On the Exact Solution of the Minimal Controllability Problem

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Abstract

This paper studies the *minimal controllability problem* (MCP), i.e., the problem of, given a linear time-invariant system, finding the sparsest input vector that ensures system's controllability. We show that the MCP can be exactly solved by (polynomially) reducing it to the minimum set covering problem, when the system dynamics matrix A is simple and the left-eigenbasis associated to A is known. In addition, we also show that the approximated solutions to the minimum set covering problem lead to feasible (sub-optimal) solutions to the MCP; hence, bounds on the optimality gap can be immediately obtained from existing literature on approximation algorithms to the minimum set covering problem. Further, we prove that under the same assumption the decision version of the minimal controllability problem is NP-complete, i.e., there exists an input vector that ensures system's controllability and at most a prescribed number of nonzero entries. Another contribution is the fact that we analyze the relation of the MCP with its *structural* counterpart, the *minimal structural controllability problem* (MSCP) which is known to admit a polynomial complexity solution procedure. We provide an illustrative example where the solution to the MCP is found using the main results and reductions developed in this paper; in particular, we show that the solution to the MSCP is not a solution to MCP even when the dynamic matrix considered to the MCP is simple; hence, disproving the general belief that a solution to MSCP is a solution to MCP when the dynamic matrix is simple, and the eigenbasis structure known.

I. INTRODUCTION

The problem of guaranteeing that a dynamic system can be driven toward a desired steady state regardless of its initial position is a fundamental question that has been studied in control systems and it is referred to as *controllability*. Several applications, for instance, control processes, control of large-flexible structures, systems biology and power systems [1]–[3] rely on the notion of controllability to safeguard their proper functioning. Furthermore, as the systems become larger (i.e., the dimension of their state space), we (often) aim to identify a relatively small subset of state variables that ensure the controllability of the system, for instance due to economic constraints [4]. Consequently, it is natural to pose the following question: *Which state variables need to be directly actuated to ensure the controllability of a dynamical system?*

The *minimal controllability problem* (MCP) relates with the aforementioned question in the sense that it is the problem of identifying the sparsest input vector that ensures system's controllability, see [4]. In [4], this problem was stated as follows: determine the sparsest input vector $b \in \mathbb{C}^n$ (assuming that it exists) such that, given a dynamic matrix $A \in \mathbb{C}^{n \times n}$, associated with a linear time-invariant system (LTI),

$$\dot{x}(t) = Ax(t) + bu(t), \tag{1}$$

with state $x \in \mathbb{C}^n$ and input $u \in \mathbb{C}$, is controllable. We refer to the system in (1) by the pair (A, b), and if (1) is controllable, we say that the pair (A, b) is controllable. Whereas in [4] polynomial algorithms provide an approximate solution to the MCP, hereafter we provide a systematic method to determine a solution to the MCP.

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Problem Statement

Given a left-eigenbasis[1] associated with the dynamics simple[2] matrix A, determine the input vector $b^* \in \mathbb{C}^n$ such that

$$b^* = \arg \min_{b \in \mathbb{R}^n} \quad \|b\|_0$$
$$\text{s.t.} \quad (A, b) \text{ controllable,}$$

(2)

where $\|b\|_0$ is the zero (quasi) norm that counts the number of non-zero entries in b. ○

In [4], some variations of the problem above were explored; for instance, when the inputs are assumed to be *dedicated*, i.e., an input actuates a single state variable. This case can be retrieved from the above problem, when A is simple, see Lemma 4 in [4].

Remark 1. *Problem* (2) *only has a feasible solution if the maximum algebraic multiplicity of the eigenvalues equals one [5]; more precisely, let the maximum algebraic multiplicity of the eigenvalues of A be q, then the pair (A, B) is controllable, where B is an input matrix, only if $\operatorname{rank}(B) = q$. In particular, if a matrix is simple then it has a unique left-eigenbasis (up to non-zero scalar multiplication) whose the zero/nonzero pattern is kept unchanged.* ◇

A. Related Work

The understanding of which state variables need to be actuated to ensure certain properties of the system has been an active research area [6]. Initially, the goal was to establish stability and/or asymptotic stability of the dynamics with respect to some reference point, for instance, consensus or agreement value [7], [8]. The trend has changed to ensure that the system is controllable, since (often) we want to ensure that a control law exists such that an arbitrary goal or desired state is achieved in finite time.

This paper follows up and subsumes some of the existing literature where the dynamic matrix is assumed to be the Laplacian, symmetric (modeling undirected graphs) and/or irreducible (modeling directed graphs with the digraph representation being a strongly connected component). More precisely, in [9] the controllability of circulant networks is analyzed by exploring the Popov-Belevitch-Hautus eigenvalue criterion, where the eigenvalues are characterized using the Cauchy-Binet formula. The controllability in multi agents with Laplacian dynamics was initially explored in [10]. Later, in [11], [12] the controllability for Laplacian dynamics is studied, and necessary and sufficient conditions are given in terms of partitions of the graph. In [13], the controllability is explored for paths and cycles, and later extended by the same authors to the controllability of grid graphs by means of reductions and symmetries of the graph [14], and considering dynamics that are scaled Laplacians. In [15], [16] the controllability is studied for strongly regular graphs and distance regular graphs. Recently, in [17], [18] additional insights on the controllability of Laplacian dynamics are given in terms of the uncontrollable subspace. In addition, in [19] the controllability of isotropic and anisotropic networks is analyzed. Further, [17] concludes by pointing out that further study for non-symmetric dynamics and the controllability is required, as well as determining if polynomial algorithms exist to determine such structures. We notice that *in several of the former cases the matrices are simple*, so by showing that the minimal controllability problem, given a left-eigenbasis, can be reduced to the minimum set covering problem, we address a closely related question (since we do not restrict to Laplacian dynamics); hence, the present paper can provide useful insights in addressing that question, when the eigenvectors structure is known.

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[1]The *eigenbasis* is the set of linear independent eigenvectors that span the entire space spanned by the columns of a square matrix. Consequently, an eigenbasis exists only if the sum of the geometric multiplicities equals the dimension of the space (or, alternatively, the sum of algebraic multiplicities). A *left-eigenbasis* corresponds to the eigenbasis that consists of left-eigenvectors.

[2]A matrix is said to be *simple* if all its eigenvalues are distinct.

A paradigm shift was introduced in [4], where it is shown that determining the minimum number of state variables that need to be actuated to ensure controllability of the dynamical systems is NP-hard. Consequently, efficient algorithms to determine the state variables that need to be actuated to achieve controllability are unlikely to exist. Thus, in [4] approximation polynomial algorithms that provide guarantees on the optimality gap were provided. In this paper, we study how exact solutions to the minimal controllability problem can be obtained. Whereas in [4] was shown that it is possible to polynomially reduce any instance of a problem that is know to be difficult (NP-hard) to an instance of the minimal controllability problem, here we aim to reduce the minimal controllability problem to a well known discrete optimization problem. Notice this is not an easy task since we are reducing a static continuous optimization problem with possibly an infinite number of solutions to a discrete optimization problem with finite number of solutions, for which we show that the obtained solutions permit to characterize the infinite dimensional space of the solutions to the minimal controllability problem.

Alternatively, in [20] instead of determining the sparsest input matrix ensuring the controllability, the aim is to determine the sparsest input matrix that ensures *structural controllability*, which we refer to as the *minimal structural controllability problem* (MSCP) – see Section II for formal definitions and problem statement. Briefly, the MSCP focus on the structure of the dynamics, i.e., the location of zeros/nonzeros, and the obtained sparsest input matrix is such that for almost all matrices satisfying the structure of the dynamics and the input matrix, the system is controllable [21]. Finally, in the present paper we provide an example where the solution to the minimal structural controllability problem is not necessarily a solution to the minimal controllability problem when the dynamic matrix is simple; hence, *disproving the general belief that a solution to MSCP is a solution to MCP in such cases.*

B. Paper Contributions

The main contributions of the present paper are as follows: (i) we characterize the exact solutions to the minimal controllability problem, by reducing it to the minimum set covering problem, when the left-eigenbasis of a dynamics simple matrix is known; (ii) we show that for a given matrix almost all input vectors satisfying a specified structure are solutions to the minimal controllability problem; this differs from the definition of structural controllability in the sense that for a fixed dynamics matrix almost all numerical realizations of the input with certain structure ensure controllability of the system; (iii) we show that a solution to the minimal structural controllability problem is not necessarily a solution to the minimal controllability problem, even when the dynamic matrix is simple, despite of the fact that it provides an approximation that holds almost always; (iv) we show that that approximated solutions to the set covering problem yield feasible (but sub-optimal) solutions to the minimal controllability problem; and (v) we discuss the strategies and their limitations of the proposed methods.

