \) choose \( \xi_n \in \mathcal{S}(a, b, \omega) \), \( L_n > 0 \) such that

\begin{equation}
\int_{Q} \left[L_n W(\xi_n(y)) + \frac{1}{L_n}(\omega)^2(x_0, \nabla \xi_n(y))\right] \, dy \leq K(x_0, a, b, \omega) + \frac{1}{n}.
\end{equation}

By Lemma 2.7 i) it follows that

\begin{equation}
\int_{Q} \frac{C}{L_n} \|\nabla \xi_n(y)\|^2 \, dy \leq \int_{Q} \frac{1}{L_n}(\omega)^2(x_0, \nabla \xi_n(y)) \, dy \leq K(x_0, a, b, \omega) + \frac{1}{n},
\end{equation}

and so

\begin{equation}
\int_{Q} \frac{1}{L_n} \|\nabla \xi_n(y)\|^2 \, dy \leq \frac{K(x_0, a, b, \omega) + \frac{1}{n}}{C} \leq \text{const.} \left(1 + \|a\|^q + \|b\|^q + \|\omega\|_{-2}^2\right).
\end{equation}

Hence, if \( |x - x_0| < \delta \), by (2.5) we have

\begin{equation}
K(x, a, b, \omega) - K(x_0, a, b, \omega) \leq \int_{Q} \frac{1}{L_n}(\omega)^2(x, \nabla \xi_n(y)) \, dy - \int_{Q} \frac{1}{L_n}(\omega)^2(x_0, \nabla \xi_n(y)) \, dy + \frac{1}{n} \leq \\

\leq \int_{Q} \left[(\omega)^2(x, \nabla \xi_n(y)) - (\omega)^2(x_0, \nabla \xi_n(y))\right] \, dy + \frac{1}{n} \leq \\

\leq \int_{Q} \varepsilon C \|\nabla \xi_n(y)\|^2 \, dy + \frac{1}{n} \leq \varepsilon C \left(1 + \|a\|^q + \|b\|^q + \|\omega\|_{-2}^2\right) + \frac{1}{n}.
\end{equation}

Let \( n \to \infty \) to obtain

\begin{equation}
K(x, a, b, \omega) - K(x_0, a, b, \omega) \leq \varepsilon C \left(1 + \|a\|^q + \|b\|^q + \|\omega\|_{-2}^2\right).
\end{equation}

In a similar way we obtain

\begin{equation}
K(x_0, a, b, \omega) - K(x, a, b, \omega) \leq \varepsilon C \left(1 + \|a\|^q + \|b\|^q + \|\omega\|_{-2}^2\right).
\end{equation}

The main result of this paper is the following

\begin{theorem}
Let (H1)-(H5) hold and let
\end{theorem}
\[ J_{\varepsilon}(u) = \int_{\Omega} \left[ \frac{1}{\varepsilon} W(u(x)) + \varepsilon h^2(x, \nabla u(x)) \right] \, dx. \]

Then the $\Gamma(L^1(\Omega))$-limit of the sequence $J_{\varepsilon}(.)$ is given by

\[ J_0(u) = \begin{cases} \int_{\Omega} K(x, a, b, \nabla u) \, dH_{N-1}(x) & \text{if } u \in \{a, b\} \text{ a.e., } u \in BV \\ +\infty & \text{otherwise} \end{cases} \]

We divide the proof of Theorem 2.9 into two parts. The first part, corresponding to item i) of Definition 2.1, will be shown in Section 3 and the second part is proven in Section 4.

3. A lower bound for the $\Gamma$-limit.

In this section we prove that the $\Gamma(L^1(\Omega))$-limit of $J_{\varepsilon}(.)$ is bounded below by $J_0(.)$.

**Proposition 3.1.** Let (H1)-(H5) hold and let $u \in L^1(\Omega; \mathbb{R}^P)$ be given. If $\varepsilon_n \to 0^+$ and if $u_n \in H^1(\Omega; \mathbb{R}^P)$ is such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^P)$ then

\[ \liminf_{n \to \infty} \int_{\Omega} \left[ \frac{1}{\varepsilon_n} W(u_n(x)) \right] \, dx \geq J_0(u). \]

The proof relies on the following lemma which allows us to modify a sequence near the boundary without increasing its total energy.

**Lemma 3.2.** Assume that (H1), (H2) and (H3) hold and let

\[ u_0(y) = \begin{cases} b & \text{if } y \cdot v(x) > 0 \\ a & \text{if } y \cdot v(x) < 0. \end{cases} \]

Let $\rho$ be a symmetric mollifier and set $v_n = \rho_{1/\varepsilon_n} * u_0$ where $\rho_{1/\varepsilon_n}(x) = \frac{1}{\varepsilon_n} \rho_x(\frac{x}{\varepsilon_n})$ and \{\varepsilon_n\} is a sequence of real numbers such that $\varepsilon_n \to 0^+$. If \{u_n\} is a sequence in $H^1(Q_v; \mathbb{R}^P)$ converging in $L^1(Q_v; \mathbb{R}^P)$ to $u_0$ then there exists a subsequence \{\varepsilon_{n_k}\} and a sequence \{w_k\} in $H^1(Q_v; \mathbb{R}^P)$ such that $w_k \to u_0$ in $L^1(Q_v; \mathbb{R}^P)$, $w_k = v_{n_k}$ on $\partial Q_v$ and
\[
\lim \sup_{k \to +\infty} \int_{Q_{f}} f \left[ \frac{1}{W(w_k(y))} + h^2 \mathbf{y} \cdot \nabla w_k(y) \right] \, dy \leq \\
\lim \inf_{n \to +\infty} \int_{Q_{f}} f \left[ -W(u_n(y)) + e_n h^2 \mathbf{y} \cdot \nabla u_n(y) \right] \, dy.
\]

**Proof. Step 1.** Assume, without loss of generality, that

\[
\lim \sup_{n \to +\infty} \int_{Q_{f}} f \left[ -W(u_n(y)) + e_n h^2 \mathbf{y} \cdot \nabla u_n(y) \right] \, dy = +\infty.
\]

We begin by showing that \( u_n \to u_0 \) in \( L^q(Q;\mathbb{R}^p) \). Indeed, after extracting a subsequence, we have

\[
u_n(y) \to u_0(y) \quad \text{a.e. and by (3.1),}
\]

\[
\int_{Q_{f}} J W_{f_n}(y) \, dy = \lim \inf_{n \to +\infty} \int_{Q_{f}} J W_{f_n}(y) \, dy \leq 0
\]

By (H2),

\[
\lim \sup_{n \to +\infty} \int_{Q_{f}} \|u_n(y) - u_0(y)\|^{q} C(W(u_n(y)) + 1) \, dy = 0.
\]

Also, as \( q \leq 2 \), we conclude that \( u_n \to u_0 \) in \( L^2(Q;\mathbb{R}^p) \) as \( n \to +\infty \).

**Step 2.** For simplicity assume that \( v = e_N \) and denote \( \mathcal{Q}_v = Q \). Notice that

\[
\ast \quad W = \left\{ \begin{array}{cc}
\text{if } y_N < -e_n \\
\text{if } y_N > e_n
\end{array} \right.
\]

and

\[
v_n \in \mathcal{A}(a,b,e_N), \quad \|Vv_n\|_{L^\infty} = O(d/e_n) \quad \text{and} \quad \text{supp} Vv_n \subset \{ |y_N| < e_n \}.
\]

Also, \( v_n \to u_0 \) in \( L^q(Q;\mathbb{R}^p) \). Let \( T_n = Q \setminus \Omega_n^{\text{R}} \) where \( o^\ast = (1 - \|u_n - v_n\|_1^2)^{1/2} \) so that
Let $M = c (1 + 2 \|u\|_0^d)$ where $c$ is the constant appearing in the growth condition of $W$ and define $k_n \in \mathbb{Z}^+$ as $k_n = \left\lceil \frac{2M}{\varepsilon_n \sqrt{\|u_n - v_n\|_2}} \right\rceil + 1$, where $\lceil y \rceil$ denotes the integer part of $y$. We divide $T_n$ into $k_n$ slices of measure $\frac{\text{meas } T_n}{k_n}$, $T_n = \bigcup_{j=1}^{k_n} S_j^n$ and $S_j^n$ are of the form $\lambda_{j+1}^n Q \setminus \lambda_j^n Q$ where $0 < \lambda_j^n < 1$, $\lambda_1^n = \alpha_n$ and $\lambda_{k_n+1}^n = 1$. Consider a family of smooth cut-off functions $\varphi_j^n \in C_0^\infty (\alpha_n Q \cup \bigcup_{k=1}^{j-1} S_k^n)$ such that $0 \leq \varphi_j^n \leq 1$, $\varphi_j^n = 1$ on $\alpha_n Q \cup \bigcup_{k=1}^{j-1} S_k^n$ and $\|\nabla \varphi_j^n\|_{L^\infty} = O \left( \frac{k_n}{\|u_n - v_n\|_2^{1/3}} \right)$ for $j = 1, \ldots, k_n$. Using these functions $\varphi_j^n$ we will consider convex combinations of $u_n$ with $v_n$ across the slices $S_j^n$. We claim that there exists $m \in \mathbb{N}$ such that, for all $n > m$ there exists $j \in \{1, \ldots, k_n\}$ such that

$$\int_{S_j^n} \left[ \frac{1}{\varepsilon_n} W(\varphi_j^n(y)u_n(y) + (1 - \varphi_j^n(y))v_n(y)) + \varepsilon_n h^2(y, \varphi_j^n(y)\nabla u_n(y) - \nabla v_n(y)) + \nabla v_n(y) + (u_n(y) - v_n(y)) \right] \nabla \varphi_j^n(y) \nabla \varphi_j^n(y) \right] \, dy \leq \sqrt{\|u_n - v_n\|_2}.$$ (3.3)

Assuming (3.3) holds, for each $n > m$ we obtain a slice $S^{(n)} \in \{S_j^n : j = 1, \ldots, k_n\}$, $S^{(n)} = S^{(n)}_{j(n)}$ such that, setting

$$w_n(y) = \varphi^{(n)}(y)u_n(y) + (1 - \varphi^{(n)}(y))v_n(y) = \begin{cases} v_n & \text{if } y \in B^{(n)} \\ u_n & \text{if } y \in A^{(n)} \end{cases}$$

where $B^{(n)} = \bigcup_{j=j(n)+1}^{k_n} S_j^n$ and $A^{(n)} = Q \setminus (B^{(n)} \cup S^{(n)})$, then

$$\lim_{n \to \infty} \int_Q \left[ \frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y, \nabla u_n(y)) \right] \, dy \geq \limsup_{n \to \infty} \int_{A^{(n)}} \left[ \frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y, \nabla u_n(y)) \right] \, dy = \limsup_{n \to \infty} \left[ \int_Q \left[ \frac{1}{\varepsilon_n} W(w_n(y)) + \varepsilon_n h^2(y, \nabla w_n(y)) \right] \, dy - \int_{B^{(n)}} \left[ \frac{1}{\varepsilon_n} W(v_n(y)) + \varepsilon_n h^2(y, \nabla v_n(y)) \right] \, dy \right]$$

$$- \int_{S^{(n)}} \left[ \frac{1}{\varepsilon_n} W(\varphi^{(n)}(y)u_n(y) + (1 - \varphi^{(n)}(y))v_n(y)) + \varepsilon_n h^2(y, \varphi^{(n)}(y)\nabla u_n(y) - \nabla v_n(y)) + \nabla v_n(y) \right]$$

$$+ (u_n(y) - v_n(y)) \nabla \varphi^{(n)}(y) \nabla \varphi^{(n)}(y) \right] \, dy \leq \sqrt{\|u_n - v_n\|_2}.$$ (3.3)
\[+(u_n(y) - v_n(y)) \otimes \nabla \varphi^{(n)}(y)) \] dy\].

By (3.3) and Step 1 the last term is bounded by \(\sqrt{\|u_n - v_n\|_2}\) which goes to zero as \(n \to +\infty\). We show that the second term also goes to zero. Indeed, by (H1), (H2), (H3), (3.2) and since \(\{v_n\}\) is bounded in \(L^\infty\) we have

\[
\int_{B_0^0} \left[ \frac{1}{\varepsilon_n} W(v_n(y)) + \varepsilon_n h^2(y, \nabla v_n(y)) \right] dy \leq \int_{T_n} \left[ \frac{1}{\varepsilon_n} W(v_n(y)) + \varepsilon_n h^2(y, \nabla v_n(y)) \right] dy \leq
\]

\[
\leq \frac{1}{\varepsilon_n} \int_{T_n \cap \{y_n < \varepsilon_n\}} C dy + \int_{T_n \cap \{y_n < \varepsilon_n\}} \varepsilon_n C \|\nabla v_n(y)\|_2^2 dy + \int_{T_n} C \varepsilon_n dy \leq
\]

\[
\leq C \left[ 2(\text{meas} T_n)^{(N-1)/N} + \varepsilon_n \text{meas} T_n \right] \to 0 \text{ as } n \to +\infty.
\]

Hence,

\[
\lim_{n \to +\infty} \int_Q \left[ \frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y, \nabla u_n(y)) \right] dy \geq \limsup_{n \to +\infty} \int_Q \left[ \frac{1}{\varepsilon_n} W(w_n(y)) + \varepsilon_n h^2(y, \nabla w_n(y)) \right] dy.
\]

On the other hand,

\[
\lim_{n \to +\infty} \int_Q \|w_n(y) - u_0(y)\| dy = \lim_{n \to +\infty} \left[ \int_{A_0} \|u_n(y) - u_0(y)\| dy + \int_{B_0} \|v_n(y) - u_0(y)\| dy + \int_{S_0^0} \|\varphi(y) u_n(y) + (1 - \varphi(y)) v_n(y) - u_0(y)\| dy \right] \leq
\]

\[
\leq \lim_{n \to +\infty} \left[ \int_Q \|u_n(y) - u_0(y)\| dy + \int_Q \|v_n(y) - u_0(y)\| dy \right] = 0
\]

since \(u_n \to u_0\) and \(v_n \to u_0\) in \(L^1(Q; \mathbb{R}^p)\).

