Minimum Number of Below Average Triangles in a Weighted Complete Graph

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Abstract

Let G be an edge weighted graph with n nodes, and let $A(3, G)$ be the average weight of a triangle in G. We show that the number of triangles with weight at most equal to $A(3, G)$ is at least $(n - 2)$ and that this bound is sharp for all $n \geq 7$. Extensions of this result to cliques of cardinality $k > 3$ are also discussed.

Keywords: k-clique, average dominating algorithms

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1 Introduction

Let G be a complete graph on n vertices and c be a weight function associated with the edges of G. For $k \geq 1$, a *k-clique* K is a subgraph of G induced by k of its nodes. The weight $c(K)$ of a k-clique is the sum of the weights of its edges. For a fixed value of k, we denote by $\mathcal{K}$ the problem of finding a maximum weight k-clique in G. This problem is, of course, solvable in time polynomial in the size of G, but our interest is in looking at combinatorial properties of the distribution of the weights of all k cliques contained in G. More precisely, let $A_{\text{clique}}(k, G)$ denote the average weight of all k-cliques in G with respect to c. This paper addresses the question of finding a set of weights c that minimizes the number of k-cliques K with $c(K) \leq A_{\text{clique}}(k, G)$.

Our main motivation for studying this apparently innocuous problem lies in the study of the domination number of algorithms. Loosely speaking, an algorithm has a domination number of $d(n)$ if the solution outputted by the algorithm on any instance of size n is not worse than at least $d(n)$ other solutions (more precise definitions will be given in Section 2). This concept was introduced by Glover and Punnen [7] and has been the topic of several recent papers [2, 9, 13]. Part of the interest in domination analysis comes from the study of neighborhood search algorithms such as the 2-opt heuristic for the Traveling Salesman Problem. It appears that there might be a link between the quality of a neighborhood search and the domination number of the simple ascent heuristic based on that neighborhood.

An *average dominating* algorithm is an algorithm that, for each instance of the problem, produces a solution that dominates the average weight of all the solutions in the instance. Grover in [8] lists many such algorithms such as the 2-opt heuristic for the Traveling Salesman Problem and the 1-Flip for the Equipartition Problem. An average dominating algorithm for the Quadratic Assignement Problem (of which $\mathcal{K}$ is a special case) is Gutin and Yeo's Greedy Expectation Algorithm (GEA) [9]. One straightforward method for calculating a lower bound for the domination number of an average dominating algorithm is to find the minimum, over all instances, of the number of solutions that are dominated by the average. This technique is used in [13] to find the domination number of the 2-opt heuristic for the Traveling Salesman Problem.

Herein, this technique is used to show that, for $k = 3$, the lower bound for the domination number of average dominating algorithms such as the GEA—found by counting the minimum number of below average 3-cliques that are in a complete weighted graph—is $(n - 2)$ for $n \geq 7$, where n is the number of nodes in the graph. Further, if a complete weighted graph on n nodes has exactly the minimum number of triangles dominated by the average where

$n \geq 9$ and $n \equiv 1, 2, 3, 4 \pmod{6}$, then all of these triangles have one edge in common. The first result can be generalized to k larger than 3: given a value $k = 3, 4 \ldots$, there exists an infinite family of positive integers N such that if $n \in N$, the domination number of any average dominating algorithm for $\mathcal{K}$ is bounded below by

$$\binom{n-2}{k-2} .$$

Presented here are only the proofs for the initial cases. A complete presentation of the results can be found in [3]. We note that the essential ingredient of the proof for these results is the partition of the set of all k-cliques into classes and showing that each class contains at least one below average k-clique. A similar partitioning technique was used by Gutin and Yeo [10] for the Traveling Salesman Problem.

In Section 2, we give a precise definition of the domination number of an algorithm. Basic definitions and results used in the remainder of the paper related to Balanced Incomplete Block Designs are also listed. Section 3 covers the case $k = 3$, showing that the minimum number of triangles below the average is $(n-2)$ and that any cost function achieving that number will have all those triangles sharing an edge (when $n \geq 9$). Section 3 is divided in two: Subsection 3.1 covers the cases $n = 7$ and $n = 8$ and is used as the base step for the inductive proof used in Subsection 3.2. The latter covers the general case of $n = 1, 2, 3, 4 \pmod{6}$ for $n \geq 9$. The remaining cases of $n = 5, 6 \pmod{6}$ can be found in [3]. Finally, in Section 4, we discuss partial results for the case $k \geq 4$.