C. Notation

We denote vectors by small font letters such as v, w, b and its corresponding entries by subscripts; for example, v_j corresponds to the j-th entry in the vector v. A collection of vectors is denoted by $\{v^j\}_{j \in \mathcal{J}}$, where the superscript indicates an enumeration of the vectors using indices from a set (usually denoted by calligraphic letter) such as $\mathcal{I}, \mathcal{J} \subset \mathbb{N}$. The number of elements of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. Real-valued matrices are denoted by capital letters, such as A, B and $A_{i,j}$ denotes the entry in the i-th row and j-th column in matrix A. We denote by I_n the n-dimensional identity matrix. Given a matrix A, $\sigma(A)$ denotes the set of eigenvalues of A, also known as the *spectrum* of A. Given two matrices $M_1 \in \mathbb{C}^{n \times m_1}$ and $M_2 \in \mathbb{C}^{n \times m_2}$, the matrix $[M_1 \ M_2]$ corresponds to the $n \times (m_1 + m_2)$ concatenated complex matrix. The structural pattern of a vector/matrix (i.e., the zero/non-zero pattern) or a *structural vector/matrix* have their entries in $\{0, \star\}$, where $\star$ denotes a non-zero entry, and they are denoted by a vector/matrix with a bar on top of it. In other words, $\bar{A}$ denotes a matrix with $\bar{A}_{i,j} = 0$ if $A_{i,j} = 0$ and $\bar{A}_{i,j} = \star$

otherwise. We denote by A^{T} the transpose of A. The function $\cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denotes the usual inner product in $\mathbb{C}^n$, i.e., $v \cdot w = v^\dagger w$, where $v^\dagger$ denotes the adjoint of v (the conjugate of v^{T}). With some abuse of notation, $\cdot : \{0, \star\}^n \times \{0, \star\}^n \to \{0, \star\}$ also denotes the map where $\bar{v} \cdot \bar{w} \neq 0$, with $\bar{v}, \bar{w} \in \{0, \star\}^n$ if and only if there exists $i \in \{1, \ldots, n\}$ such that $\bar{v}_i = \bar{w}_i = \star$. Additionally, $\|v\|_0$ denotes the number of non-zero entries of the vector v in either $\{0, \star\}^n$ or $\mathbb{R}^n$. Given a subspace $\mathcal{H} \subset \mathbb{C}^n$ we denote by $\mathcal{H}^c$ the subspace $\mathbb{C}^n \setminus \mathcal{H}$. In addition, inequalities involving vectors are to be interpreted component-wise. With abuse of notation, we will use inequalities involving structural vectors as well – for instance, we say $\bar{v} \geq \bar{w}$ for two structural vectors $\bar{v}$ and $\bar{w}$ if and the only if the following two conditions hold: i) if $\bar{w}_i = 0$, then $\bar{v}_i \in \{0, \star\}$ and ii) if $\bar{w}_i = \star$ then $\bar{v}_i = \star$. Some other operations over vectors are also considered: given $v \in \mathbb{C}^n$, the vector $v_{1:k}$ (with k a natural number satisfying $1 \leq k \leq n$) represents a vector in $\mathbb{C}^k$ with its entries equal to the first k entries of v. Given $v \in \mathbb{C}^n$ and $\bar{b} \in \{0, \star\}^m$, where $m \leq n$, we denote by $v|_{\bar{b}}$ the real vector of dimension $\|\bar{b}\|_0 \leq m$ where its entries matches those of v at the positions where $\bar{b}$ is non-zero, and such that the order of their appearance is the same as in v. By convention, in the algorithms we use the predicate $==$ to denote the semantic equality predicate between two variables; for instance, for variables a and b, $a == b$ returns true *if and only if* the value of variable a equals the value of variable b.

II. Preliminaries and Terminology

In this section, we review the *minimum set covering problem*, and its decision version, referred to as the *set covering problem* [22]. Next, we review the eigenvalues and eigenvectors Popov-Belevitch-Hautus (PBH) controllability criteria for LTI systems [23], and some notions from structural systems theory required to compare the minimal controllability problem with the minimal structural controllability problem.

A (computational) problem is said to be *reducible in polynomial time* to another if there exists a procedure to transform the former to the latter using a polynomial number of operations on the size of its inputs. Such reduction is useful in determining the qualitative complexity class [24] a particular problem belongs to. The following result may be used to check for NP-completeness of a given problem.

Lemma 1 ([24]). *If a problem $\mathcal{P}_A$ is NP-complete, $\mathcal{P}_B$ is in NP and $\mathcal{P}_A$ is reducible in polynomial time to $\mathcal{P}_B$, then $\mathcal{P}_B$ is NP-complete.* ◇

Now, consider the set covering (decision) problem: Given a collection of sets $\{\mathcal{S}_j\}_{j=1,\ldots,p}$, where $\mathcal{S}_j \subset \mathcal{U}$, is there a collection of at most k sets that covers $\mathcal{U}$, i.e., $\bigcup_{j \in \mathcal{K}} \mathcal{S}_j = \mathcal{U}$, where $\mathcal{K} \subset \{1, \ldots, p\}$ and $|\mathcal{K}| \leq k$?

This is the decision problem associated with the *minimum set covering problem*, a well known NP-hard problem, given as follows.

Definition 1 ([22]). *(Minimum Set Covering Problem) Given a set of m elements $\mathcal{U} = \{1, 2, \ldots, m\}$ and a set of n sets $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ such that $\mathcal{S}_i \subset \mathcal{U}$, with $i \in \{1, \cdots, n\}$, and $\bigcup_{i=1}^{n} \mathcal{S}_i = \mathcal{U}$, the minimum set covering problem consists of finding a set of indices $\mathcal{I}^* \subseteq \{1, 2, \ldots, n\}$ corresponding to the minimum number of sets covering $\mathcal{U}$, i.e.,*

$$\mathcal{I}^* = \underset{\mathcal{I} \subseteq \{1,2,\ldots,n\}}{\arg \min} \quad |\mathcal{I}|$$
$$s.t. \quad \mathcal{U} = \bigcup_{i \in \mathcal{I}} \mathcal{S}_i \ .$$

◇

In particular, the set covering problem is used in the present paper to show the NP-completeness of the MCP, by considering the following result.

Proposition 1 ([24]). *Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be two NP-hard problems, and $\mathcal{P}_A^d$ and $\mathcal{P}_B^d$ be their decision versions, respectively. If a problem $\mathcal{P}_A$ is polynomially reducible to $\mathcal{P}_B$ (or equivalently, their decision versions) and $\mathcal{P}_B$ is polynomially reducible to $\mathcal{P}_A$ (or equivalently, their decision versions), then both $\mathcal{P}_A^d$ and $\mathcal{P}_B^d$ are NP-complete.* ◇

Now consider an arbitrary LTI system, as given by (1), then two possible and well known controllability tests are stated next.

Theorem 1 ([23]). *(PBH test for controllability using eigenvalues) Given* (1)*, the system is controllable if and only if*

$$rank\left(\left[\begin{array}{cc} A - \lambda I_n & b \end{array}\right]\right) = n \text{ for all } \lambda \in \mathbb{C}.$$
◇

In fact, it suffices to verify the criterion of Theorem 1 for each $\lambda \in \sigma(A)$ only.