**Step 3.** It remains to show (3.3). We begin by proving that

\[
\limsup_{n \to +\infty} \sum_{j=1}^{k_n} \int_{s_j^n} W(\varphi_j^n(y) u_n(y) + (1 - \varphi_j^n(y)) v_n(y)) dy \leq M. \tag{3.4}
\]

Indeed, by (H2) and since \(u_n \to u_0\) and \(v_n \to u_0\) in \(L^q(Q; \mathbb{R}^p)\)

\[
\limsup_{n \to +\infty} \sum_{j=1}^{k_n} \int_{s_j^n} W(\varphi_j^n(y) u_n(y) + (1 - \varphi_j^n(y)) v_n(y)) dy \leq \int_{T_n} W(\varphi_j^n(y) u_n(y) + (1 - \varphi_j^n(y)) v_n(y)) dy \leq
\]

\[
\leq \limsup_{n \to +\infty} \int_{T_n} C(1 + \|\varphi_j^n(y) u_n(y) + (1 - \varphi_j^n(y)) v_n(y)\|_p^q) dy \leq
\]

11.
\[
\leq \limsup_{n \to +\infty} \int_{\mathcal{T}_n} \left[ C(1 + \|u_n(y)\|^{q_l} + \|v_n(y)\|^{q_l}) \right] dy \leq C(1 + 2\|u_0\|^{q_l}) = M.
\]

To show (3.3) we argue by contradiction. If (3.3) were false then for all \(m \in \mathbb{N}\) there would exist \(n > m\) such that for all \(j \in \{1, \ldots, k_n\}\)

\[
\int_{S_j^n} [\frac{1}{\varepsilon_n} W(\varphi_j^n(y)u_n(y) + (1 - \varphi_j^n(y))v_n(y)) + \varepsilon_n h^2(y, \varphi_j^n(y)(\nabla u_n(y) - \nabla v_n(y)) + \nabla v_n(y) + (u_n(y) - v_n(y))\nabla \varphi_j^n(y))] dy > \sqrt{\|u_n - v_n\|_2}.
\]

Then by (H3), for this subsequence

\[
\int_{S_j^n} W(\varphi_j^n(y)u_n(y) + (1 - \varphi_j^n(y))v_n(y)) dy > \varepsilon_n \sqrt{\|u_n - v_n\|_2} - \varepsilon_n^2 \int_{S_j^n} [h^2(y, \varphi_j^n(y)(\nabla u_n(y) - \nabla v_n(y)) + \nabla v_n(y) + (u_n(y) - v_n(y))\nabla \varphi_j^n(y))] dy \geq
\]

\[
\geq \varepsilon_n \sqrt{\|u_n - v_n\|_2} - C\varepsilon_n^2 \text{meas} S_j^n - C \varepsilon_n^2 \int_{S_j^n} \|\nabla u_n(y)\|^{2} dy - C \varepsilon_n^2 \int_{S_j^n} \|\nabla v_n(y)\|^{2} dy - C \frac{\varepsilon_n^2 k_n^2}{\|u_n - v_n\|_2^{2/3}} \int_{S_j^n} \|u_n(y) - v_n(y)\|^{2} dy.
\]

Summing the above inequality from \(j = 1\) to \(j = k_n\) we obtain

\[
\sum_{j=1}^{k_n} \int_{S_j^n} W(\varphi_j^n(y)u_n(y) + (1 - \varphi_j^n(y))v_n(y)) dy \geq \sum_{j=1}^{k_n} \int_{T_n} h^2(y, \varphi_j^n(y)(\nabla u_n(y) - \nabla v_n(y)) + \nabla v_n(y) + (u_n(y) - v_n(y))\nabla \varphi_j^n(y))] dy \geq
\]

\[
\geq k_n \varepsilon_n \sqrt{\|u_n - v_n\|_2} - C\varepsilon_n^2 \text{meas} T_n - C \varepsilon_n^2 \int_{T_n} \|\nabla u_n(y)\|^{2} dy - C \varepsilon_n^2 \int_{T_n} \|\nabla v_n(y)\|^{2} dy - C \varepsilon_n^2 k_n \|u_n - v_n\|_2^{2/3} \int_{T_n} \|u_n(y) - v_n(y)\|^{2} dy.
\]

By choice of \(k_n\)

\[
k_n \varepsilon_n \sqrt{\|u_n - v_n\|_2} \geq 2M
\]

and

\[
C \varepsilon_n k_n \|u_n - v_n\|_2^{2/3} \int_{T_n} \|u_n(y) - v_n(y)\|^{2} dy \leq C [2M \|u_n - v_n\|_2^{1/6} + \varepsilon_n \|u_n - v_n\|_2^{2/3}] \to 0 \text{ as } n \to +\infty.
\]
and by properties of $v_n$,
\[
C \int \| IVv_n(y) \|_2^2 \, dy \leq C \cdot \frac{1}{n} \int \frac{1}{\gamma} \, dy = C e^{\gamma \text{meas}(\Omega_{n-1})} \xrightarrow{n \to \infty} 0.
\]

Therefore, since $\int \| H V u_n(y) \|_2^2 \, dy$ is bounded, it follows that
\[
\forall \quad \bigg \{ \sup_{n < n_0} \int f \left( W(u_n(y)) + (l(p_n(y))v_n(y)) \right) \, dy \bigg \} \leq 2M
\]
contradicting (3.4).

Proof of Proposition 3.1. Step 1. We begin by proving the proposition in the case where $u = XA00a + (1-3CA00)w$ with $p = +\infty$. As $J_0(u) = +\infty$ it suffices to show that for any sequence $\varepsilon_n \to 0^+$ and for any $u_n \in H^1(\Omega; \mathbb{R}^p)$ such that $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^p)$ we have
\[
J \left[ W(u_n(x)) + e_n h_2(x, Vu_n(x)) \right] \xrightarrow{n \to \infty} +\infty.
\]

We argue by contradiction. Suppose that there exists a subsequence (which we continue to denote by $E_n$ and $u_n$ for convenience) such that
\[
J \left[ W(u_n(x)) + E_n h_2(x, Vu_n(x)) \right] \leq \text{const},
\]

Then, by the growth condition on $h$, we have
\[
f \left[ \frac{1}{\gamma} W(u_n(x)) + E_n II Vu_n(x) \right] \xrightarrow{n \to \infty} +\text{const}.
\]

which, by the Cauchy-Schwartz inequality, implies
\begin{equation}
J \left[ W(u_n(x)) \right] \| Vu_n(x) \| \xrightarrow{n \to \infty} +\text{const}.
\end{equation}

Let
\[
f(r) := \inf u^\gamma \| u \| \forall W(u),
\]

where
\[
a+b \quad \gamma
\]

If we set
\[ r_0 := \frac{a-b}{2}, \]
then by (H2) there exists \( r_1 > r_0 \) such that
\[
\int_{r_0}^{r_1} f(r) \, dr > \frac{K}{2}
\]
where
\[
K = 2 \inf \left\{ \int_{-1}^{1} \sqrt{W(g(s))} |g'(s)| \, ds \mid g \text{ is piecewise } C^1, g(-1) = a, g(1) = b \right\}.
\]
Let
\[
M := \max_{\text{u-clsr}} \sqrt{W(u)}
\]
and define
\[
\varphi(v) := \inf \left\{ \int_{-1}^{1} T(\gamma(s)) \|\gamma'(s)\| \, ds \mid \gamma \text{ is a piecewise } C^1, \gamma(-1) = a, \gamma(1) = v \right\}
\]
where
\[
T(u) := \min \{ \sqrt{W(u)}, M \}.
\]
This function was studied by Fonseca and Tartar [FT] where they showed that
i) \( \varphi : \mathbb{R}^p \rightarrow [0, +\infty) \) is a Lipschitz function
ii) if \( u \in H^1(\Omega; \mathbb{R}^p) \) then \( \varphi \circ u \in H^1(\Omega; \mathbb{R}^p) \) and \( \|\nabla(\varphi \circ u)(x)\| \leq \sqrt{W(u(x))} \|\nabla u(x)\| \text{ a.e. } x \in \Omega. \)
Hence, since \( \varphi \circ u \rightarrow \varphi \circ a = (1-\chi_A)\varphi(b) \) in \( L^1 \) strong, from the lower semicontinuity formula (2.2),
ii) and (3.5) we have
\[
\varphi(b)\text{Per}_\Omega(A) = \|D(1-\chi_A)\varphi(b)\| \leq \lim \inf_{n \to \infty} \int_\Omega \|\nabla(\varphi \circ u_n)(x)\| \, dx \leq \text{const.}
\]
contradicting the fact that \( \text{Per}_\Omega(A) = +\infty. \)

**Step 2.** We now turn to the case where \( u = \chi_A(x)a + (1-\chi_A(x))b \) with \( \text{Per}_\Omega(A) < +\infty. \)
Assume, without loss of generality, that
\[
\lim \inf_{n \to \infty} \int_\Omega \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \right] \, dx = \lim \inf_{n \to \infty} \int_\Omega \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \right] \, dx < +\infty.
\]
We must show that
\[
\lim_{n \to \infty} \int_\Omega \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \right] \, dx \geq \int_{\Omega \cap \partial'A} K(x,a,b,v(x)) \, dH_{N-1}(x). \tag{3.6}
\]
Using the blow up method introduced by Fonseca and Müller [FM1] we reduce the problem to verifying the pointwise inequality (3.8) below. As the integrands \( \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \) form a sequence of nonnegative functions bounded in \( L^1 \) there exists a subsequence (still denoted by \( \varepsilon_n \) and \( u_n \)) and a nonnegative Radon measure \( \mu \) such that
\[
\frac{1}{\varepsilon_n} W(u_n(.)) + \varepsilon_n h^2(., \nabla u_n(.)) \to \mu \text{ weakly * in the sense of measures}
\]
i.e. for all \( \varphi \in C_0(\Omega) \)
\[
\int_{\Omega} \varphi(x) \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \right] \, dx \to \int_{\Omega} \varphi \, d\mu \text{ as } n \to \infty. \tag{3.7}
\]
Using the Radon-Nikodym theorem we may write \( \mu \) as a sum of two mutually singular nonnegative measures \( \mu = \mu_s H_{N-1}(\Omega \cap \partial^* A) + \mu_s \). We claim that
\[
\mu_s(x_0) \geq K(x_0, a, b, v(x_0)) \text{ for } H_{N-1} \text{ a.e. } x_0 \in \Omega \cap \partial^* A. \tag{3.8}
\]
Assuming that (3.8) holds we consider an increasing sequence of smooth cut-off functions, \( \varphi_k \in C_0(\Omega) \), with \( 0 \leq \varphi_k \leq 1 \) and \( \sup_k \varphi_k(x) = 1 \) in \( \Omega \) and we obtain
\[
\lim_{n \to \infty} \int_{\Omega} \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \right] \, dx \geq \lim_{n \to \infty} \int_{\Omega} \varphi_k(x) \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \right] \, dx \to \int_{\Omega} \varphi_k(x) \, d\mu((x)) \geq \int_{\Omega} \varphi_k(x) \mu_s(x) \, dH_{N-1}(\Omega \cap \partial^* A) \geq \int_{\Omega} \varphi_k(x) K(x, a, b, v(x)) \, dH_{N-1}(x).
\]
Letting \( k \to +\infty \) and using the Monotone Convergence Theorem we conclude (3.6).

**Step 3.** It remains to show inequality (3.8). By Theorems 2.4 and 2.5 for \( H_{N-1} \) a.e. \( x \in \Omega \cap \partial^* A \) we have
\[
i \lim_{\delta \to 0^+} \frac{1}{\delta} \int_{\{y \in B(x, \delta) : (y-x).v(x) > 0\}} |u(y) - bl \, dy = 0
\]
\[
\lim_{\delta \to 0^+} \frac{1}{\delta} \int_{\{y \in B(x, \delta) : (y-x).v(x) < 0\}} |u(y) - al \, dy = 0
\]
and
\[
ii) \mu_s(x) = \lim_{\delta \to 0^+} \frac{\mu(x + \delta Q_v(x))}{H_{N-1}(\Omega \cap \partial^* A)(x + \delta Q_v(x))}
\]
Choose a point $x \in \Omega \cap \partial^* A$ such that i) and ii) hold. Let $0 < \eta < 1$ and let $\varphi \in C_0^\infty(\Omega_V(x))$ be such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $\eta \Omega_V(x)$. Using ii) and (2.4) we have

$$
\mu_a(x) \geq \limsup_{\delta \to 0^+} \frac{\mu(x+\delta \Omega_V(x))}{\delta N} \geq \limsup_{\delta \to 0^+} \frac{1}{\delta N} \int_{x+\delta \Omega_V(x)} \varphi \left( \frac{y-x}{\delta} \right) \, \mathrm{d}\mu(y) = \limsup_{\delta \to 0^+} \frac{1}{\delta N} \lim_{n \to +\infty} \int_{x+\delta \Omega_V(x)} \varphi \left( \frac{y-x}{\delta} \right) \left[ \frac{1}{\varepsilon_n} W(u_n(y)) + \varepsilon_n h^2(y, \nabla u_n(y)) \right] \, \mathrm{d}y = \limsup_{\delta \to 0^+} \frac{1}{\delta N} \lim_{n \to +\infty} \int_{\Omega_V(x)} \varphi(y) \left[ \frac{1}{\varepsilon_n} W(u_n(x+\delta y)) + \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y)) \right] \, \mathrm{d}y \geq \limsup_{\delta \to 0^+} \frac{1}{\delta} \limsup_{n \to +\infty} \int_{\eta \Omega_V(x)} \left[ \frac{1}{\varepsilon_n} W(u_n(x+\delta y)) + \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y)) \right] \, \mathrm{d}y. \quad (3.9)
$$