Theorem 2 ([23]). *(PBH test for controllability using eigenvectors) Given* (1)*, the system is not controllable if and only if there exists a left-eigenvector v of A such that $v^\dagger b = 0$.* ◇

Now, we introduce the structural counterpart of the MCP, the *minimal structural controllability problem* (MSCP). In addition, some notions in structural systems theory and relevant graph theoretic constructs are needed to state the results. For a more exhaustive introduction to this topic we refer the reader to [20], [21].

We start by stating the structural counterpart of controllability as follows.

Definition 2 ([21]). *(Structural controllability) Given an LTI system* (1) *with sparseness given by $(\bar{A}, \bar{b})$, with $\bar{A} \in \{0, \star\}^{n \times n}$ and $\bar{b} \in \{0, \star\}^n$, the pair $(\bar{A}, \bar{b})$ is said to be structurally controllable if there exists a controllable pair (A, b), with the same sparseness as $(\bar{A}, \bar{b})$.* ◇

In fact, a stronger characterization of structural controllability holds as stated in the following remark.

Proposition 2 ([25]). *For a structurally controllable pair $(\bar{A}, \bar{b})$, the numerical realizations (A, b) with the same sparseness as $(\bar{A}, \bar{b})$ that are non-controllable are described by a proper variety in $\mathbb{C}^{n \times n} \times \mathbb{C}^n$. In other words, almost all realizations respecting the structural pattern of a structurally controllable pair are controllable.* ◇

The MSCP is posed as follows: given the structural matrix $\bar{A}$ associated with the dynamics matrix A, find $\bar{b} \in \{0, \star\}^n$ such that

$$\bar{b} = \underset{\bar{b}' \in \{0, \star\}^{n \times p}}{\arg\min} \|\bar{b}'\|_0$$
$$\text{s.t.} \quad (\bar{A}, \bar{b}') \text{ is structurally controllable.}$$
(3)

In order to state formally the solution to problem (3) as obtained in [20], we introduce the following graph theoretic notions. A digraph consists of a set of *vertices* $\mathcal{V}$ and a set of *directed edges* $\mathcal{E}$ of the form (v_i, v_j), where $v_i, v_j \in \mathcal{V}$. Given the matrices A and b in (1) and their structure denoted by $\bar{A}$ and $\bar{b}$ respectively, we define the following digraphs: 1) the *state digraph*, denoted by $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, which is the digraph that comprises only the state variables as vertices (i.e., $\mathcal{X} = \{x_1, \ldots, x_n\}$ as *state vertices*) and a set of directed edges between the state vertices (i.e., $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : x_i, x_j \in \mathcal{X} \text{ and } \bar{A}_{j,i} \neq 0\}$); 2) the *system digraph*, denoted by $\mathcal{D}(\bar{A}, \bar{b}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, where $\mathcal{U} = \{u_1\}$ denote the *input vertices* and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_1, x_j) : u_1 \in \mathcal{U}, x_j \in \mathcal{X} \text{ and } \bar{b}_{j,1} \neq 0\}$ the edges from the input vertices to the state vertices. A digraph $\mathcal{D}$ is said to be *strongly connected* if there exists a directed path between any two vertices. A subgraph of $\mathcal{D}$ is a digraph whose vertex and edge sets are subsets of those of $\mathcal{D}$. A *strongly connected component* (SCC) is a maximal subgraph (i.e., there is no other subgraph, containing it, with the same property) $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ such that for every $u, v \in \mathcal{V}_S$ there exists a path from u

to v and from v to u. We can create a *directed acyclic graph* (DAG) $\mathcal{D}^* = (\mathcal{V}^*, \mathcal{E}^*)$ by visualizing each SCC of $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ as a virtual vertex, in which a directed edge between two vertices (SCCs) exists *if and only if* there exists a directed edge connecting the corresponding SCCs in the digraph $\mathcal{D}$. The SCCs in the DAG may be further categorized as follows.

Definition 3 ([20]). *An SCC is said to be linked if it has at least one incoming/outgoing edge from/to another SCC. In particular, an SCC is non-top linked if it has no incoming edge to its vertices from the vertices of another SCC.* ◇

Corollary 1 ([20], [26]). *Let $\mathcal{D}(\bar{A}) = (\mathcal{X} \equiv \{x_1, \ldots, x_n\}, \mathcal{E}_{\mathcal{X}, \mathcal{X}})$ be the state digraph such that $\{(x_i, x_i) : i = 1, \ldots, n\} \subset \mathcal{E}_{\mathcal{X}, \mathcal{X}}$. The pair $(\bar{A}, \bar{b})$ is structurally controllable if and only if every non-top linked SCC of $\mathcal{D}(\bar{A})$ has an incoming edge from the input vertex in $\mathcal{D}(\bar{A}, \bar{b})$ to one of its state variables.* ◇

Now, note that, by Definition 2, a pair (A, b) is controllable only if the corresponding structural pair $(\bar{A}, \bar{b})$ is structurally controllable. Therefore, it is natural to first characterize all the sparsest structures of input vectors that ensure structural controllability, i.e., solutions to (3). In particular, as a consequence of Proposition 2 we have the following result which links the MCP to its structural counterpart.

Proposition 3 ([20]). *Given A, a solution $b \in \mathbb{C}^n$ for the MCP and a numerical realization $b' \in \mathbb{C}^n$ of a solution to the MSCP associated with the structural matrix $\bar{A}$, we have*

$$\|b\|_0 \geq \|b'\|_0.$$

More generally, for each b that solves the MCP, there exists a solution $\bar{b}'$ of the MSCP such that b' satisfy

$$\bar{b} \geq \bar{b}',$$

where $\bar{b}$ and $\bar{b}'$ denotes the structural vector associated with b and b', respectively.

Conversely, given a structural matrix $\bar{A}$ and a solution $\bar{b}'$ to the single-input minimal structural controllability problem, for almost all numerical instances A satisfying the structural pattern of $\bar{A}$, then almost all numerical instances satisfying the structural pattern of $\bar{b}'$ are solutions to the MCP associated with A. ◇

Finally, we emphasize that the solution to the MSCP has been fully explored in [20] and can be determined recurring to polynomial complexity algorithms; more precisely, $\mathcal{O}(n^3)$ where n is the dimension of the state space.

III. MAIN RESULTS

In this section, we show that the minimal controllability problem (MCP), given the left-eigenbasis of the dynamics simple matrix A, is polynomially reducible to the minimum set covering problem (Section III-A). This reduction is achieved by exploiting the PBH eigenvector criterion (Theorem 2) for controllability. More precisely, the reduction is obtained in two steps: first we provide a necessary condition on the structure $\bar{b}$ of the sparsest input vector b (see Lemma 2), which is obtained by formulating a minimum set covering problem (see Algorithm 1) associated with the structure (i.e., location of non-zero entries) of the left-eigenvectors of the dynamics matrix A. Second, we show that a possible numerical realization of $\bar{b}$ which solves the MCP may be generated using a polynomial construction, using Algorithm 2. More specifically, such sparsity of $\bar{b}$ matches the one determined by Algorithm 1. Further, both algorithms (Algorithm 1 and Algorithm 2) have polynomial complexity in the number of state variables (see Lemma 3). These results combined provide a systematic solution to the MCP (see Theorem 3), and are intermediary results to another main result of the paper, stated in Theorem 5, i.e., the decision version of the MCP, given the left-eigenbasis of the dynamics simple matrix A, is NP-complete. Additionally, we also provide density arguments which show that almost all numerical realizations of the structural vectors $\bar{b}$ (with maximum sparsity given by Lemma 2) are a solution to the MCP, see Corollary 2.

Complementary to the main result of this paper, we show (see Corollary 3) that approximated solutions to the minimum set covering problem yield feasible sub-optimal (in general) solutions to the MCP, with guaranteed optimality gap.

A. On the Exact Solution of the MCP

We start by considering a set of key intermediate results, upon which we will obtain one of the main result of this paper (Theorem 3), i.e., the MCP is reducible to the minimum set covering problem under the assumption that the left-eigenbasis of a dynamics simple matrix is known.

The first set of results provide necessary conditions on the structure that an input vector b must satisfy to ensure controllability of (A, b) (see Lemma 2) and a polynomial complexity algorithmic procedure (Algorithm 1) that reduces the problem of obtaining such necessary structural patterns to a minimum set covering problem. Subsequently, we provide Algorithm 2 with polynomial complexity (see Lemma 3), which is shown to provide solutions to the MCP by generating appropriate numerical instances of the structural input vectors.

Lemma 2. *Given a collection of non-zero vectors $\{\bar{v}^j\}_{j \in \mathcal{J}}$ with $\bar{v}^j \in \{0, \star\}^n$, the procedure of finding $\bar{b}^* \in \{0, \star\}^n$ such that*

$$\bar{b}^* = \underset{\bar{b} \in \{0, \star\}^n}{\arg \min} \quad \|\bar{b}\|_0 \tag{4}$$
$$s.t. \quad \bar{v}^j \cdot \bar{b} \neq 0, \text{ for all } j \in \mathcal{J}$$

can be polynomially (in $|\mathcal{J}|$ and n) reducible to a minimum set covering problem with universe $\mathcal{U}$ and a collection $\mathcal{S}$ of sets by applying Algorithm 1. ◇

Proof: Consider the sets $\mathcal{S}$ and $\mathcal{U}$ obtained in Algorithm 1. The following equivalences hold: let $\mathcal{I} \subset \{1, \cdots, n\}$ be a set of indices and $\bar{b}_{\mathcal{I}}$ the structural vector whose i-th component is non-zero if and only if $i \in \mathcal{I}$. Then, the collection of sets $\{\mathcal{S}_i\}_{i \in \mathcal{I}}$ in $\mathcal{S}$ covers $\mathcal{U}$ if and only if $\forall j \in \mathcal{J}$, $\exists k \in \mathcal{I}$ such that $j \in \mathcal{S}_k$, which is the same as $\forall j \in \mathcal{J}$, $\exists k \in \mathcal{I}$ such that $\bar{v}_k^j \neq 0$ and $\bar{b}_k \neq 0$, this can be rewritten as $\forall j \in \mathcal{J}$, $\exists k \in \mathcal{I}$ such that $\bar{v}_k^j \bar{b}_k \neq 0$ and therefore $\forall j \in \mathcal{J} \quad \bar{v}^j \cdot \bar{b} \neq 0$. In summary, $\bar{b}_{\mathcal{I}}$ is a feasible solution to the problem in (2). In addition, it can be seen that by such reduction, the optimal solution $\bar{b}^*$ of (2) corresponds to the structural vector $\bar{b}_{\mathcal{I}^*}$, where $\{\mathcal{S}_i\}_{i \in \mathcal{I}^*}$ is the minimal collection of sets that cover $\mathcal{U}$, i.e., $\mathcal{I}^*$ solves the minimum set covering problem associated with $\mathcal{S}$ and $\mathcal{U}$. Hence, the result follows by observing that Algorithm 1 has polynomial complexity, namely $\mathcal{O}(\max\{|\mathcal{J}|, n\}^3)$. ∎

Next, given a structural vector $\bar{b} \in \{0, \star\}^n$ and a set of vectors $\{v^j\}_{j \in \mathcal{J}}$, where $v^j \in \mathbb{C}^n$, we use Algorithm 2 to find a numerical realization of $\bar{b}$, say b, that verifies $v^j \cdot b \neq 0$, for all $j \in \mathcal{J}$. Note that, such b exists only if it satisfies $\bar{b} \geq \bar{b}^*$ for some solution $\bar{b}^*$ of (2).

We now establish the correctness of Algorithm 2 and its polynomial computational complexity on the size of its input.

Lemma 3. *Algorithm 2 is correct and has complexity $\mathcal{O}(\max\{|\mathcal{J}|, n\}^3)$, where $|\mathcal{J}|$ is the size of the collection of vectors given as input and n the size of vectors used as input to the algorithm.* ◇

Proof: Correctness: First, observe that if Step 1 succeeds, there exists a collection of $\|\bar{b}\|_0 \times 1$ real vectors $\{b'^j\}_{j \in \mathcal{J}}$ without zero entries, such that $v'^j \cdot b'^j \neq 0$ for $j \in \mathcal{J}$. Now, we have to show that there exists a common b' without zero entries such that $v'^j \cdot b' \neq 0$ for $j \in \mathcal{J}$. This is done in two steps: first we ensure that $v'^j \cdot b' \neq 0$ for $j \in \mathcal{J}$ (in Step 3) and second we ensure that b' has no zero entries, while keeping $v'^j \cdot b' \neq 0$ for $j \in \mathcal{J}$ (Step 4). Notice that Step 4 is not executed if $\bar{b}$ is the sparsest input vector, determined in Lemma 3.

Step 3 is correct, by proceeding inductively as follows: Let $b'^{(j)}$ and $b'_{(j)}$ denote the value of b' at the beginning and ending of iteration j respectively. If $j = 1$ then $\mathcal{J}_e = \{1\}$ and $b'_{(1)} = \alpha_1 v'^1$, which implies that $b'_{(1)} \cdot v'^1 \neq 0$. Now, let us assume

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Algorithm 1 Polynomial reduction of the structural optimization problem (2) to a set-covering problem

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Input: $\{\bar{v}^j\}_{j \in \mathcal{J}}$, a collection of $|\mathcal{J}|$ vectors in $\{0, \star\}^n$.

Output: $\mathcal{S} = \{\mathcal{S}_i\}_{i \in \{1,\ldots,n\}}$ and $\mathcal{U}$, a set of n sets and the universe of the sets, respectively.

1: **for** $i = 1, \ldots, n$

      set $\mathcal{S}_i = \{\}$

   end for

2: **for** $j = 1, \ldots, |\mathcal{J}|$

      for $i = 1, \ldots, n$

         if $\bar{v}_i^j \neq 0$ **then**

            $\mathcal{S}_i = \mathcal{S}_i \cup \{j\}$

         end if

      end for

   end for

3: **set** $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$

4: **set** $\mathcal{U} = \bigcup\limits_{i=1}^{n} \mathcal{S}_i$
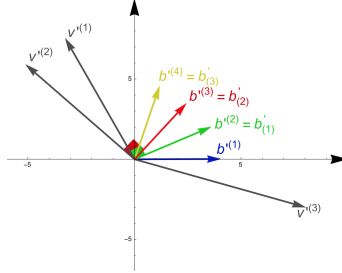

Fig. 1. Illustration of Step 4 of Algorithm 2, without loss of generality and for illustrative purposes we assume that the eigenvectors have real entries, considering $\{v^j\}_{j \in \{1,2,3\}}$ and $\bar{b}$ such that $\|\bar{b}\|_0 = 2$. In the first iteration of Step 4, the second entry of $b'^{(1)}$ is zero, hence a vector with the second entry different from zero is selected. Without loss of generality, let that vector to be $v'^{(1)}$ and in the second iteration we have $b'^{(2)}$ that has no zero entry but it is orthogonal to $v'^{(1)}$. Therefore, the component of $v'^{(1)}$ has to be re-scaled, originating $b'^{(3)}$. Although $b'^{(3)}$ is no longer orthogonal to $v'^{(1)}$, it is orthogonal to $v'^{(2)}$, hence the component of $v'^{(1)}$ has to be re-scaled once again. Thus, we obtain $b'^{(4)}$ which comprises only non-zero entries and it is not orthogonal with respect to $v'^{(1)}, v'^{(2)}$ and $v'^{(3)}$, terminating Step 4.

that at iteration β of Step 3 we have $b'^{(\beta)}$, such that $b'^{(\beta)} \cdot v'^j \neq 0$ for all $j \in \mathcal{J}_e = \{1, \ldots, \beta - 1\}$. Thus, we need to show that $b'_{(\beta)}$ is such that $b'_{(\beta)} \cdot v'^j \neq 0$ for all $j \in \mathcal{J}_e = \{1, \ldots, \beta\}$. Notice that $b'_{(\beta)} = b'^{(\beta)} + s\varepsilon_1 v'^\beta$, where s corresponds to the number of times the inner for-loop was executed. In fact, due to the linearity of the inner product, we have that $v'^j \cdot b'_{(\beta)} = 0$ for some $j \in \mathcal{J}_e$ at most $|\mathcal{J}_e|$ times. Hence after the execution of the inner-most loop (consisting of $|\mathcal{J}_e| + 1$ iterations at most), we have s such that the vector $b'_{(\beta)}$ satisfies $v'^j \cdot b'_{(\beta)} \neq 0$ for $j \in \mathcal{J}_e = \{1, \ldots, \beta\}$. Hence, at the end of step $|\mathcal{J}|$, we have $b'_{(|\mathcal{J}|)}$ that is not orthogonal to v'^j for each $j \in \mathcal{J}$.