Let

$$w_n,\delta(y) = u_n(x+\delta y), \quad u_0(y) = \begin{cases} b & \text{if } y \cdot v(x) > 0 \\ a & \text{if } y \cdot v(x) < 0 \end{cases}$$

Note that, since $u_n \to u$ in $L^1$ and by i), we have

$$\lim_{\delta \to 0^+} \lim_{n \to +\infty} \| w_n,\delta - u_0 \|_{L^1(\Omega_V(x))} = \lim_{\delta \to 0^+} \lim_{n \to +\infty} \left[ \frac{1}{\Omega_V(x) \cap \{ y : y \cdot v(x) > 0 \}} \int \| u_n(x+\delta y) - b \| \, \mathrm{d}y + \frac{1}{\Omega_V(x) \cap \{ y : y \cdot v(x) < 0 \}} \int \| u_n(x+\delta y) - a \| \, \mathrm{d}y \right] = \lim_{\delta \to 0^+} \left[ \frac{1}{\Omega_V(x) \cap \{ y : y \cdot v(x) > 0 \}} \int \| u(x+\delta y) - b \| \, \mathrm{d}y + \frac{1}{\Omega_V(x) \cap \{ y : y \cdot v(x) < 0 \}} \int \| u(x+\delta y) - a \| \, \mathrm{d}y \right] = 0.$$

Since $\nabla w_n,\delta(y) = \delta \nabla u_n(x+\delta y)$, from (3.9) we get

$$\mu_a(x) \geq \limsup_{\delta \to 0^+} \limsup_{n \to +\infty} \left[ \int_{\eta \Omega_V(x)} \left[ \frac{\delta}{\varepsilon_n} W(w_n,\delta(y)) + \frac{\varepsilon_n}{\delta} (h^\ast)^2(x+\delta y, \nabla w_n,\delta(y)) \right] \, \mathrm{d}y + \int_{\eta \Omega_V(x)} \left[ \delta \varepsilon_n h^2(x+\delta y, \frac{1}{\delta} \nabla w_n,\delta(y)) - \frac{\varepsilon_n}{\delta} (h^\ast)^2(x+\delta y, \nabla w_n,\delta(y)) \right] \, \mathrm{d}y \right] \geq \left. \int_{\eta \Omega_V(x)} \left[ \delta \varepsilon_n h^2(x+\delta y, \frac{1}{\delta} \nabla w_n,\delta(y)) - \frac{\varepsilon_n}{\delta} (h^\ast)^2(x+\delta y, \nabla w_n,\delta(y)) \right] \, \mathrm{d}y \right|.$$
\[
= \frac{\varepsilon_n}{\delta} \int_{\eta_{Q_v(\delta)}} |18^2 h^2(x+\delta y, \frac{1}{\delta} \nabla w_n, \delta(y)) - (h_\infty)^2(x+\delta y, \nabla w_n, \delta(y))| \, dy =
\]
\[
= \frac{\varepsilon_n}{\delta} \int_{\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| \leq \delta L\}} |18^2 h^2(x+\delta y, \frac{1}{\delta} \nabla w_n, \delta(y)) - (h_\infty)^2(x+\delta y, \nabla w_n, \delta(y))| \, dy +
\]
\[
+ \frac{\varepsilon_n}{\delta} \int_{\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| > \delta L\}} |18^2 h^2(x+\delta y, \frac{1}{\delta} \nabla w_n, \delta(y)) - (h_\infty)^2(x+\delta y, \nabla w_n, \delta(y))| \, dy = I_1 + I_2
\]

where, by (H3) and Lemma 2.7 i),

\[
I_1 \leq \text{const.} \frac{\varepsilon_n}{\delta} \text{meas}(\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| \leq \delta L\}) \delta^2 \leq \text{const.} \varepsilon_n \delta \to 0 \text{ as } \delta \to 0 \text{ and } n \to +\infty
\]

and, by (H4) (with \(t=1/\delta\)), Hölder's inequality and (H3),

\[
I_2 \leq \frac{\varepsilon_n}{\delta} \int_{\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| \leq \delta L\}} C^c \|\nabla w_n, \delta(y)\|^{2-m} \delta^m \, dy \leq
\]
\[
\leq C^c \delta^{m-1} \int_{\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| \leq \delta L\}} \varepsilon_n \|\nabla w_n, \delta(y)\|^{2-m} \, dy \leq
\]
\[
\leq C \delta^{m-1} \varepsilon_n^{m/2} \left[ \int_{\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| \leq \delta L\}} \varepsilon_n \|\nabla w_n, \delta(y)\|^{2} \, dy \right]^{1-m/2} =
\]
\[
= C \delta^{m-1} \varepsilon_n^{m/2} \left[ \epsilon_n \delta^2 + \int_{\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| > \delta L\}} \delta^2 \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y)) \, dy \right]^{1-m/2} =
\]
\[
= C (\delta \varepsilon_n)^{m/2} \left[ \epsilon_n \delta + \int_{\eta_{Q_v(\delta)} \cap \{\|\nabla w_{n, \delta}\| > \delta L\}} \delta^2 \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y)) \, dy \right]^{1-m/2} \to 0 \text{ as } \delta \to 0, \ n \to +\infty
\]

since by (3.9) \(\int_{\eta_{Q_v(\delta)}} \delta^2 \varepsilon_n h^2(x+\delta y, \nabla u_n(x+\delta y)) \, dy\) remains bounded. So (3.10) reduces to

\[
\mu_\delta(x) \geq \limsup_{\delta \to 0^+} \limsup_{n \to +\infty} \int_{\eta_{Q_v(\delta)}} \left[ \frac{\delta}{\varepsilon_n} W(w_n, \delta(y)) + \frac{\varepsilon_n}{\delta} (h_\infty)^2(x+\delta y, \nabla w_n, \delta(y)) \right] \, dy =
\]
\[
= \limsup_{\delta \to 0^+} \limsup_{n \to +\infty} \int_{\eta_{Q_v(\delta)}} \left[ \frac{\delta}{\varepsilon_n} W(w_n, \delta(y)) + \frac{\varepsilon_n}{\delta} (h_\infty)^2(x, \nabla w_n, \delta(y)) \right] \, dy +
\]
Fix $\epsilon > 0$. By Lemma 2.7 ii), (H3) and (3.9) we have for $\delta$ small enough,

\[
\int_{Q_\delta(0)} eC \|V_{n}(x+8y)\|^{2} \, dy = eC \|n \|_{L^2(Q(x))} \leq eC \|e_{n8} + e_{n8} h_{2}(x+8y, V_{n}(x+8y)) \|_{L^2}.
\]

Hence,

\[
\|V_{n}(x)\|_{L^2} \leq \limsup_{\delta \to 0} \sup_{n} f \left[ \int_{Q_{\delta}(x)} \delta (w_{n}(v)) + \epsilon_{n} (\delta)^{2}(x, V_{n}(v)) \right] \, dy + O(\epsilon). \tag{3.12}
\]

Let

\[
\limsup_{\delta \to 0} \sup_{n} \int_{Q_{\delta}(0)} f \left[ \int_{Q_{\delta}(x)} \delta (w_{n}(v)) + \epsilon_{n} (\delta)^{2}(x, V_{n}(v)) \right] \, dy + O(\epsilon) =
\]

where $8k \to 0+$ as $k \to +\infty$. Choose $n(k)$ large enough so that, setting $a_{k} = \frac{\epsilon}{k}$, we have $0 < a_{k} < 1/k$, II wnoc, and

\[
\limsup_{n \to \infty} \int_{Q_{\delta}(0)} \delta (w_{n}(v)) + \epsilon_{n} (\delta)^{2}(x, V_{n}(v)) \, dy + O(\epsilon) =
\]

Thus, defining $v_{k}(y) = Wn^{\wedge}(y)$, it follows from (3.12), (3.13), and (3.14) that
\[ \mu_a(x) \geq \lim_{k \to +\infty} \int_{Q(x)} \left[ \frac{1}{\alpha_k} W(v_k(y)) + \alpha_k (h^\infty)^2(x, \nabla v_k(y)) \right] dy + O(\epsilon) \]

where \( v_k \to u_0 \) in \( L^1(Q(x)) \) and \( \alpha_k \to 0^+ \) as \( k \to +\infty \). Changing variables we obtain

\[ \mu_a(x) \geq \lim_{k \to +\infty} \eta^N \int_{Q(x_0)} \left[ \frac{1}{\alpha_k} W(v_k(\eta z)) + \alpha_k (h^\infty)^2(x, \nabla v_k(\eta z)) \right] dz + O(\epsilon) = \]

\[ = \eta^{N-1} \lim_{k \to +\infty} \int_{Q(x_0)} \left[ \frac{1}{\tilde{\alpha}_k} W(\tilde{u}_k(z)) + \tilde{\alpha}_k (h^\infty)^2(x, \nabla \tilde{u}_k(z)) \right] dz + O(\epsilon) \] (3.15)

where \( \tilde{\alpha}_k = \frac{\alpha_k}{\eta} \), \( \tilde{\alpha}_k \to 0^+ \) and \( \tilde{u}_k(z) = v_k(\eta z) \). Applying Lemma 3.2 to \( h^\infty \) and to the sequences \( \tilde{u}_k \) and \( \tilde{\alpha}_k \) we conclude that there exists a subsequence \( \{\tilde{\alpha}_i\} \subset \{\tilde{\alpha}_k\} \) and a sequence \( \{\xi_i\} \in H^1(Q(x_0) ; \mathbb{R}^p) \) such that \( \xi_i \to u_0 \) in \( L^1(Q(x_0) ; \mathbb{R}^p) \), \( \xi_i \in \mathcal{A}(a, b, v(x)) \) and

\[ \lim \inf_{i \to +\infty} \int_{Q(x_0)} \left[ \frac{1}{\tilde{\alpha}_i} W(\tilde{u}_i(z)) + \tilde{\alpha}_i (h^\infty)^2(x, \nabla \tilde{u}_i(z)) \right] dz \leq \]

\[ \leq \lim_{k \to +\infty} \int_{Q(x_0)} \left[ \frac{1}{\alpha_k} W(\tilde{u}_k(z)) + \alpha_k (h^\infty)^2(x, \nabla \tilde{u}_k(z)) \right] dz. \] (3.16)

Thus, by (3.15) and (3.16) we have

\[ \mu_a(x) \geq \eta^{N-1} \lim \inf_{i \to +\infty} \int_{Q(x_0)} \left[ \frac{1}{\tilde{\alpha}_i} W(\tilde{u}_i(z)) + \tilde{\alpha}_i (h^\infty)^2(x, \nabla \tilde{u}_i(z)) \right] dz + O(\epsilon) \geq \]

\[ \geq \eta^{N-1} K(x, a, b, v(x)) + O(\epsilon). \]

(3.8) now follows if we let \( \eta \to 1^+ \) and \( \epsilon \to 0^+ \).  

4. An upper bound for the \( \Gamma \)-limit.

We now prove the second part of Theorem 2.9.

Proposition 4.1. Under the hypotheses (H1)-(H5) given any \( u \in L^1(\Omega ; \mathbb{R}^p) \) there exist sequences \( \epsilon_n \to 0^+ \) and \( u_n \in H^1(\Omega ; \mathbb{R}^p) \) such that \( u_n \to u \) in \( L^1(\Omega ; \mathbb{R}^p) \) and
It is clear that it suffices to consider the case where $u = \chi_A(x)a + (1 - \chi_A(x))b$ with $\text{Per}_\Omega(A) < +\infty$, since

$$\lim_{n \to +\infty} \int_{\Omega} \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(x, \nabla u_n(x)) \right] \, dx = 0$$

implies that

$$\lim_{n \to +\infty} \int_{\Omega} W(u_n(x)) \, dx = 0$$

and so, as $u_n \to u$ in $L^1(\Omega; \mathbb{R}^p)$ and due to the continuity of $W$, we conclude that

$$u(x) \in \{a, b\} \text{ a.e. } x \in \Omega.$$ 

Also, as in Step 1 of the proof of Proposition 3.1, we obtain $\text{Per}_\Omega(A) < +\infty$. We begin by considering the simpler case where $u = \chi_A(x)a + (1 - \chi_A(x))b$ has planar interface and $h$ and $K$ do not depend explicitly on $x$.