Nevertheless, $b'_{(|\mathcal{J}|)}$ may still have some zero entries as a result of a specific linear combination leading it to lie on some hyperplane orthogonal to the canonical vectors of $\mathbb{C}^p$. In that case, Step 4 is executed which fixes the issue. To show that Step 4 is correct, we proceed inductively as follows: First, observe that if $b'_{(|\mathcal{J}|)}$ given at the end of Step 3 has no zero entries, Step 4 is not executed and the result follows straightforwardly. Now, let us assume that $b' = b'_{(|\mathcal{J}|)}$ has some zero entries, and

---

Algorithm 2 Determines a numerical realization of an input vector b with specified input structure $\bar{b}$

Input: $\{v^j\}_{j \in \mathcal{J}}$, a collection of $|\mathcal{J}|$ real vectors, and $\bar{b} \in \{0, \star\}^n$.

Output: $b \in \mathbb{C}^n$ solution to (5).

1: If $\bar{b}$ is not such that $\bar{v}^j \cdot \bar{b} \neq 0$ for all $j \in \mathcal{J}$ then no solution exists and **exit**;

2: Let b' to be a real $p \times 1$ vector where $p = \|\bar{b}\|_0$ and let $v'^j = v^j|_{\bar{b}}$ with $j \in \mathcal{J}$ (see Notation section).

3: $\varepsilon_1 = 0.1$, $b' = 0$ and $\alpha_j = 1$ for $j \in \mathcal{J}$

    for $j = 1, \ldots, |\mathcal{J}|$

    $\mathcal{J}_e = \mathcal{J}_e \cup \{j\}$

    $b' = b' + \alpha_j v'^j$

        for $i \in \mathcal{J}_e$

            if $b' \cdot v'^i = 0$

                then $b' = b' + \varepsilon_1 v'^i$

                else exit for

            end if

        end for

    end for

    $\varepsilon_2 = 0.1$

4: **for** $k = 1, \ldots, p$

        if $b'_k == 0$ **then**

            find m such that $v'^m_k \neq 0$

            for $l = 1, \ldots, p + |\mathcal{J}| + 1$

                $b' = b' + \varepsilon_2 v'^m$

                if $(\|b'_{1:k}\|_0 == k)$ **and** $(b' \cdot v'^j \neq 0, j \in \mathcal{J})$

                    then exit for

                end if

            end for

        end if

    end for

5: **set** b with the same structure as $\bar{b}$ and where each non-zero entry has the value in b' by order of its appearance.

---

consider the following inductive argument on k for $b'_{1:k}$, where k corresponds to the iteration of the outer-most for-loop in Step 4. Consider $k = 1$ if $\|\bar{b}_{1:1}\|_0 = 1$, then the first entry of $\bar{b}$ is non-zero as desired. Otherwise, if $\|\bar{b}_{1:1}\|_0 = 0$, then the first entry in b' is zero. Therefore, we re-scale a single vector v'^m with $m \in \mathcal{J}$ that has the first entry different from zero, i.e., $b'_{(1)} = b'^{(1)} + l\varepsilon_2 v'^m$. Therefore, the new vector $b'_{(1)}$ no longer has its first entry equal to zero as desired. Now, let us consider the induction step. Suppose that at iteration k of the outer-most for-loop in Step 4, $\|b'_{1:k}\|_0 = k - 1$ and $b' \cdot v'^j \neq 0$ for $j \in \mathcal{J}$, then we want to show that at the end of iteration k we have $\|b'_{1:k}\|_0 = k$ and $b' \cdot v'^j \neq$ for $j \in \mathcal{J}$. Let $b'_{(k)} = b'^{(k)} + l\varepsilon_2 v'^m$ for some $l \in \{1, \ldots, p + |\mathcal{J}| + 1\}$, see Figure 1. Similarly to Step 3, notice that due to the linearity of the inner product, $b'_{(k)}$

can be orthogonal to a canonical vector at most p times and orthogonal to a vector v'^j at most $|\mathcal{J}|$, which implies that there exists an $l \in \{1, \ldots, p + |\mathcal{J}| + 1\}$ such that $\|b'_{(k)}\|_0 = k$ and $b' \cdot v'^j \neq 0$ for $j \in \mathcal{J}$. Hence, the correctness of Step 4 follows. In other words, at the end of Step 4 we obtain b' that is not orthogonal to any vector v'^j and has all its entries different from zero. By performing Step 5 it follows immediately that b is a solution to (5).

Complexity: The computation complexity is dominated by the execution of the two nested for-loops (in Steps 3 and 4) which incurs in $\mathcal{O}(\max\{|\mathcal{J}|, n\}^2)$, where n is an upper bound to $p = \|\bar{b}\|_0$. Since the inner-most operations in the for-loops are at most linear in $\max\{|\mathcal{J}|, n\}$, it follows that Algorithm 2 runs in $\mathcal{O}(\max\{|\mathcal{J}|, n\}^3)$. ∎

Algorithm 2 is used to prove the next lemma (Lemma 4), which shows that the sparsest vector pattern given by Lemma 2 leads to a numerical realization that is a solution to the MCP, by recalling Theorem 2 (the PBH eigenvector controllability test), as stated in (2).

Lemma 4. *Given $\{v^i\}_{i \in \mathcal{J}}$ with $v^i \in \mathbb{C}^n$, the procedure of finding $b^* \in \mathbb{C}^n$ such that*

$$b^* = \arg\min_{b \in \mathbb{C}^n} \quad \|b\|_0 \tag{5}$$
$$s.t. \quad v^i \cdot b \neq 0, \text{ for all } i \in \mathcal{J},$$

is polynomially (in $|\mathcal{J}|$ and n) reducible to a minimum set covering problem (provided by Algorithm 1), with numerical entries determined using Algorithm 2. ◇

Proof: By Lemma 2, given $\{\bar{v}^i\}_{i \in \mathcal{J}}$, problem (5) is polynomially (in $|\mathcal{J}|$ and n) reducible to a minimum set covering problem. Now, given a solution $\bar{b}$ to (2), Algorithm 2 can be used to obtain a numerical instantiation b with the same structure as $\bar{b}$ such that $v^i \cdot b \neq 0$ for all $i \in \mathcal{J}$, which incurs polynomial complexity (in $|\mathcal{J}|$ and n) by Lemma 3. Furthermore, it is readily seen that any feasible solution b' to (5) satisfies $\|b'\|_0 \geq \|\bar{b}\|_0 = \|b\|_0$. Hence, b obtained by the above recipe is a solution to (5) and the desired assertion follows by observing that all steps in the above construction, except the minimum set covering problem, yielding $\bar{b}$ has polynomial complexity (in $|\mathcal{J}|$ and n). ∎

Now, we state one of the main results of the paper.

Theorem 3. *Let $\{v^i\}_{i \in \{1, \ldots, n\}}$ be the left-eigenvectors in the left-eigenbasis of the dynamic matrix A in (1). The solution to the MCP (2) can be determined by solving the set covering problem obtained using Algorithm 1, and entries determined using Algorithm 2.* ◇

Proof: By invoking the PBH eigenvector test in Theorem 2, notice that the problem in (5) is a restatement of the MCP in (2). ∎

Now, we show that almost all possible numerical realizations of the input vector structure determined in Lemma 2 are solutions to (5), hence to the MCP presented in (2). In fact, notice that in Algorithm 2 both $\varepsilon_1, \varepsilon_2$ can be arbitrary positive scalars, which leads to the conclusion that there exist infinitely many numerical realizations of the sparsest input vector (determined as in Lemma 2) that are solutions to (5), and hence the MCP.

Theorem 4. *Given $\{v^i\}_{i \in \mathcal{J}}$ with countable $\mathcal{J}$ such that $v^i \neq 0$ for all $i \in \mathcal{J}$. Then the set $\Omega = \{b \in \mathbb{C}^n : b \sim \bar{b}, v^i \cdot b = 0 \text{ for some } i \in \mathcal{J}\}$ has zero Lebesgue measure, where the notation $b \sim \bar{b}$ denotes that b is structurally similar to $\bar{b}$, i.e., a numerical instantiation of $\bar{b}$.*

In particular, taking $\{v^i\}_{i \in \mathcal{J}}$ to be the set of left-eigenvectors of A, almost all numerical realizations b of $\bar{b}$ (solution to problem (5)) are solutions to the MCP. ◇

Proof: Let $\{v^i\}_{i \in \mathcal{J}}$, with countable $\mathcal{J}$, be given and let $\bar{b}$ be a solution to problem (5). For $b \in \mathbb{C}^n$, the equation $v^i \cdot b = 0$ represents a hyperplane $\mathcal{H}^i \subset \mathbb{C}^n$ (provided $v^i \neq 0$ for all i), thus the equation $v^i \cdot b \neq 0$ defines the space $\mathbb{C}^n \setminus \mathcal{H}^i$. Therefore, the set of b that satisfies $v^i \cdot b \neq 0$ for all $i \in \mathcal{J}$, is given by $\bigcap_{i \in \mathcal{J}} \left(\mathbb{C}^n \setminus \mathcal{H}^i \right) = \mathbb{C}^n \setminus \left(\bigcup_{i \in \mathcal{J}} \mathcal{H}^i \right)$ and the set Ω of values which does not verify the equations is the complement, i.e., $\left(\mathbb{C}^n \setminus \bigcup_{i \in \mathcal{J}} \mathcal{H}^i \right)^{\mathsf{c}} = \bigcup_{i \in \mathcal{J}} \mathcal{H}^i$, which is a set with zero Lebesgue measure in $\mathbb{C}^n$, since $|\mathcal{J}|$ is countable.

Now, if $\{v^i\}_{i \in \mathcal{J}}$ is taken to be the set of left-eigenvectors of A and $\bar{b}$ the corresponding solution to problem (5), each member of the set Ω constitutes a solution to (5) and hence the MCP. Since, by the preceding arguments, Ω has Lebesgue measure zero in $\mathbb{C}^n$, it follows readily that almost all numerical instances of $\bar{b}$ are solutions to the MCP. ∎

Subsequently, we obtain the following result.

Corollary 2. *Given $\bar{b}$ determined in Lemma 3, almost all possible numerical realizations $b \in \mathbb{C}^n$ (not only those determined by Algorithm 2) of $\bar{b}$ are solutions for the optimization problem in (5).* ◇

Remark 2. *Corollary 2 differs from the converse result in Proposition 3 in the sense that the dynamic simple matrix A is fixed, i.e., a numerical instance with specified structure, and density arguments are provided to the numerical realizations of the input vector with specified structure.* ◇

B. Computational Complexity

Let $E^l_{\lambda_i}$ be the left-eigenspace associated with the eigenvalue λ_i. Then, the letf-eigenspace associated with a matrix A, with $\lambda_i \in \sigma(A) = \{\lambda_1, \ldots, \lambda_p\}$ is given $E = E^l_{\lambda_1} \oplus \ldots \oplus E^l_{\lambda_p}$, where $\oplus$ is the direct sum of the subspaces and $\alpha_1, \ldots, \alpha_p$ are the corresponding algebraic multiplicities. Then, E is a left-eigenbasis if $\alpha_1 + \ldots + \alpha_p = n$, which implies that $\dim(E^l_{\lambda_i}) = \alpha_i$. Therefore, for each $\alpha_i > 1$ it follows there exist α_i linear independent left-eigenvectors that span $E^l_{\lambda_i}$; hence, for the purpose of the computations involved in Algorithm 1 the collection of left-eigenvectors that span $E^l_{\lambda_i}$ can have arbitrary structures. Therefore, $|\mathcal{J}|$ in Algorithm 1 (see also Lemma 2) grows super-polynomially in n. In Example 1, we explore an extreme case where the algebraic multiplicity of an $n \times n$ matrix is n, where we identify a sequence of $n \times n$ matrices with $n = 1, 2, \ldots$ for which the number of distinct eigenvectors grows exponentially with n.

Example 1. *Consider A to be the $n \times n$ matrix given by*

$$\mathbb{1}_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n} .$$

Thus, a subset of possible (and distinct) left-eigenvectors associated with the eigenvalue 0 is given by

$$\Xi_n = \left\{ [x_1, x_2, \ldots, x_n]^{\mathsf{T}} : x_i \in \{-1, 0, 1\} \text{ and } \sum_{i=1}^{n} x_i = 0 \right\}. \tag{6}$$

In other words, it consists of all possible combinations of $x_i \in \{-1, 0, 1\}$ summing up to zero. This is illustrated in Figure 2, where for the matrix $\mathbb{1}_n$ we have m (exponential number of) left-eigenvectors in Ξ_n. ○

Consequently, if $|\mathcal{J}|$ grows super-polynomially in n, the reduction is not polynomial in n. This, in fact, motivates our restriction that the set of dynamics matrices A have simple eigenvalues, in which case $|\mathcal{J}| = n$, see also Remark 1. With this, another main result of this paper can now be stated as follows.

Theorem 5. *The MCP, for the case when the left-eigenbasis of a dynamics simple matrix A in (1) is known, is NP-complete.* ◇
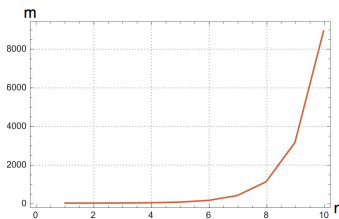
Fig. 2. Number of m left-eigenvectors of the matrix $\mathbb{1}_n$ in Ξ_n given by (6).

Proof: From [4], we have that the MCP is NP-hard, and, hence, in particular, the minimum set covering problem can be polynomially reduced to it. Therefore, we just need to show that the MCP (assuming that A comprises only simple eigenvalues and the left-eigenbasis is known) can be reduced polynomially to the minimum set covering problem.

To this end, note that, given the set $\{\bar{v}^i\}_{i \in \mathcal{J}}$ of left-eigenvectors of A, the MCP is equivalent to problem (5), the latter being polynomially (in $|\mathcal{J}|$ and n) reducible to the minimum set covering problem (see Lemma 4). Since $|\mathcal{J}| = n$, the overall reduction to the minimum set covering problem is polynomial in n and the result follows by invoking Proposition 1. ∎

Notice that the above result would not have been possible using standard tools in computational complexity, for instance, using Lemma 1. The reason being that the MCP cannot be verified polynomially, i.e., its belonging to the NP cannot be easily tested, since only approximation algorithms are available to verify standard controllability tests: either the Kalman controllability test [23], or the PBH criteria presented in Theorem 1 and Theorem 2.

Additionally, Theorem 5 leads to the following interesting observation.

Remark 3. *By Proposition 3 (the converse part), it follows that a solution of the MCP almost always coincides with a numerical realization of a solution to an associated minimal structural controllability. Combining this with the fact that the MCP is NP-complete when the eigenvalues of A are simple (see Theorem 5), it follows that the set of NP-problems has zero Lebesgue measure.* ◇

As stated in Theorem 5, the condition that the system matrices A be restricted to have simple eigenvalues, is in fact necessary in a sense for the proposed reduction of the MCP to the minimum set covering problem to be polynomial in n. This fact is explored in the next remark.