Lemma 4.2. Let (H1)-(H5) hold, let $\Omega = Q_v$ and

$$u(y) = \begin{cases} b & \text{if } y \cdot v > 0 \\ a & \text{if } y \cdot v < 0. \end{cases}$$

Then there exist sequences $\varepsilon_n \to 0^+$ and $u_n \in \mathcal{A}(a, b, v)$ such that $u_n \to u$ in $L^1(Q_v; \mathbb{R}^p)$ and

$$\lim_{n \to +\infty} \int_{Q_v} \left[ \frac{1}{\varepsilon_n} W(u_n(x)) + \varepsilon_n h^2(\nabla u_n(x)) \right] \, dx = K(a, b, v) = J_0(u).$$

Proof. Assume, without loss of generality, that $v = e_N$ so that

$$u(y) = \begin{cases} b & \text{if } y \cdot e_N > 0 \\ a & \text{if } y \cdot e_N < 0. \end{cases}$$

Denote $Q_v$ by $Q$ and let $Q'$ be the projection of $Q$ on $\mathbb{R}^{N-1}$: $Q' = \{ y \in Q : y_N = 0 \}$. Let $L_n > 0$ and $\xi_n \in \mathcal{A}(a, b, e_N)$ be such that

$$\lim_{n \to +\infty} \int_{Q} \left[ L_n W(\xi_n(y)) + \frac{1}{L_n} (h^m)^2(\nabla \xi_n(y)) \right] \, dy = K(a, b, e_N). \quad (4.1)$$

For $n$ fixed, define
\[ v^n_k(y) = v^n_k(y', y_N) = \begin{cases} 
\xi_n(y', y_N) & \text{if } y_N > \varepsilon/2 \\
\xi_n(y', y_N) & \text{if } -\varepsilon/2 \leq y_N \leq \varepsilon/2 \\
a & \text{if } y_N < -\varepsilon/2.
\end{cases} \]

Clearly \( v^n_k \in \mathfrak{A}(a, b, e_N) \) for all \( n \in \mathbb{N}, \varepsilon > 0 \). Also,

\[
\|v^n_k - u\|_{L^1(Q')} = \|v^n_k - b\|_{L^1(Q')} \geq \int_Q \|\xi_n(y', y_N) - b\| \, dy = \varepsilon \int_Q \|\xi_n(y) - b\| \, dy \to 0 \text{ as } \varepsilon \to 0^+.
\]

for \( n \) fixed. Likewise \( \|v^n_k - u\|_{L^1(Q)} = \|v^n_k - u\|_{L^1(Q)} \to 0 \text{ as } \varepsilon \to 0^+ \) and so \( v^n_k \to u \) in \( L^1(Q; \mathbb{R}^p) \) as \( \varepsilon \to 0^+ \). On the other hand

\[
\int_Q \left[ \frac{L_n}{\varepsilon} W(v^n_k(y)) + \frac{\varepsilon}{L_n} h^2(\nabla v^n_k(y)) \right] \, dy = \int_Q \left[ \frac{L_n}{\varepsilon} W(v^n_k(y)) + \frac{\varepsilon}{L_n} (h^m)^2(\nabla v^n_k(y)) \right] \, dy + \int_Q \left[ \frac{\varepsilon}{L_n} h^2(\nabla v^n_k(y)) - \frac{\varepsilon}{L_n} (h^m)^2(\nabla v^n_k(y)) \right] \, dy =: I_1 + I_2
\]

where

\[
I_1 = \int_{-\varepsilon/2}^{\varepsilon/2} \left[ \frac{L_n}{\varepsilon} W(v^n_k(y)) + \frac{\varepsilon}{L_n} (h^m)^2(\nabla v^n_k(y)) \right] \, dy = \int_{-\varepsilon/2}^{\varepsilon/2} \left[ \frac{L_n}{\varepsilon} W(\xi_n(y', y_N)) + \frac{1}{L_n \varepsilon} (h^m)^2(\nabla \xi_n(y', y_N)) \right] \, dy
\]
\[ \int_{Q} \left( J \left[ L \cdot w(y) \right] + \mathcal{L} \left( (h^{\infty})^2 (V^\wedge(y)) \right) \right) dy = \] (4.2)

and

\[ \left| I_2 \right| \leq \int_{Q} \frac{e}{L_n} \left| h^2 \left( \nabla V_n^\wedge(y) \right) - (h^\infty)^2 \left( \nabla V_n^\wedge(y) \right) \right| dy = \]

\[ \int_{Q} e^2 (h^\infty)^2 \left( \nabla V_n^\wedge(y', y^\wedge_n) \right) dy = \]

\[ = \frac{1}{L_n e} \int_{Q \cap \{ ||V_n^\wedge(y') ||_\infty > L \}} e^2 (h^\infty)^2 \left( \nabla V_n^\wedge(y', y^\wedge_n) \right) dy + \]

\[ \frac{J}{L_n e} \int_{Q \cap \{ ||V_n^\wedge(y') ||_\infty > L \}} e^2 (h^\infty)^2 \left( \nabla V_n^\wedge(y', y^\wedge_n) \right) dy =: I_2^1 + I_2^2. \]

By (H3) and Lemma 2.7 i)

\[ I_2^1 \leq \frac{1}{U e} \ \text{meas} Q \ C e^2 = \text{const.} \frac{e}{L_n} \]

and by (H4), Hörmander's inequality and Lemma 2.7 i),

\[ I_2^2 \leq \frac{C}{L_n e} \int_{Q \cap \{ ||V_n^\wedge(y') ||_\infty > L \}} \left[ \int_{Q \cap \{ ||V_n^\wedge(y') ||_\infty > L \}} \left( \nabla V_n^\wedge(y') \right) dy \right] \left( \nabla V_n^\wedge(y') \right) dy \]

\[ \leq e^{\pi \cdot i} \left[ J \wedge \left( \nabla V_n^\wedge(y') \right) \right] dy \]

\[ \leq C \left( J \left[ L \cdot w(y) \right] + \mathcal{L} \left( (h^{\infty})^2 (V^\wedge(y)) \right) \right) dy \]

where by (4.1) \( \int_{Q} \left( (h^\infty)^2 (V^\wedge(y)) \right) dy \) is a bounded sequence. Choose \( E(n) \) such that \( e(n) \to 0 \) as \( n \to +\infty \), \( u L^1_{\infty} \to 0 \) and
\[ \int_Q \left[ L_n W(\xi_n(y)) + \frac{1}{L_n} (x-\xi_n)^2(\nabla \xi_n(y)) \right] \, dy - \int_Q \left[ L_n W(\nu_{\xi_n}(y)) + \frac{\varepsilon(n)}{L_n} (x-\nu_{\xi_n}(y))^2(\nabla \nu_{\xi_n}(y)) \right] \, dy \leq \frac{1}{n}. \]

Let \( u_n = v_{\xi_n}^n \) and \( \delta_n = \frac{\varepsilon(n)}{L_n} \). Then \( \delta_n \to 0^+ \), \( u_n \to u \) in \( L^1(Q; \mathbb{R}^p) \) and
\[
\lim_{n \to \infty} \int_Q \left[ \frac{1}{\delta_n} W(u_n(y)) + \delta_n h^2(\nabla u_n(y)) \right] \, dy = \lim_{n \to \infty} \int_Q \left[ L_n W(\xi_n(y)) + \frac{1}{L_n} (x-\xi_n)^2(\nabla \xi_n(y)) \right] \, dy
= K(a, b, \varepsilon_N) = \int_{Q \cap \{ y_N = 0 \}} K(a, b, \varepsilon_N) \, dH_{N-1}(y).
\]

**Lemma 4.3.** Let (H1)-(H5) hold and let
\[
u(x) = \begin{cases} b & \text{if } (x-a_0) \cdot v > 0 \\ a & \text{if } (x-a_0) \cdot v < 0 \end{cases}
\]
for some \( a_0 \in \mathbb{R}^N \). Define
\[ \mathcal{A}(a_0, a, b, v, \eta) := \{ u \in H^1(a_0 + \eta Q; \mathbb{R}^p) : u(x) = b \text{ if } (x-a_0) \cdot v = \eta/2, u(x) = a \text{ if } (x-a_0) \cdot v = -\eta/2 \}
\text{ and } u \text{ is periodic with period } \eta \text{ in the directions of } v_1, \ldots, v_{N-1} \}.
\]
Given a sequence \( \varepsilon_n \to 0^+ \) there exists a subsequence \( \{ \varepsilon_{n_k} \} \) and a sequence \( \{ v_k \} \) in \( \mathcal{A}(a_0, a, b, v, \eta) \) such that \( v_k \to u \) in \( L^1(a_0 + \eta Q; \mathbb{R}^p) \) and
\[
\lim_{k \to \infty} \int_{a_0 + \eta Q_v} \left[ \frac{1}{\varepsilon_{n_k}^2} W(v_k(x)) + \eta \varepsilon_{n_k} h^2(\nabla v_k(x)) \right] \, dx = \eta^{N-1} K(a, b, v).
\]

**Proof.** For simplicity, we assume that \( v = \varepsilon_N \) and we denote \( Q_v \) by \( Q \).

**Case 1.** Suppose first that \( a_0 = 0 \) and \( \eta = 1 \). By Lemma 4.2, consider \( \alpha_k \to 0^+ \) and \( u_k \in \mathcal{A}(a, b, \varepsilon_N) \) such that \( u_k \to u \) in \( L^1(Q; \mathbb{R}^p) \) and
\[
\lim_{k \to \infty} \int_Q \left[ \frac{1}{\alpha_k} W(u_k(x)) + \alpha_k h^2(\nabla u_k(x)) \right] \, dx = K(a, b, \varepsilon_N).
\]

Fix \( k \in \mathbb{N} \) and define
\[ v_{k,n}(x) := \begin{cases} 
  b & \text{if } \frac{\varepsilon_n}{2\alpha_k} < x_N < \frac{1}{2} \\
  u_k(x', \frac{\alpha_k}{\varepsilon_n} x_N) & \text{if } |x_N| < \frac{\varepsilon_n}{2\alpha_k} \\
  a & \text{if } -\frac{1}{2} < x_N < -\frac{\varepsilon_n}{2\alpha_k}.
\] 

Clearly \( v_{k,n} \in \mathcal{A}(a,b,\varepsilon_N) \) and

\[
\int_{Q} \left[ \frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n h^2(\nabla v_{k,n}(x)) \right] \, dx = \]

\[
= \int_{Q} \left[ \frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n (h^\omega)^2(\nabla v_{k,n}(x)) \right] \, dx + \int_{Q} \left[ \varepsilon_n h^2(\nabla v_{k,n}(x)) - \varepsilon_n (h^\omega)^2(\nabla v_{k,n}(x)) \right] \, dx =:
\]

\[
= I_1 + I_2
\]  

(4.4)

where

\[
I_1 = \int_{Q} \int_{\Omega} \left[ \frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n (h^\omega)^2(\nabla v_{k,n}(x)) \right] \, dx = \]

\[
- \int_{Q} \int_{\Omega} \left[ \frac{1}{\alpha_k} W(u_k(x)) + \alpha_k (h^\omega)^2(\nabla u_k(x)) \right] \, dx =
\]

\[
= \int_{Q} \int_{\Omega} \left[ \frac{1}{\alpha_k} W(u_k(x)) + \alpha_k (h^\omega)^2(\nabla u_k(x)) \right] \, dx + \int_{Q} \int_{\Omega} \left[ \alpha_k (h^\omega)^2(\nabla u_k(x)) - \alpha_k h^2(\nabla u_k(x)) \right] \, dx =
\]

\[
= I_1^1 + I_1^2.
\]  

(4.5)

By (H4), Hölder's inequality and (H3)

\[
I_1^2 = \int_{Q \cap \{\|\nabla u_k\| \leq L\}} \alpha_k [(h^\omega)^2(\nabla u_k(x)) - h^2(\nabla u_k(x))] \, dx + \int_{Q \cap \{\|\nabla u_k\| > L\}} \alpha_k [(h^\omega)^2(\nabla u_k(x)) - h^2(\nabla u_k(x))] \, dx
\]

\[
\leq C \alpha_k + \int_{Q \cap \{\|\nabla u_k\| > L\}} \alpha_k C \|\nabla u_k(x)\|^{2-m} \, dx \leq
\]

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\[ \leq O(\alpha_k) + (\alpha_k)^{m/2} C \left[ \int_{Q \cap \{||\nabla u_k(x)|| > L\}} \alpha_k ||\nabla u_k(x)||^2 \, dx \right]^{1-m/2} \leq \]

\[ \leq O(\alpha_k) + (\alpha_k)^{m/2} C \left[ \int_{Q} \alpha_k C^* + \alpha_k h^2(\nabla u_k(x)) \, dx \right]^{1-m/2} = O(\alpha_k) \quad (4.6) \]

since, by (4.3) \{ \int_{Q} \alpha_k h^2(\nabla u_k(x)) \, dx \} remains bounded. On the other hand

\[ \| I_2 \| \leq \int_{Q} \frac{\alpha_k^2}{\epsilon_n} \left( \frac{\alpha_k \nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)}{\epsilon_n} \right) \cdot \frac{\alpha_k^2}{\epsilon_n} (h^m)^2(\nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)) \, dx = \]

\[ = \frac{\alpha_k^2}{\epsilon_n} \int_{Q \cap \{||\nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)|| \leq L \epsilon_n / \alpha_k\}} \frac{\epsilon_n^2}{\alpha_k^2} h^2(\nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)) \cdot (h^m)^2(\nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)) \, dx =: I_1^2 + I_2^2. \quad (4.7) \]

By (H3) and Lemma 2.7 i),

\[ I_2^1 \leq \frac{\alpha_k^2}{\epsilon_n} \int_{Q \cap \{||\nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)|| \leq L \epsilon_n / \alpha_k\}} \frac{\epsilon_n^2}{\alpha_k^2} (1 + ||\nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)||^2) \, dx \leq C \frac{\alpha_k^2 \epsilon_n^2}{\epsilon_n \alpha_k^2} = O(\epsilon_n) \quad (4.8) \]

and, by (H4),

\[ I_2^1 \leq C \frac{\alpha_k^2}{\epsilon_n} \int_{Q} \frac{\epsilon_n^m}{\alpha_k^m} ||\nabla u_k(x', \frac{\alpha_k}{\epsilon_n} x_N)||^{2-m} \, dx \leq C \frac{\epsilon_n^m}{\alpha_k^m} \int_{Q} ||\nabla u_k(x)||^{2-m} \, dx = \epsilon_n^m h(\alpha_k). \quad (4.9) \]

Hence, by (4.4)-(4.9),
\[
\limsup_{n \to +\infty} \int_{Q} \left[ \frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n h^2(\nabla v_{k,n}(x)) \right] \, dx \leq \int_{Q} \left[ \frac{1}{\alpha_k} W(u_k(x)) + \alpha_k h^2(\nabla u_k(x)) \right] \, dx + \\
+ O(\alpha_k) \text{ for all } k
\]

and, by definition of \(K(a,b,e_N)\)

\[
K(a,b,e_N) \leq \liminf_{n \to +\infty} \int_{Q} \left[ \frac{1}{\varepsilon_n} W(v_{k,n}(x)) + \varepsilon_n h^2(\nabla v_{k,n}(x)) \right] \, dx.
\]

Also,

\[
\|v_{k,n} - u\|_{L^1(Q;\mathbb{R}^p)} = \int_{Q \setminus \{l(x) < \varepsilon_n/2\alpha_k\}} |u_k(x') - u(x)| \, dx = \frac{\varepsilon_n}{\alpha_k} \int_{Q} |u_k(x) - u(x)| \, dx.
\]