Remark 4. *The proposed reduction from the MCP to the minimum set covering problem is polynomial in $\max(|\mathcal{J}|, n)$, where $|\mathcal{J}|$ denotes the number of left-eigenvectors. Nevertheless, because the number of left-eigenvectors can grow exponentially it follows that the proposed reduction cannot be used to show that the decision version of the (general) MCP is NP-complete; further, one needs to take in consideration Remark 1. However, this does not imply that the decision version of the MCP for arbitrary dynamics matrices (i.e., when A is not restricted to have simple eigenvalues) is not NP-complete, which remains an open question.* ◇

C. Approximations to the Solution of the Minimal Controllability Problem

A practical consequence of Algorithm 2 and its correctness (see Lemma 3) is that any approximated solution of the minimum set covering problem given by Algorithm 1 leads to a feasible sub-optimal (in general) solution to the MCP. This is formally stated as follows.

Corollary 3. *Let A be as in (1) with $\{v^i\}_{i \in \mathcal{J}}$ its set of left-eigenvectors and $\bar{b}' \in \{0, \star\}^n$ be such that $\bar{v}^j \cdot \bar{b} \neq 0$, for all $j \in \mathcal{J}$, corresponding to an approximate solution of the minimum set covering problem generated by Algorithm 1 on the input set*

$\{\bar{v}^i\}_{i\in\mathcal{J}}$, where $\bar{v}^i$ denotes the structural vector associated with v^i for each $i \in \mathcal{J}$. If Algorithm 2 is instantiated with $\{v^i\}_{i\in\mathcal{J}}$ and the sparsity pattern $\bar{b}'$, then its output $b' \in \mathbb{C}^n$ is a numerical realization such that (A, b') is controllable. ◇

Polynomial approximation algorithms to the minimum set covering problem have previously been explored in the literature. For instance, the minimum set covering problem can be casted as a *submodular* optimization problem [27], where polynomial (in the number of sets) greedy algorithms with some guarantees on the optimality gap are available. Alternatively, in [28] a polynomial approximation to the minimum set covering problem is provided with distance bounded by $c\log(s)$, where $c \in \mathbb{R}^+$ and s denotes the number of sets in an optimal solution. In particular, these approximation schemes give similar approximability bounds provided in [4]. Nonetheless, additional information on the structure of the left-eigenvectors (or equivalently, the structure of the sets in the minimum set covering problem obtained using Algorithm 1) can be leveraged to obtain better approximations, for instance, see [29].

IV. DISCUSSION OF RESULTS

In this section, we discuss the results presented in the previous section. More precisely, we explore the computational aspects of the PBH criteria in Section IV-A in relation to the computational difficulty in determining solutions to the MCP. Second, we discuss the limitations of the existing numerical methods to determine the left-eigenbasis and their implications when compared with the assumptions used in this paper.

A. The implications of the minimal controllability problem on the PBH controllability tests

In the previous section, we have seen that the number of left-eigenvectors associated with A can grow exponentially. Therefore, we now present some observations regarding the use of PBH controllability tests.

○ *PBH eigenvalue test* (see Theorem 1) provides a polynomial method to check the controllability of an LTI system, since for each eigenvalue λ of A (n in total) only the computation of the rank of $[A - \lambda I_n \ b]$ is required. Nevertheless, it does not provide any immediate information about which entries of b should be different from zero and with what specific values such that the rank condition is ensured. Therefore, a naive usage of the PBH eigenvalue test would lead to a strictly combinatorial procedure for solving the MCP.

○ *PBH eigenvector test* (see Theorem 2) provides an insight about the design of b, since only its sparseness has to be determined, for instance, by solving a set covering problem (given in Algorithm 1), regardless of the size of the collection of the left-eigenvectors considered. Thus, given a specified structural vector, a numerical realization can be determined using Algorithm 2. Although, it may still be difficult to compute the optimal solution in general, since the minimum set covering problem is NP-hard. On the other hand it should not be used (in general) to verify controllability, since it requires to compute the entire set of left-eigenvectors (recall it can grow exponentially), and posterior verification if each one is orthogonal to the columns of the input vector (see Theorem 2).

B. Numerical and Computational Analysis

In the previous sections we assumed that the left-eigenbasis of the dynamics matrix is known. Now, we discuss the case where only the dynamics matrix is known. In this case, we need the following result from [30].

Theorem 6 ([30]). *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with simple eigenvalues. The deterministic arithmetic complexity of finding the eigenvalues and the eigenvectors of A is bounded by $\mathcal{O}\left(n^3\right) + t(n, m)$ operations, where $t(n, m) = \mathcal{O}\left(\left(n\log^2 n\right)\left(\log m + \log^2 n\right)\right)$,*

for a required upper bound of $2^{-m}\|A\|$ on the absolute output error of the approximation of the eigenvalues and eigenvectors of A and for any fixed matrix norm $\|\cdot\|$.   ◇

More precisely, Theorem 6 states that in practice, only a numerical approximation of the left-eigenbasis is possible in polynomial time. In this case, let $\epsilon = 2^m\|A\|$ be as in Theorem 6, then the results stated in Lemma 2 and Lemma 4 (see also Algorithm 1 and Algorithm 2) can only be used in an ϵ-*approximation* of the left-eigenbasis of the dynamics matrix. Therefore, the ϵ-approximation of the left-eigenbasis may lead to the following issues:

(i) an entry in the left-eigenvectors is considered as zero, where in fact it can be some non-zero value that (in norm) is smaller then ϵ. Consequently, the sets generated using Algorithm 1 (see also Lemma 2) do not contain the indices associated with those non-zero entries. Thus, additional sets need to be considered to the minimum set covering, which implies that the structure of the input vector may contain more nonzero entries than the sparsest input vector that is a solution to the MCP. In other words, we obtain an over-approximation of the sparsest input vector that is a solution to the MCP.

(ii) an entry in the left-eigenvectors in the ϵ-*approximation* of the left-eigenbasis is nonzero. Then, it does not represent an issue when computing the structure of the input vector as described in Lemma 2 (see also Algorithm 1), but it can represent a problem when determining the numerical realization by resorting to Algorithm 2. Nonetheless, by Corollary 2 it follows that such issue is unlikely to occur.

In order to undertake a deeper understanding of which entries fall in the first issue presented above, several methods to compute eigenvectors can be used and solutions posteriorly compared, see [31] for a survey of the different methods and computational issues associated with those.

V. ILLUSTRATIVE EXAMPLE

To illustrate the main results of this paper, consider the dynamical matrix A given by

$$A = \begin{bmatrix} 3 & \frac{3}{2} & -\frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 2 & 0 & 0 & 0 \\ -2 & -\frac{3}{2} & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 2 & \frac{3}{2} & \frac{3}{2} & 1 & \frac{9}{2} \end{bmatrix},$$ (7)

where the digraph $\mathcal{D}(\bar{A})$, with $\bar{A}$ corresponding to the structure of A, is depicted in Figure 3. Note that $\sigma(A) = \{1, 2, 3, 4, 5\}$, hence it consists of simple eigenvalues which implies that the results in Section III-A are applicable.

A. Minimal controllability problem

Applying the developments in Section III-A, to obtain the solution for the MCP, the first step is to compute the left-eigenvectors of A. The left-eigenvectors of A are $v^1 = [\,1\;1\;0\;0\;1\,]^\mathsf{T}$, $v^2 = [\,0\;0\;1\;0\;1\,]^\mathsf{T}$, $v^3 = [\,0\;0\;0\;1\;0\,]^\mathsf{T}$, $v^4 = [\,0\;1\;0\;0\;0\,]^\mathsf{T}$ and $v^5 = [\,1\;0\;1\;1\;0\,]^\mathsf{T}$.