Thus, for all \(k\), choose \(\varepsilon_{n_k}\) such that, setting \(v_k := v_{k,n_k}\), we have \(\varepsilon_{n_k} \leq 1\) and

\[
\limsup_{n \to +\infty} \int_{Q} \left[ \frac{1}{\varepsilon_{n_k}} W(v_{k,n}(x)) + \varepsilon_{n_k} h^2(\nabla v_{k,n}(x)) \right] \, dx = \int_{Q} \left[ \frac{1}{\varepsilon_{n_k}} W(v_k(x)) + \varepsilon_{n_k} h^2(\nabla v_k(x)) \right] \, dx + \\
+ O(1/k).
\]

Then,

\[
\|v_k - u\|_{L^1(Q;\mathbb{R}^p)} \leq \|u_k - u\|_{L^1(Q;\mathbb{R}^p)} \to 0 \text{ as } k \to +\infty
\]

and

\[
\lim_{k \to +\infty} \int_{Q} \left[ \frac{1}{\varepsilon_{n_k}} W(v_k(x)) + \varepsilon_{n_k} h^2(\nabla v_k(x)) \right] \, dx = K(a,b,e_N).
\]

**Case 2.** We now take \(\Omega = a_0 + \eta Q\) for some \(a_0 \in \mathbb{R}^N\) and \(\eta > 0\) and we define

\[
h_{\eta}(A) = h\left(\frac{A}{\eta}\right).
\]

Setting

\[
u_0(x) = \begin{cases} b & \text{if } x \cdot e_N > 0 \\ a & \text{if } x \cdot e_N < 0 \end{cases}
\]

by case 1, given \(\varepsilon_n \to 0^+\), there exist a subsequence \(\{\varepsilon_{n_k}\}\) and a sequence \(\{v_k\}\) in \(\mathcal{A}^\infty(a,b,e_N)\) such that \(v_k \to u_0\) in \(L^1(Q;\mathbb{R}^p)\) and
\[
\lim_{k \to +\infty} -\int_{Q} \left[ \frac{1}{\eta} W(v_k(y)) + e_{\eta_k} h^2(Vv_k(y)) \right] \, dy = K_{\eta}(a, b, e_N),
\]

where

\[
K_{\eta}(a, b, e_N) = \inf \left\{ \int_{Q} \left[ LW(\xi(x)) + (h^{-})^2(V\xi(x)) \right] \, dx : \xi \in \mathcal{T}(a, b, CN), L > 0 \right\}.
\]

Note that, due to the homogeneity of \( h^\infty \),

\[
K_{\eta}(a, b, e_N) = \inf \left\{ \int_{Q} \left[ LW(E(x)) + (h^{-})^2(V\xi(x)) \right] \, dx : \% \in \mathcal{T}(a, b, CN), L > 0 \right\} = \frac{1}{\eta} K(a, b, e_N). \tag{4.10}
\]

For \( x \in a_0 + \omega \), let

\[
u_k(x) := \frac{x - a_0}{\eta_k}. \tag{4.10}
\]

Clearly \( u_k \# (a_0, a, b, e_N, \eta_k) \),

\[
J \int_{u_k(x) - u(x)} \, dx = \int_{\omega + \eta \omega} lv_k(x) - u(x) \, dx = r'' \int_{\omega + \eta \omega} lv_k(x) - u(a_0 + \omega \xi) \, dx = \frac{1}{\eta} K(a, b, e_N).
\]

Proof of Proposition 4.1.

**Step 1.** Assume first that \( u \) has planar interface i.e.

\[
u(x) b \quad \text{if } (x - a_0, v > 0)
\]

\[
u(x) a \quad \text{if } (x - a_0, v < 0).
\]
Without loss of generality, assume that $a_0 = 0$ and $v = e_N$. Let $\Omega = \{x \in \Omega : x_N = 0\}$. In order to ensure that property (H5) and Proposition 2.8 ii) are satisfied uniformly we will work on compact subsets of $\Omega$. Fix $\varepsilon > 0$ and let $\Omega_\varepsilon' \subset \subset \Omega'$ be such that

$$H_{N-1}(\Omega' \setminus \Omega_\varepsilon') = O(\varepsilon).$$

(4.11)

Since $\Omega_\varepsilon'$ is compact we can find $\delta > 0$ such that $\Omega_\varepsilon' \times [-\delta/2, \delta/2] \subset \subset \Omega$ and (H5) and Proposition 2.8 ii) are satisfied uniformly in $\Omega_\varepsilon' \times [-\delta/2, \delta/2]$ i.e.

$$x, y \in \Omega_\varepsilon' \times [-\delta/2, \delta/2], \|x - y\| < \delta \Rightarrow h^2(x, A) - h^2(y, A) \leq C(1 + \|A\|^2), \text{ for all } A \in M^{p \times N}$$

(4.12)

and

$$x, y \in \Omega_\varepsilon' \times [-\delta/2, \delta/2], \|x - y\| < \delta \Rightarrow |K(x, a, b, v) - K(y, a, b, v)| < C(1 + \|a\|^2 + \|b\|^2 + \|a - b\|^2).$$

(4.13)

We may write

$$\Omega_\varepsilon' = \bigcup_{i=1}^p (a_i + \eta Q_i) \cup \omega$$

(4.14)

where $H_{N-1}(\omega) = O(\varepsilon)$, $Q_i := a_i + \eta Q$ are cubes with disjoint interiors, $0 < \eta < \delta$ and

$$\bigcup_{i=1}^p (a_i + \eta Q_i) =: \bigcup_{i=1}^p Q_i \subset \subset \Omega.$$

Since $\partial \Omega$ is Lipschitz it is possible to pick $\eta$ as above and so that

$$H_{N-1}\left(\operatorname{proj}_{\{x_N=0\}}(\Omega \cap \{x_N < \eta/2\}) \setminus \bigcup_{i=1}^p Q_i\right) =: H_{N-1}(P) = O(\varepsilon).$$

(4.15)

We claim that given any sequence $\alpha_n \to 0^+$ there exists a subsequence $\{\alpha_n^1\}$ and a sequence $\{u_k\}$ in $H^1(\Omega; \mathbb{R}^p)$ such that $u_k \to u$ in $L^1(\Omega; \mathbb{R}^p)$ and

$$\lim_{k \to \infty} \int_{\Omega} \frac{1}{\alpha_n^1} W(u_k(x)) + \alpha_n^1 h^2(x, \nabla u_k(x)) \, dx = \int_{\Omega} K(x, a, b, e_N) \, dH_{N-1}(x).$$

(4.16)

By Lemma 4.3 given a sequence $\alpha_n \to 0^+$ there exist a subsequence $\{|\alpha_n\}^1$ and a sequence $\{u_k^{(1)}\}$ in $B(a_1, a, b, e_N, \eta)$ such that $u_k^{(1)} \to u$ in $L^1(Q_1; \mathbb{R}^p)$ and

$$\lim_{k \to \infty} \int_{Q_1} \left[ \frac{1}{\alpha_k^{(1)}} W(u_k^{(1)}(x)) + \alpha_k^{(1)} h^2(a_1, \nabla u_k^{(1)}(x)) \right] \, dx = \eta^{N-1} K(a_1, a, b, e_N).$$

(4.17)

By Lemma 3.2 there exists a subsequence $\{\beta_k^{(1)}\}$ of $\{|\alpha_n\}^1$ and a sequence $\{w_k^{(1)}\}$ in $H^1(Q_1; \mathbb{R}^p)$ such that $w_k^{(1)} \to u$ in $L^1(Q_1; \mathbb{R}^p)$, $w_k^{(1)}(x) = v_k^{(1)}(x) / \eta$ for $x \in \partial Q_1$ (the $v_j$ are mollifications of $u$) and

$$\limsup_{k \to \infty} \int_{Q_1} \left[ \frac{1}{\beta_k^{(1)}} W(w_k^{(1)}(x)) + \beta_k^{(1)} h^2(a_1, \nabla w_k^{(1)}(x)) \right] \, dx \leq \infty.$$
\[ \liminf_{k \to +\infty} \int_{Q_1} \left[ \frac{1}{\alpha_k} W(u_k^{(1)}(x)) + \alpha_k h^2(a_1, \nabla u_k^{(1)}(x)) \right] \, dx = \eta^{N-1} K(a_1, a, b, e_N). \quad (4.18) \]

By Proposition 3.1,
\[ \liminf_{k \to +\infty} \int_{Q_1} \left[ \frac{1}{\beta_k} W(w_k^{(1)}(x)) + \beta_k h^2(a_1, \nabla w_k^{(1)}(x)) \right] \, dx \geq \eta^{N-1} K(a_1, a, b, e_N) \]
which, together with (4.18), implies
\[ \lim_{k \to +\infty} \int_{Q_1} \left[ \frac{1}{\beta_k} W(w_k^{(1)}(x)) + \beta_k h^2(a_1, \nabla w_k^{(1)}(x)) \right] \, dx = \eta^{N-1} K(a_1, a, b, e_N). \quad (4.19) \]

By Lemma 4.3 there exists a subsequence \( \{\alpha^{(2)}_k\} \) of \( \{\beta_k^{(1)}\} \) and a sequence \( \{u_k^{(2)}\} \) in \( \mathcal{B}(a_2, a, b, e_N, \eta) \) such that \( u_k^{(2)} \to u \) in \( L^1(Q_2; \mathbb{R}^p) \) and
\[ \lim_{k \to +\infty} \int_{Q_2} \left[ \frac{1}{\alpha_k^{(2)}} W(u_k^{(2)}(x)) + \alpha_k^{(2)} h^2(a_2, \nabla u_k^{(2)}(x)) \right] \, dx = \eta^{N-1} K(a_2, a, b, e_N). \]

Once again, by applying Lemma 3.2, we conclude that there is a subsequence \( \{\beta^{(2)}_k\} \) of \( \{\alpha^{(2)}_k\} \) and a sequence \( \{w_k^{(2)}\} \) in \( H^1(Q_2; \mathbb{R}^p) \) such that \( w_k^{(2)} \to u \) in \( L^1(Q_2; \mathbb{R}^p) \), \( w_k^{(2)}(x) = \frac{u_k^{(2)}(x)}{\eta} \) for \( x \in \partial Q_2 \) and
\[ \lim_{k \to +\infty} \int_{Q_2} \left[ \frac{1}{\beta_k^{(2)}} W(w_k^{(2)}(x)) + \beta_k^{(2)} h^2(a_2, \nabla w_k^{(2)}(x)) \right] \, dx = \eta^{N-1} K(a_2, a, b, e_N). \]

By induction we repeat the above argument in order to obtain subsequences \( \{\beta_k^{(p)}\} \subset \{\beta_k^{(p-1)}\} \subset \ldots \subset \{\beta_k^{(2)}\} \subset \{\beta_k^{(1)}\} \) and sequences \( \{w_k^{(j)}\} \) in \( H^1(Q_j; \mathbb{R}^p) \) such that \( w_k^{(j)} \to u \) in \( L^1(Q_j; \mathbb{R}^p) \), \( w_k^{(j)}(x) = \frac{u_k^{(j)}(x-a_j)}{\eta} \) for \( x \in \partial Q_j \) and
\[ \lim_{k \to +\infty} \int_{Q_j} \left[ \frac{1}{\beta_k^{(j)}} W(w_k^{(j)}(x)) + \beta_k^{(j)} h^2(a_j, \nabla w_k^{(j)}(x)) \right] \, dx = \eta^{N-1} K(a_j, a, b, e_N) \]
for \( j = 1, \ldots, p \). Consider the sequence \( \{\beta_k^{(p)}\} \) and for all \( j = 1, \ldots, p \) let \( \{\xi_k^{(j)}\} \) be the corresponding subsequence of \( \{w_k^{(j)}\} \) such that
Define the sequence \( u_{k,\varepsilon}(x) \) as follows,
\[
 u_{k,\varepsilon}(x) = \begin{cases} 
 \xi^{(j)}_{sk}(x) & \text{if } x \in Q_j \\
 b & \text{if } x_N > \eta / 2 \\
 a & \text{if } x_N < - \eta / 2 
\end{cases}
\]
and in \( \Omega \cap \left\{ |x_N| < \frac{\eta}{2} \right\} \setminus \bigcup_{i=1}^{P} Q_i \) we define \( u_{k,\varepsilon}(x) \) using the periodicity of \( \{\nu^p_{i}\} \). Clearly \( u_{k,\varepsilon} \in H^1(\Omega;\mathbb{R}^P) \). As \( \|\nu^p_{i}\|_{\infty} \leq \text{const.} \) and since
\[
\text{meas}\left( \left( \Omega \cap \left\{ |x_N| < \frac{\eta}{2} \right\} \right) \setminus \bigcup_{i=1}^{P} Q_i \right) = O(\varepsilon)
\]
we have
\[
\|u_{k,\varepsilon} - u\|_{L^1(\Omega;\mathbb{R}^P)} = O(\varepsilon) + \sum_{i=1}^{P} \|u_{k,\varepsilon} - u\|_{L^1(Q_i;\mathbb{R}^P)}
\]
and so,
\[
\lim_{\varepsilon \to 0} \left[ \lim_{k \to \infty} \|u_{k,\varepsilon} - u\|_{L^1(\Omega;\mathbb{R}^P)} \right] = 0.
\]
Also,
\[
\begin{align*}
\int_{\Omega} & \left[ \frac{1}{\beta^p_k} W(u_{k,\varepsilon}(x)) + \beta^p_k h^2(x, \nabla u_{k,\varepsilon}(x)) \right] \, dx = \\
&= \sum_{i=1}^{P} \int_{Q_i} \left[ \frac{1}{\beta^p_k} W(\xi^{(i)}_{sk}(x)) + \beta^p_k h^2(a_i, \nabla \xi^{(i)}_{sk}(x)) \right] \, dx + \\
&+ \sum_{i=1}^{P} \int_{Q_i} \beta^p_k \left[ h^2(x, \nabla \xi^{(i)}_{sk}(x)) - h^2(a_i, \nabla \xi^{(i)}_{sk}(x)) \right] \, dx + \\
&+ \int_{(\Omega \cap \{ |x_N| < \eta / 2 \}) \setminus \bigcup_{i=1}^{P} Q_i} \left[ \frac{1}{\beta^p_k} W(u_{k,\varepsilon}(x)) + \beta^p_k h^2(x, \nabla u_{k,\varepsilon}(x)) \right] \, dx + \\
&+ \int_{(\Omega \cap \{ |x_N| > \eta / 2 \})} \beta^p_k h^2(x,0) \, dx =: I_1 + I_2 + I_3 + I_4,
\end{align*}
\]
where, by (4.20),
\[ \lim_{k \to +\infty} I_1 = \eta^{N-1} \sum_{i=1}^{p} K(a_i, a, b, eN) \]

and, since \( \beta^{(p)}_k \to 0^+ \),
\[ \lim_{k \to +\infty} I_4 = 0. \]

Also, as \( u_{k, \epsilon} \) is the periodic extension of \( v_{\beta_k}^{(p)} \) on \( \Omega \cap \left\{ |x_N| < \frac{\eta}{2} \right\} \) \( \cup \bigcup_{i=1}^{p} Q_i \) and
\[ \nu_{\beta_k}(x) = \begin{cases} 
  b & \text{if } x_N > \beta_k^{(p)} \\
  a & \text{if } x_N < \beta_k^{(p)}
\end{cases} \]
with \( \|\nabla v_{\beta_k}^{(p)}\|_{\infty} = O\left(\frac{1}{\beta_k^{(p)}}\right) \) if \( |x_N| < \beta_k^{(p)} \), we have, by (4.15),
\[ I_3 \leq \int \frac{1}{\beta_k^{(p)}} + \frac{1}{\beta_k^{(p)}} \, dx = O(\epsilon). \]

As \( \eta < \delta \), by (4.12) and (H3),
\[ \limsup_{k \to +\infty} I_2 \leq \sum_{i=1}^{p} \int_{Q_i} \beta_k^{(p)} \epsilon C (1+\|\nabla \xi_k^{(i)}(x)\|^2) \, dx \leq \]
\[ \leq \limsup_{k \to +\infty} \sum_{i=1}^{p} \int_{Q_i} \beta_k^{(p)} \epsilon C [1+h^2(a_i, \nabla \xi_k^{(i)}(x))] \, dx = O(\epsilon) \]

since by (4.20) \( \int_{Q_i} \beta_k^{(p)} h^2(a_i, \nabla \xi_k^{(i)}(x)) \, dx \) remains bounded. Finally, we note that, by (4.11), (4.13) and (4.14),
\[ \left| \int_{Q} K(x, a, b, eN) \, dH_{N-1}(x) - \eta^{N-1} \sum_{i=1}^{p} K(a_i, a, b, eN) \right| \leq \]
\[ \leq \int_{\Omega \setminus \bigcup_{i=1}^{p} Q_i} K(x, a, b, eN) \, dH_{N-1}(x) + \sum_{i=1}^{p} \int_{Q_i} \left| K(x, a, b, eN) - K(a_i, a, b, eN) \right| \, dH_{N-1}(x) = O(\epsilon), \]
so to obtain the desired approximating sequence it suffices to let \( \epsilon \to 0^+ \) and use a diagonalization procedure.

**Step 2.** Now suppose that \( u \) has polygonal interface i.e. \( u = \chi_A a + (1-\chi_A)b \) where \( A \subset \Omega \) is of the form \( A = A' \cap \Omega, \partial^* A \cap \Omega = \partial^* A' \cap \Omega \) with \( A' \) a polyhedral set (i.e. \( A' \) is a
bounded, strongly Lipschitz domain and \( 3A^t = Hi u \ldots u HM \cdot Hi \) are closed subsets of hyperplanes of the type \( \{ x \in \mathbb{R}^N : x \cdot Vi = oq \} \). Notice that Step 1 corresponds to the case where \( A^* \) is a large cube. We claim that for any sequence \( \epsilon \to 0^+ \) there exists a subsequence \( \{ \epsilon_n \} \) and a sequence \( \{ u_n \} \) in \( H^K(\mathbb{R}^N) \) such that \( u_n \to u \) in \( L^H(\mathbb{R}^N) \) and

\[
\lim_{n \to 0} \int \left[ W_{Cun} (x) + \epsilon_n V(x, Vu_n(x)) \right] \, dx = \int K(x, a, b, v(x)) \, dH_{N,1}(x). \tag{4.21}
\]

Recall that \( 3^* A_n Q = 3^* A^1_n Cl = \bigcap_{i=1}^M (Hi \cap Cl) \). Let \( I = \{ i \in \{1, \ldots, M\} : U^\wedge \cap (H, nCl) > 0 \} \). If \( \text{card } I = 0 \) then \( u(x) = a \) a.e. in \( \mathbb{R}^N \) or \( u(x) = b \) a.e. in \( Q \) so it suffices to take \( \epsilon^a = \epsilon_n \) and \( u_n = u \), for all \( n \). If \( \text{card } I = 1 \) then \( 9^* A_n \ell I \) reduces to one planar interface and we are back to Step 1. Using an induction procedure, assume that the result is true if \( \text{card } I = k, k^M-1 \) and we prove it is still true if \( \text{card } I = k \). Assume that

\[
d^* A \cap Q = (Hi \cap 0) \cup \ldots \cup (H_k \cap Q).
\]

Consider \( S := \{ x \in \mathbb{R}^N : \text{dist} (x, Hi) = \text{dist} (x, H_2 \cup \ldots \cup H_M) \} \). Then \( S \) is locally the graph of a Lipschitz function and for every \( x \in S \) there exists \( e > 0 \) such that

\[
B(x_0, e) \cap \{ x \in S : \text{dist} (x, Hi) > \text{dist} (x, H_2 \cup \ldots \cup H_M) \}
\]

is connected. Also

\[
d^* A \cap S = \bigcup_{i=1}^k \{ x \in (Hi \cup H_2 \cup \ldots \cup H_M) : \text{dist} (x, Hi) < \text{dist} (x, H_2 \cup \ldots \cup H_M) \}
\]

and so \( H_i = \{ x \in Q : \text{dist} (x, Hi) < \text{dist} (x, H_2 \cup \ldots \cup H_M) \} \) is connected. Also

\[
Q = \{ x \in Q : \text{dist} (x, Hi) < \text{dist} (x, H_2 \cup \ldots \cup H_M) \}
\]

Clearly \( C_{\ell} \) is open and \( C_{\ell} \cap n (H_2 \cup \ldots \cup H_k) = 0 \). Since \( C_{\ell} \) is the intersection of a strongly Lipschitz domain with \( Q \) and \( dC_{\ell} \) is locally Lipschitz it follows that \( C_{\ell} \) is also a strongly Lipschitz domain. We would like to apply the induction hypothesis to \( C_{\ell} \) and to \( C_{\ell} \setminus Q := Q_{\ell} \). But, although \( \text{card } \left( 9^* A \cap n \right) = (H_2 \cup \ldots \cup H_k) \) \( \cap Q \) consists of \( p \) flat interfaces and \( dQ_{\ell} \) is locally Lipschitz, it may happen that \( Q_{\ell} \) is no longer connected. We write

\[
\Omega_2 = \bigcup_{i=1}^q \Omega_i
\]

where \( a^* \) are open, connected, strongly Lipschitz domains with \( \cap \Omega_i \cap \Omega_j = 0 \) for \( i \neq j \). It is easy to verify that if \( i \neq j \) then \( \exists j \in \cap \Omega_i \cap \Omega_j = 0 \). Thus we only need to match the deformations across the interfaces \( dO_i \cap n dC_{\ell} \cap n Cl \). Fix \( \epsilon > 0 \) and let

\[
U_\epsilon = \{ x \in \mathbb{R}^N : \text{dist} (x, S) < \epsilon \},
\]

\[
U_\epsilon' = \{ x \in \mathbb{R}^N : \text{dist} (x, S) < \epsilon, \text{dist} (x, Hi) < \text{dist} (x, H_2 \cup \ldots \cup H_k) \},
\]

\[
U_{\epsilon,1} = \{ x \in \mathbb{R}^N : \text{dist} (x, S) < \epsilon, \text{dist} (x, Hi) > \text{dist} (x, H_2 \cup \ldots \cup H_k) \}.
\]

As \( \text{Pern}(A) < \infty \), choose \( k = k(\epsilon) \) such that
\[ \sum_{i \leq k} \operatorname{Per}_\Omega(\partial^* A \cap \omega_i) < \delta \] (4.22)

and due to Proposition 2.8 i), we can also request that

\[ \sum_{i \leq k} \int_{\partial^* A \cap \omega_i} K(x,a,b,v(x)) \, d\mathcal{H}_{N-1}(x) < \delta. \] (4.23)

Since \( \Omega_1 \) contains only one interface, by Step 1, given any sequence \( \varepsilon_n \to 0^+ \) there exists a subsequence \( \{\varepsilon_n^{(1)}\} \) and a sequence \( \{v_n\} \) in \( H^1(\Omega_1; \mathbb{R}^p) \) such that \( v_n \to u \) in \( L^1(\Omega_1; \mathbb{R}^p) \) and

\[ \lim_{n \to \infty} \int_{\Omega_1} \left[ \frac{1}{\varepsilon_n^{(1)}} W(v_n(x)) + \varepsilon_n^{(1)} h^2(x, \nabla v_n(x)) \right] \, dx = \int_{\partial^* A \cap \Omega_1} K(x,a,b,v(x)) \, d\mathcal{H}_{N-1}(x). \]

\( \partial^* A \cap \omega_1 \) contains at most \( M-1 \) flat interfaces so we can use the induction hypothesis to obtain a subsequence \( \{\varepsilon_n^{(2)}\} \) of \( \{\varepsilon_n^{(1)}\} \) and a sequence \( \{u_n^{(1)}\} \) in \( H^1(\omega_1; \mathbb{R}^p) \) such that \( u_n^{(1)} \to u \) in \( L^1(\omega_1; \mathbb{R}^p) \) and

\[ \lim_{n \to \infty} \int_{\omega_1} \left[ \frac{1}{\varepsilon_n^{(2)}} W(u_n^{(1)}(x)) + \varepsilon_n^{(2)} h^2(x, \nabla u_n^{(1)}(x)) \right] \, dx = \int_{\partial^* A \cap \omega_1} K(x,a,b,v(x)) \, d\mathcal{H}_{N-1}(x). \]

We continue this process inductively in order to obtain subsequences \( \{\varepsilon_n^{(k+1)}\} \subset ... \subset \{\varepsilon_n^{(1)}\} \subset \{\varepsilon_n\} \) and sequences \( \{u_n^{(i)}\} \) in \( H^1(\omega_i; \mathbb{R}^p) \) such that \( u_n^{(i)} \to u \) in \( L^1(\omega_i; \mathbb{R}^p) \) and

\[ \lim_{n \to \infty} \int_{\omega_i} \left[ \frac{1}{\varepsilon_n^{(i+1)}} W(u_n^{(i)}(x)) + \varepsilon_n^{(i+1)} h^2(x, \nabla u_n^{(i)}(x)) \right] \, dx = \int_{\partial^* A \cap \omega_i} K(x,a,b,v(x)) \, d\mathcal{H}_{N-1}(x) \]

for all \( i = 1, ..., k \). Consider the sequence \( \{\varepsilon_n^{(k+1)}\} =: \{\alpha_n\} \) and for all \( i = 1, ..., k \) let \( \{\xi_n^{(i)}\} \) be the corresponding subsequence of \( \{u_n^{(i)}\} \) such that

\[ \lim_{n \to \infty} \int_{\omega_i} \left[ \frac{1}{\alpha_n} W(\xi_n^{(i)}(x)) + \alpha_n h^2(x, \nabla \xi_n^{(i)}(x)) \right] \, dx = \int_{\partial^* A \cap \omega_i} K(x,a,b,v(x)) \, d\mathcal{H}_{N-1}(x) \] (4.24)

and let \( \{v_n^{(1)}\} \) be the corresponding subsequence of \( \{v_n\} \) such that

\[ \lim_{n \to \infty} \int_{\Omega_1} \left[ \frac{1}{\alpha_n} W(v_n^{(1)}(x)) + \alpha_n h^2(x, \nabla v_n^{(1)}(x)) \right] \, dx = \int_{\partial^* A \cap \Omega_1} K(x,a,b,v(x)) \, d\mathcal{H}_{N-1}(x). \] (4.25)

Define \( \tilde{u}(x) = \chi_A(x) a + (1-\chi_A(x)) b \) so that the restriction of \( \tilde{u} \) to \( \Omega \) is \( u \) and let
\[ w_n(.) = \frac{1}{(\alpha_n)^N} \rho(\frac{\cdot}{\alpha_n}) \ast \tilde{u}. \]

Notice that \( w_n \) is bounded in \( L^\infty \) and
\[
\|\nabla w_n\|_\infty = O\left(\frac{1}{\alpha_n}\right); \quad \text{supp} \ \nabla w_n \subset \{ x \in \mathbb{R}^N : \text{dist} (x, \Sigma(\tilde{u})) < \alpha_n \}. \tag{4.26}
\]