Hence, the structures of the left-eigenvectors of A are $\bar{v}^1 = [\,\star\;\star\;0\;0\;\star\,]^\mathsf{T}$, $\bar{v}^2 = [\,0\;0\;\star\;0\;\star\,]^\mathsf{T}$, $\bar{v}^3 = [\,0\;0\;0\;\star\;0\,]^\mathsf{T}$, $\bar{v}^4 = [\,0\;\star\;0\;0\;0\,]^\mathsf{T}$ and $\bar{v}^5 = [\,\star\;0\;\star\;\star\;0\,]^\mathsf{T}$. The next step is to build the sets, using Algorithm 1, for the minimum set covering problem, based on $\bar{v}_i$, $i = 1, \ldots, 5$, where the j-th set corresponds to the set of indices of the left-eigenvector which have a non-zero entry on the j-th position, and we obtain $\mathcal{S}_1 = \{1, 5\}$, $\mathcal{S}_2 = \{1, 4\}$, $\mathcal{S}_3 = \{2, 5\}$, $\mathcal{S}_4 = \{3, 5\}$ and $\mathcal{S}_5 = \{1, 2\}$.

The universe of the minimum set covering problem, computed with Algorithm 1, is

$$\mathcal{U} = \bigcup_{i=1}^{n} \mathcal{S}_i = \{1, 2, 3, 4, 5\}\,.$$

Now, it is easy to see that a solution of this minimum set covering problem is the set of indices $\mathcal{I}^* = \{2, 3, 4\}$, since $\mathcal{U} = \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$ and there is no pair of sets, i.e., $\mathcal{I}' = \{i, i'\}$ with $i, i' \in \{1, \ldots, 5\}$ such that $\mathcal{U} = \mathcal{S}_i \cup \mathcal{S}_{i'}$. Therefore, the structure of the vector b, a solution for the MCP, has to be

$$\bar{b} = [\, 0 \; \star \; \star \; \star \; 0 \,]^\mathsf{T}. \tag{8}$$

Finally, in order to design b, we have to solve the following system with three unknowns which ensures that b is non-orthogonal to each left-eigenvector of A, i.e.,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ b_2 \\ b_3 \\ b_4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and hence we must have that $b_2, b_3, b_4 \neq 0$ and $b_3 + b_4 \neq 0$. By inspection, a possible choice is $b = [\, 0 \; 1 \; 1 \; 1 \; 0 \,]^\mathsf{T}$, although a systematic way can be executed using Algorithm 2, instantiated with the set of left-eigenvectors of A given by $\{v^j\}_{j \in \{1, \ldots, 5\}}$ and the structure of b given by $\bar{b}$ in (8). We can now verify that the controllability matrix of the system has full rank. The controllability matrix is given by

$$\mathcal{C} = [\, b \; Ab \; A^2 b \; A^3 b \; A^4 b \,]$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 3 & 4 \\ -1 & 4 & -6 & 9 & 22 \\ 14 & 8 & -39 & 27 & 103 \\ 137 & 16 & -216 & 81 & 472 \end{bmatrix},$$

and rank$(\mathcal{C}) = 5$, therefore the LTI system is controllable.

B. Minimal structural controllability problem

We now consider the solution to the MSCP associated with $\bar{A}$ associated with A given in (7). First notice that by Corollary 1, it follows that the system is structurally controllable through a single input. This happens because the decomposition in SCCs is consists in two non-top linked SCCs, see Figure 3-b). Therefore, to achieve structural controllability, we just need to connect a single input to a state variable in each non-top linked SCC (in this case two state variables, i.e., x_2 and x_4), which corresponds to have $\bar{b} = [\, 0 \; \star \; 0 \; \star \; 0 \,]^\mathsf{T}$.
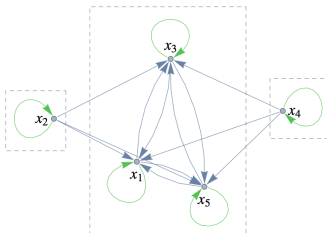

Fig. 3. This figure depicts the state digraph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X}, \mathcal{X}})$, for A in (7), where the green edges correspond to edges of the form (x_i, x_i), $i = 1, \ldots, n$, and the the dashed rectangles denote the decomposition of $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X}, \mathcal{X}})$ into SCCs, where the left and right dashed rectangles correspond to non-top link SCCs. Notice that $\bar{b} = [\, 0 \; \star \; 0 \; \star \; 0 \,]^\mathsf{T}$ ensures a structurally controllable system, as prescribed in Corollary 1. In addition, notice that no other $\bar{b}'$ with less non-zero elements exists such that $(\bar{A}, \bar{b}')$ is structurally controllable, therefore $\bar{b}$ is a solution for the MSCP.

C. Discussion of results

In our illustrative example the structural controllability solution to the MSCP provides a strict lower bound on the number of state variables we should actuate with the input, i.e., the sparsity of the input vector (in accordance to Proposition 3). More precisely, we achieve structural controllability by actuating two variables (specifically x_2, x_4), but in order to ensure controllability for the given numerical instance A in (7), an additional state variable, for instance, x_3, needs to be actuated. This verifies that structural controllability is a necessary, but not sufficient, condition to achieve controllability. In particular, considering the converse part of Proposition 3, we note that the numerical values of the matrix A fall into the set of zero Lebesgue measure (see also Proposition 2), where the solution associated with the MSCP does not provide a solution for the MCP.

To sharpen the intuition behind these results and observations, we perturbed the matrix A by adding a random uniform noise on the interval $[-10^{-10}, 10^{-10}]$ to each of its non-zero entries, which leads to a new matrix that we denote by A' (with the same structure as A). The structure of the left-eigenvector of the matrix A' now becomes: $\bar{v}'^1 = [\,\star\;\star\;\star\;\star\;\star\,]^\mathsf{T}$, $\bar{v}'^2 = [\,\star\;\star\;\star\;\star\;\star\,]^\mathsf{T}$, $\bar{v}'^3 = [\,0\;0\;0\;\star\;0\,]^\mathsf{T}$, $\bar{v}'^4 = [\,0\;\star\;0\;0\;0\,]^\mathsf{T}$ and $\bar{v}'^5 = [\,\star\;\star\;\star\;\star\;\star\,]^\mathsf{T}$.

Now, building the sets for the minimum set covering problem, using Algorithm 1, based on $\bar{v}'_i$, $i = 1, \ldots, 5$, where the j-th set corresponds to the set of indices of the left-eigenvector which have a non-zero entry on the j-th position, we have $\mathcal{S}'_1 = \{1, 2, 5\}$, $\mathcal{S}'_2 = \{1, 2, 4, 5\}$, $\mathcal{S}'_3 = \{1, 2, 5\}$, $\mathcal{S}'_4 = \{1, 2, 3, 5\}$ and $\mathcal{S}'_5 = \{1, 2, 5\}$, and the universe of the minimum set covering problem is $\mathcal{U} = \{1, 2, 3, 4, 5\}$. Finally, by inspection we can see that a solution of this minimum set covering problem is the set of indices $\mathcal{I}'^* = \{2, 4\}$. Hence, in this case, the solution to the MCP coincides with a numerical realization of the solution to the MSCP associated with $\bar{A}$. This further illustrates the conclusions of Proposition 2 and Proposition 3.

VI. Conclusions and Further Research

In this paper, we showed how to obtain exact solutions to the MCP, given the left-eigenbasis of the dynamics simple matrix A, by (polynomially) reducing it to a minimum set covering problem. From a practical point of view, feasible solutions may be obtained using efficient (polynomial complexity) approximation algorithms known for the minimum set covering problem with guaranteed suboptimality bounds. In addition, we discussed how the complexity of the problem increases if the dynamics matrix is not simple; subsequently, it remains open if for general system matrices (i.e., not restricted to have simple eigenvalues) and multi-input systems, the MCP may still be polynomially reducible to a combinatorial problem like the minimum set covering problem. Further, we analyzed the relation between the MCP and its structural counterpart, the minimal structural controllability problem. Finally, a possible and interesting direction for future research include the use of the structure of the inputs obtained, and considering methods such as coordinate gradient descent to minimize an energy cost.

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