Since \( w_n \to u \) and \( v_n^{(1)} \to u \) in \( L^1(U_\delta^+ \cap \Omega_1; \mathbb{R}^p) \) we may use the slicing method to connect \( v_n^{(1)} \) and \( w_n \) across \( U_\delta^+ \cap \Omega_1 \) (as in the proof of Lemma 3.2 we let \( M = C(1 + 2\|u\|_q^2), \ C = \max\{c, \ c \text{ meas} \Omega\} \)
and \( c \) is the constant appearing in (H2) and we divide \( U_\delta^+ \) into \( k_n \) slices of the form \( S_i = \{ x \in U_\delta^+ : \alpha_i < \text{dist} (x, S) < \alpha_{i+1} \} \) where \( \alpha_1 = 0, \ \alpha_{k_n+1} = \delta \) and \( k_n \) is given by \( \left\lceil \frac{2M}{\alpha_n^2} \right\rceil + 1, \ \sigma_n = \|v_n^{(1)} - w_n\|_{L^1(U_\delta^+ \cap \Omega_1; \mathbb{R}^p)} \). We obtain a sequence \( \{\eta_n\} \) in \( H^1(\Omega_1; \mathbb{R}^p) \) such that \( \eta_n \to u \) in \( L^1(\Omega_1; \mathbb{R}^p) \), \( \eta_n = w_n \) on \( \partial \Omega_1 \cap \Omega \) and, by (4.26), (H1), (H2), (H3) and (4.25)
\[
\limsup_{n \to \infty} \int_{\Omega_1} \left[ \frac{1}{\alpha_n} W(\eta_n(x)) + \alpha_n h^2(x, \nabla \eta_n(x)) \right] \, dx \leq \nabla \eta_n^{(1)} - \alpha_n h^2(x, \nabla \eta_n^{(1)}(x)) \right] \, dx \]
\[
\leq \limsup_{n \to \infty} \int_{\Omega_1} \left[ \frac{1}{\alpha_n} W(v_n^{(1)}(x)) + \alpha_n h^2(x, \nabla v_n^{(1)}(x)) \right] \, dx \]
\[
+ \int_{U_\delta^+ \cap \Omega_1} \left[ \frac{1}{\alpha_n} W(w_n(x)) + \alpha_n h^2(x, \nabla w_n(x)) \right] \, dx \leq \int_{\partial^* A \cap \Omega_1} K(x,a,b,\nu(x)) \, dH_{N-1}(x) + \limsup_{n \to \infty} \frac{C}{\alpha_n} \text{meas} \{ x \in U_\delta^+ \cap \Omega_1 : \text{dist} (x, \Sigma(u)) < \alpha_n \} = \int_{\partial^* A \cap \Omega_1} K(x,a,b,\nu(x)) \, dH_{N-1}(x) + C \text{Per}_\Omega(\partial^* A \cap U_\delta^+ \cap \Omega_1). \tag{4.27}
\]

Similarly, for each \( i = 1, \ldots, k \), we connect \( \xi_n^{(i)} \) to \( w_n \) across \( U_\delta^+ \cap \omega_i \) and we obtain sequences \( \varphi_n^{(i)} \) in \( H^1(\omega_i; \mathbb{R}^p) \) such that \( \varphi_n^{(i)} \to u \) in \( L^1(\omega_i; \mathbb{R}^p) \), \( \varphi_n^{(i)} = w_n \) on \( \partial \omega_i \cap \Omega \) and
\[
\limsup_{n \to \infty} \int_{\omega_n} \left[ \frac{1}{\alpha_n} W(\varphi_n^{(i)}(x)) + \alpha_n h^2(x, \nabla \varphi_n^{(i)}(x)) \right] \, dx \leq \int_{\partial^* A \cap \omega_i} K(x,a,b,v(x)) \, dH_{N-1}(x) + C \text{ Per}_{\Omega}(\partial^* A \cap U^+ \cap \omega_i).
\]

(4.28)

Define
\[
u_n(x) := \sum_{i=1}^k \chi_{\omega_i}(x) \varphi_n^{(i)}(x) + \chi_{\Omega \setminus \Omega_n}(x) \eta_n(x) + \sum_{i=k+1}^{\infty} \chi_{\omega_i}(x) w_n(x).
\]

Then,
\[
\|u_n - u\|_{L^1(\Omega; \mathbb{R}^p)} \leq \sum_{i=1}^k \int_{\omega_i} |\varphi_n^{(i)}(x) - u(x)| \, dx + \int_{\Omega_i} |\eta_n(x) - u(x)| \, dx + C \sum_{i=k+1}^{\infty} \text{ meas}\{x \in \omega_i : \text{dist}(x, \Sigma(u)) < \alpha_n\} \leq \sum_{i=1}^k \int_{\omega_i} |\varphi_n^{(i)}(x) - u(x)| \, dx + \int_{\Omega_i} |\eta_n(x) - u(x)| \, dx + C \text{ meas}\{x \in \Omega : \text{dist}(x, \Sigma(u)) < \alpha_n\}.
\]

Hence,
\[
\lim_{n \to \infty} \|u_n - u\|_{L^1(\Omega; \mathbb{R}^p)} \leq \sum_{i=1}^k \lim_{n \to \infty} \int_{\omega_i} |\varphi_n^{(i)}(x) - u(x)| \, dx + \lim_{n \to \infty} \int_{\Omega_i} |\eta_n(x) - u(x)| \, dx + \lim_{n \to \infty} C \text{ meas}\{x \in \Omega : \text{dist}(x, \Sigma(u)) < \alpha_n\} = 0
\]

since \(\Sigma(u)\) is polyhedral. On the other hand, by (4.26), (H1), (H2) and (H3)
\[
\int_{\Omega} \left[ \frac{1}{\alpha_n} W(u_n(x)) + \alpha_n h^2(x, \nabla u_n(x)) \right] \, dx \leq \sum_{i=1}^k \int_{\omega_i} \left[ \frac{1}{\alpha_n} W(\varphi_n^{(i)}(x)) + \alpha_n h^2(x, \nabla \varphi_n^{(i)}(x)) \right] \, dx + \int_{\Omega_1} \left[ \frac{1}{\alpha_n} W(\eta_n(x)) + \alpha_n h^2(x, \nabla \eta_n(x)) \right] \, dx + \sum_{i=k+1}^{\infty} C \alpha_n \text{ meas}\{x \in \omega_i : \text{dist}(x, \Sigma(u)) < \alpha_n\} = \sum_{i=1}^k \int_{\omega_i} \left[ \frac{1}{\alpha_n} W(\varphi_n^{(i)}(x)) + \alpha_n h^2(x, \nabla \varphi_n^{(i)}(x)) \right] \, dx + \int_{\Omega_1} \left[ \frac{1}{\alpha_n} W(\eta_n(x)) + \alpha_n h^2(x, \nabla \eta_n(x)) \right] \, dx + \frac{C}{\alpha_n} \text{ meas}\{x \in \bigcup_{i=k+1}^{\infty} \omega_i : \text{dist}(x, \Sigma(u)) < \alpha_n\}.
\]

Thus, by (4.22), (4.23), (4.27) and (4.28),
\begin{align*}
\lim \sup_{n \to \infty} \int_{\Omega} \left[ \frac{1}{\alpha_n} W(u_n(x)) + \alpha_n h^2(x, \nabla u_n(x)) \right] \, dx \leq \sum_{i=1}^{k} \int_{\omega_i \cap \partial^+ A} K(x, a, b, \nabla \psi(x)) \, dH_{N-1}(x) + \\
+ C \sum_{i=1}^{k} \text{Per}_{\Omega}(\partial^+ A \cap \omega_i^+ \cap \omega_i) + \int_{\Omega \cap \partial^+ A} K(x, a, b, \nabla \psi(x)) \, dH_{N-1}(x) + C \text{Per}_{\Omega}(\partial^+ A \cap \omega_\delta \cap \Omega_i) + \\
+ C \text{Per}_{\Omega}(\partial^+ A \cap (\bigcup_{i=k+1} \omega_i)) \leq \\
\leq \int_{\Omega \cap \partial^+ A} K(x, a, b, \psi(x)) \, dH_{N-1}(x) + O(\delta).
\end{align*}

Hence we proved that for all \( \delta > 0 \) there exists a subsequence \( \{\alpha_n(\delta)\} \) of \( \{\epsilon_n\} \) and there exists a sequence \( \{u_n(\delta)\} \) such that

i) \( \lim_{n \to \infty} \|u_n(\delta) - u\|_{L^1(\Omega, \mathbb{R}^p)} = 0; \)

ii) \( \lim \sup_{n \to \infty} \int_{\Omega} \left[ \frac{1}{\alpha_n(\delta)} W(u_n(\delta)(x)) + \alpha_n(\delta) h^2(x, \nabla u_n(\delta)(x)) \right] \, dx \leq \\
\leq \int_{\Omega \cap \partial^+ A} K(x, a, b, \psi(x)) \, dH_{N-1}(x) + O(\delta). \)

Let \( \delta = 1 \) and choose \( n_1 \) such that
\[
\|u_{n_1}(1) - u\|_{L^1(\Omega, \mathbb{R}^p)} \leq 1
\]
and
\[
\int_{\Omega} \left[ \frac{1}{\alpha_1(1)(x))} W(u_{n_1}(1)(x)) + \alpha_1(1) h^2(x, \nabla u_{n_1}(1)(x)) \right] \, dx \leq \int_{\Omega \cap \partial^+ A} K(x, a, b, \psi(x)) \, dH_{N-1}(x) + 2O(1).
\]

Suppose that \( \alpha_{n_1}(1) = \alpha_{j_1} \). Now let \( \delta = \frac{1}{2} \) and choose \( n_2 \) large enough so that
\[
\alpha_{n_2}(2) = \alpha_j \text{ with } j_2 > j_1,
\]
\[
\|u_{n_2}(2) - u\|_{L^1(\Omega, \mathbb{R}^p)} \leq \frac{1}{2}
\]
and

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Continuing this process, we choose \( n^* \) large enough so that
\[
o_{nk}(k) = c^k \text{ with } j_k > j_k \cdot (x) > J2 > j_i.
\]
and
\[
\|u_n(k) - u\|_{L^1(\Omega; \mathbb{R}^p)} \leq \frac{1}{k}
\]
and
\[
\int_{\Omega} \left[ \frac{1}{\alpha_{nk}} W(u_{nk}(k)(x)) + c^k(x) \right] dx \leq \int_{\Omega} K(x, a, b, v(x)) \, dH_{N-1}(x) + 20(\frac{1}{\alpha_{nk}}).
\]
Then \( \{o^k(k)\} \) is a subsequence of \( \{a_n\} \) and defining \( v_k := i^k(k) \) we have \( v_k \rightarrow u \) in \( L^1(\Omega; \mathbb{R}^p) \) and
\[
\limsup_{k \rightarrow \infty} \int_{\Omega} \left[ \frac{1}{\alpha_{nk}} W(v_k(x)) + a_n(k) h^2(x, Vv_k(x)) \right] dx \leq \int_{\Omega} K(x, a, b, v(x)) \, dH_{N-1}(x).
\]
This, together with Proposition 3.1, gives the result.

*Step 3.* Finally consider an arbitrary \( u = X \alpha + O \cdot X A \cdot b \) with \( \text{Pern}(A) < +\infty \). By Theorem 2.6 there exist polyhedral sets \( A^* \) such that \( j^k \rightarrow X A \) in \( L^1(Q) \), \( \text{Pern}(A_k) - \rightarrow \text{Pern}(A) \) and \( \text{meas}(A_k) = \text{meas}(A) \). By Step 2 for every \( k \) there exist sequences \( E^k \rightarrow 0^k \) as \( n \rightarrow +\infty \), and \( u^{(k)}_n \)
\[
\rightarrow X \alpha_k^a + (i^k k^) b \text{ as } n \rightarrow +\infty \text{ in } L^1(\Omega; \mathbb{R}^p) \text{ such that}
\]
\[
\lim_{n \rightarrow +\infty} \int_{\Omega} \left[ \frac{1}{\alpha_{nk}} W(u_{nk}(k)(x)) \right] dx = \int_{\Omega} K(x, a, b, v(x)) \, dH_{N-1}(x).
\]
Consider \( n(k) \) such that
\[
\|u^{(k)}_n - (\chi_{A^k_2 a} + (1-\chi_{A^k_2 b}) b))\|_{L^1(\Omega; \mathbb{R}^p)} \leq 1/k
\]
and
\[
\int_{\Omega} K(x, a, b, v(x)) \, dH_{N-1}(x) \cdot \int_{\Omega} \left[ \frac{1}{\alpha_{nk}} W(u_{nk}(k)(x)) + e^{k k} h^2(x, Vv_{nk}(k)(x)) \right] dx < \frac{1}{k}
\]
with \( \rho \leq e^k \). Set \( v_k = u^k \) and \( e^k \rightarrow 0 \); then
\[ v_k \rightarrow \chi_A a + (1-\chi_A)b \text{ in } L^1(\Omega;\mathbb{R}^p) \]

and for every continuous function \( g : \Omega \times \mathbb{R}^p \rightarrow [0, +\infty) \) we have

\[
\int_{\partial^* A \cap \Omega} g(x, v(x)) \, d\mathcal{H}_{N-1}(x) \rightarrow \int_{\partial^* A \cap \Omega} g(x, v(x)) \, d\mathcal{H}_{N-1}(x).
\]

As \( K(\cdot, a, b, \cdot) \) is upper semicontinuous there exist continuous functions \( g_m : \Omega \times \mathbb{R}^p \rightarrow [0, +\infty) \) such that

\[
K(x, a, b, \xi) \leq g_m(x, \xi) \leq C|\xi|,
\]

and

\[
K(x, a, b, \xi) = \inf_m g_m(x, \xi)
\]

for every \((x, \xi) \in \Omega \times \mathbb{R}^p\), where we have extended \( K(x, a, b, \cdot) \) as a homogeneous function of degree one (see [FM2], Lemma 2.15 and Step 3, Section 5). Thus for all \( m \)

\[
\limsup_{k \rightarrow +\infty} \int_{\Omega} \left[ \frac{1}{\alpha_k} W(v_k(x)) + \alpha_k h^2(x, \nabla v_k(x)) \right] \, dx
\]

\[
= \limsup_{k \rightarrow +\infty} \int_{\partial^* A \cap \Omega} K(x, a, b, v(x)) \, d\mathcal{H}_{N-1}(x) \leq \limsup_{k \rightarrow +\infty} \int_{\partial^* A \cap \Omega} g_m(x, v(x)) \, d\mathcal{H}_{N-1}(x)
\]

\[
= \int_{\partial^* A \cap \Omega} g_m(x, v(x)) \, d\mathcal{H}_{N-1}(x) .
\]

Taking the limit when \( m \rightarrow +\infty \) and using Lebesgue's Monotone Convergence Theorem we deduce that

\[
\limsup_{k \rightarrow +\infty} \int_{\Omega} \left[ \frac{1}{\alpha_k} W(v_k(x)) + \alpha_k h^2(x, \nabla v_k(x)) \right] \, dx \leq \int_{\partial^* A \cap \Omega} K(x, a, b, v(x)) \, d\mathcal{H}_{N-1}(x)
\]

which, by Proposition 3.1 concludes the proof. \( \square \)

5. A constrained penalized minimization problem.

In this section we assume the following additional hypotheses:

(H6) \( W \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^p); \)

(H7) there exist constants \( \alpha, \delta > 0 \) such that

\[ l|u - a| < \delta \Rightarrow \alpha l|u - a|^q \leq W(u) \leq \frac{1}{\alpha} l|u - a|^q \]

and
\[ ||u - b|| < \delta \Rightarrow \alpha ||u - b||^q \leq W(u) \leq \frac{1}{\alpha} ||u - b||^q; \]

**H8** \( h^2(x, \cdot) \) is quasiconvex for all \( x \in \Omega \).

In (H8) we could just as well have taken \( h(x, \cdot) \) quasiconvex since by Jensen's inequality the quasiconvexity of \( h^2(x, \cdot) \) follows trivially.

We consider the following minimization problem:

**(P) minimize**

\[
E(u) = \int_{\Omega} W(u(x)) \, dx
\]

subject to the constraint \( \frac{1}{\text{meas}(\Omega)} \int_{\Omega} u(x) \, dx = m \), where \( m = \theta a + (1-\theta)b \) for some \( \theta \in (0,1) \).

Clearly any piecewise constant function of bounded variation of the form \( u = \chi_A a + (1-\chi_A)b \) with \( \text{meas}(A) = \theta \text{meas}(\Omega) \) is a solution of (P), so there exist an infinite number of solutions to this problem. In order to single out one of them, and keeping in mind that the Wulff set is the preferred shape for some types of materials for which the surface energy density is anisotropic (see [Fo], [FM3], [T1], [T2], [W]), we consider the family of anisotropic singular perturbations

\[
E_\epsilon(u) = \int_{\Omega} \left[ W(u(x)) + \epsilon^2 h^2(x, \nabla u(x)) \right] \, dx
\]

and the corresponding minimization problems

**(P\(\epsilon\)) minimize** \( E_\epsilon(u) \) on \( \left\{ u \in H^1(\Omega; \mathbb{R}^P) : \frac{1}{\text{meas}(\Omega)} \int_{\Omega} u(x) \, dx = m \right\} \).

Since \( h^2 \) is quasiconvex, the growth conditions on \( W \) and \( h \) guarantee the existence of a minimizer \( u_\epsilon \) of \( E_\epsilon \), by direct methods of the Calculus of Variations. We show that the solutions \( u_\epsilon \) of \( (P\epsilon) \) select the solution of \( (P) \) which minimizes the integral over the interface \( \Omega \cap \partial^*\{u=a\} \) of the surface energy density, namely:

**Theorem 5.1.** Assume hypotheses (H1)-(H7) hold and let \( \{u_n\} \) be a sequence of minimizers of \( E_n \) converging to \( u_0 \) in \( L^1(\Omega; \mathbb{R}^P) \). Then \( u_0 \) is a solution of the problem:

\[
(P) \text{ minimize } \int_{\Omega - \partial^*\{u=a\}} K(x, a, b, \nabla u(x)) \, dH_{N-1}(x) =: J_0(u) \text{ on } V := \left\{ u \in BV(\Omega; \mathbb{R}^P), u \in \{a,b\} \text{ a.e.} : \frac{1}{\text{meas}(\Omega)} \int_{\Omega} u(x) \, dx = m \right\}.
\]

**Proof.** We follow the proof of Fonseca and Tartar [FT]. We assume that \( \Omega = Q \) and that \( h \) and \( K \) are independent of \( x \). Since all subsequent constructions were based on this one it is easy to see that the theorem remains valid in the general case.
Step 1. We begin by showing that there exists a constant $C > 0$ such that $J(\mathbf{u}_n) \leq C$ for sufficiently large $n$. Indeed, let $\gamma$ be a smooth function with compact support that satisfies $\gamma(-1) = 0$, $\gamma(1) = 1$ and $0 \leq \gamma \leq 1$. Define

$$w_n(x) = \begin{cases} a & \text{if } x_N > x_n + \varepsilon_n \\ \gamma\left(\frac{x_N - x_n}{\varepsilon_n}\right)a + (1 - \gamma\left(\frac{x_N - x_n}{\varepsilon_n}\right))b & \text{if } |x_N - x_n| < \varepsilon_n \\ b & \text{if } x_N < x_n - \varepsilon_n \end{cases}$$

where $x_n$ is chosen so that

$$\text{meas}\{(x \in Q : x_N > x_n + \varepsilon_n)\} + \int\limits_{\{x \in Q : |x_N - x_n| < \varepsilon_n\}} \gamma\left(\frac{x_N - x_n}{\varepsilon_n}\right) dx = \theta.$$

It follows that $w_n \in H^1(Q; \mathbb{R}^p)$ and $\int Q w_n(x) \, dx = m$ so by (H3) we have

$$J_{\varepsilon_n}(u_n) \leq J_{\varepsilon_n}(w_n) \leq \frac{1}{\varepsilon_n} \int\limits_{\{x \in Q : |x_N - x_n| < \varepsilon_n\}} W(w_n(x)) \, dx + \varepsilon_n \int\limits_Q C (1 + \|\nabla w_n(x)\|^2) \, dx \leq \text{const.} \max\{W(v) : v \in [a,b]\} + \varepsilon_n \frac{1}{\varepsilon_n} \int\limits_{\{x \in Q : |x_N - x_n| < \varepsilon_n\}} \|\gamma'\left(\frac{x_N - x_n}{\varepsilon_n}\right)\|^2 \|a - b\|^2 \, dx \leq \text{const.} \max\{W(v) : v \in [a,b]\} + 1 + \|\gamma'\|^2 \|a - b\|^2,$$

where $[a,b]$ denotes the line segment joining $a$ and $b$.

Step 2. By Step 1 and Proposition 3.1 it follows that $u_0 \in V$ and

$$\lim \inf_{n \to +} J\left(\mathbf{u}_n\right) \geq J_0(u_0). \quad (5.1)$$

Let $u \in V$. It suffices to show that there exists a sequence $\varepsilon_n \to 0^+$ and a sequence $\{v_n\}$ in $H^1(Q; \mathbb{R}^p)$ such that $v_n \to u$ in $L^1(Q; \mathbb{R}^p)$, $\lim_{n \to +} J_{\varepsilon_n}(v_n) = J_0(u)$ and $\int Q v_n(x) \, dx = m$. Then, by (5.1) and since $u_n$ is a minimizer of $E_{\varepsilon_n}$, we have

$$J_0(u) = \lim_{n \to +} J_{\varepsilon_n}(v_n) \geq \lim \sup_{n \to +} J_{\varepsilon_n}(u_n) \geq J_0(u_0)$$

and thus $u_0$ is a solution of (P). To prove the existence of $\{v_n\}$ we modify the sequence $\{w_n\}$ constructed in the proof of Step 1 of Lemma 4.2 to obtain a new sequence satisfying

$$\lim_{n \to +} J_{\varepsilon_n}(v_n) = \lim_{n \to +} J_{\varepsilon_n}(w_n) \quad \text{and} \quad \int Q v_n(x) \, dx = m.$$

Let $L_n > 0$ and $\xi_n \in \mathscr{A}(a,b,\varepsilon_N)$ be such that

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\[
\lim_{n \to \infty} \int_{Q} \left[ L_n W(\xi_n(x)) + \frac{1}{L_n} (h^n)^2(\nabla \xi_n(x)) \right] dx = K(a, b, c_n).
\]

For \( n \) fixed, we defined
\[
w^n_\delta(x) = w^n_\delta(x', x_N) = \begin{cases} 
b & \text{if } x_N > \delta/2 \\
\xi_n(x', x_N/\delta) & \text{if } -\delta/2 \leq x_N \leq \delta/2 \\
a & \text{if } x_N < -\delta/2.
\end{cases}
\]

We showed in Lemma 4.2 that
\[
\|w^n_\delta - u\|_{L^1(Q; \mathbb{R}^n)} = O(\delta)
\] (5.2)

and
\[
\int_{Q} \left[ \frac{L_n}{\delta} W(w^n_\delta(x)) + \frac{\delta}{L_n} h^n(\nabla w^n_\delta(x)) \right] dx = \int_{Q} \left[ L_n W(\xi_n(x)) + \frac{1}{L_n} (h^n)^2(\nabla \xi_n(x)) \right] dx + \\
+ \text{const.} \frac{\delta}{L_n} + \text{const.} (\frac{\delta}{L_n})^{m/2}.
\]

Let \( \tilde{w}^{n}_\delta = w^{n}_\delta + m - \int_{Q} w^{n}_\delta(x) dx \). Then \( \int_{Q} \tilde{w}^{n}_\delta(x) dx = m \) and \( \tilde{w}^{n}_\delta \to u \) in \( L^1(Q; \mathbb{R}^n) \) as \( \delta \to 0^+ \). We claim that, for \( n \) fixed, \( \int_{Q} \tilde{w}^{n}_\delta(x) dx = m \). Indeed, since \( \nabla \tilde{w}^{n}_\delta(x) = \nabla w^{n}_\delta(x) \),
\[
\int_{Q} \frac{L_n}{\delta} [W(\tilde{w}^{n}_\delta(x)) - W(w^{n}_\delta(x))] dx = \int_{Q} \frac{L_n}{\delta} [W(\tilde{w}^{n}_\delta(x)) - W(w^{n}_\delta(x))] dx = \\
= \int_{Q \cap \{x_N < -\delta/2\}} \frac{L_n}{\delta} W(a + m - \int_{Q} w^{n}_\delta(x) dx) dx + \\
+ \int_{Q \cap \{x_N > \delta/2\}} \frac{L_n}{\delta} W(b + m - \int_{Q} w^{n}_\delta(x) dx) dx + \\
+ \int_{Q \cap \{|x_N| < \delta/2\}} \frac{L_n}{\delta} [W(w^{n}_\delta(x)) + m - \int_{Q} w^{n}_\delta(x) dx] - W(w^{n}_\delta(x))] dx.
\]

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By (H7) and (5.2),
\[
\int_{Q_n(x_N < S/2)} L_n W(a + m - \int_{\partial Q} w^p_\delta(x) \, dx) \, dx \leq \int_{Q} \int_{Q} W(b + \delta - \int_{\partial Q} \xi(x) \, dx) \, dx \leq \int_{Q} \int_{Q} W(b + \delta - \int_{\partial Q} \xi(x) \, dx) \, dx = O(\delta),
\]
and by (H6) and (5.2),
\[
\begin{align*}
\int_{Q_n(x_N > S/2)} & \int_{Q} \int_{Q} \left[ \int_{\partial Q} [W(w^p_\delta(x) + m - \int_{\partial Q} w^p_\delta(x) \, dx) - W(w_\delta(x))] \, dx \right] \, dx \leq \text{const.} \cdot \frac{1}{\delta} \cdot \text{meas} \{ x \in Q : |x_N| < S/2 \} \int_{Q} \int_{Q} \left[ \int_{\partial Q} [W(w^p_\delta(x) + m - \int_{\partial Q} w^p_\delta(x) \, dx) - W(w_\delta(x))] \, dx \right] \, dx = O(\delta).
\end{align*}
\]
Choose \( \delta(n) \) such that \( \delta(n) \to 0^+ \), \( \delta^+ \to 0^+ \), \( II\xi \to v \), \( \text{ull} \), \( \text{ull} \), \( \text{ull} \), and
\[
\begin{align*}
\text{if} \quad & \int_{Q} \int_{Q} \left[ L_n W(\xi_n(x)) + \frac{1}{L_n} (h-\delta)^2(\nabla \xi_n(x)) \right] \, dx - \int_{Q} \int_{Q} \left[ L_n W(\xi_{\delta(n)}(x)) + \frac{1}{L_n} (h-\delta)^2(\nabla \xi_{\delta(n)}(x)) \right] \, dx \leq \frac{1}{n},
\end{align*}
\]
Let \( \xi_n = \tilde{w} f_{\alpha^0} \) and \( \xi_0 = \tilde{w} \). Then \( \tilde{w} = 0^+ \), \( v_n \to 0 \), \( u_n \to 0 \) in \( L^p(Q; IR) \), \( f \), \( v_n(x) \, dx = m \) and
\[
\begin{align*}
\lim_{n \to +\infty} & \int_{Q} \int_{Q} \left[ L_n W(\xi_n(x)) + \frac{1}{L_n} (h-\delta)^2(\nabla \xi_n(x)) \right] \, dx = \\
&= \lim_{n \to +\infty} \int_{Q} \int_{Q} \left[ L_n W(\xi_n(x)) + \frac{1}{L_n} (h-\delta)^2(\nabla \xi_n(x)) \right] \, dx = K(a,b,e_\eta) = \int_{Q} \int_{Q} K(a,b,e_\eta) \, dH_{n} = J_0(u).
\end{align*}
\]
Final Comments: In the isotropic case with no explicit dependence on \( x \), i.e. \( h(Vu) = IIVu \), it was shown in [FT] that
\[
K(a,b',v) = K
\]
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where
\[
\overline{K} = 2 \inf \left\{ \frac{1}{\int W(g(s)) h^\#(g'(s)) \, ds : g \text{ is piecewise } C^1, g(-1) = a, g(1) = b} \right\}.
\]

The same result was proved by [OS] in the anisotropic scalar case. We conjecture that, in general, in the anisotropic case \( K > \overline{K} \).

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