2 Domination Analysis

When referring to an *instance* of a maximization Combinatorial Optimization Problem *(COP)* $\mathcal{P}$, we mean a pair $(\mathcal{S}, c)$ consisting of a *solution set* $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, whose elements are called *feasible solutions*, and a real-valued function $c : \mathcal{S} \to \mathbb{R}$ called the *weight function*. Different weight functions correspond to different instances. In the case of $\mathcal{K}$, the weight function is the sum of the weights of the edges in the solution. Let $\mathcal{A}$ be an algorithm for a maximization COP $\mathcal{P}$ that returns the solution S on the instance I of $\mathcal{P}$. Then, we say the domination number of $\mathcal{A}$ on the instance I is

$$dom(\mathcal{A}, \mathcal{P}, I) = |\{T \in \mathcal{S} : \; c(T) \leq c(S)\}| .$$

Given a set of instances $\mathcal{I}$ for $\mathcal{P}$, we define the domination number of $\mathcal{A}$ over $\mathcal{I}$ as

$$dom(\mathcal{A}, \mathcal{P}, \mathcal{I}) = \min_{I \in \mathcal{I}} dom(\mathcal{A}, \mathcal{P}, I) \ .$$

Denote the average of the weights of the k-clique subgraphs $A_{\text{clique}}(k, G)$. We can calculate this by noting each edge is in $\binom{n-2}{k-2}$ k-cliques, and there are $\binom{n}{k}$ total k-cliques. Thus,

$$\begin{aligned} A_{\text{clique}}(k, G) &= \frac{\binom{n-2}{k-2}}{\binom{n}{k}} \ c(G) \\ &= \frac{k(k-1)}{n(n-1)} \ c(G) \ . \end{aligned}$$

The algorithm $\mathcal{A}$ is *average dominating* if on every weighted complete graph $G = (K_n, c)$, $\mathcal{A}$ produces a k-clique with weight greater than or equal to $A_{\text{clique}}(k, G)$. Hence, a lower bound for the domination number of $\mathcal{A}$ can be found by finding

$$\min_{c:\ E(G) \to \mathbb{R}} |\{s \in \mathcal{S} : \ c(s) \leq A_{\text{clique}}(k, G)\}|$$

where $\mathcal{S}$ is the set of k-cliques in G.

Our proofs concerning how many k-cliques have a weight that is less than or equal to the average weight rely on design theory. In particular, we are interested in certain classes of balanced incomplete block designs (BIBD). The information given here is from [5] and [14], where more complete accounts of the subject can be found.

A *balanced incomplete block design* (BIBD) is a pair of sets $(V, \mathcal{B})$, where V is a set of v elements and $\mathcal{B}$ is a set of b subsets of V (called *blocks*), each having size k and such that each element of V is in exactly r blocks and every subset of two elements of V is contained in exactly λ blocks. The values v, b, r, k, and λ are the *parameters* of the BIBD. There may be more than one BIBD for any set of parameters, and so the notation (v, b, r, k, λ)-BIBD is often used to represent a BIBD with these parameters. Moreover, because the parameters v, k, and λ define the other two parameters

$$r = \frac{\lambda(v-1)}{k-1} \quad \text{and} \quad b = \frac{vr}{k}\ , \tag{1}$$

a (v, b, r, k, λ)-BIBD may be simply called a (v, k, λ)-BIBD. We make use of both notations.

The *incidence matrix* of a (v, b, r, k, λ)-BIBD is a $0,1$-matrix where the columns are indexed by the elements V and the rows are indexed by the blocks $\mathcal{B}$. The entry a_{ij} is 1 if the element of V corresponding to column i is in the block of $\mathcal{B}$ corresponding to row j; otherwise, $a_{ij} = 0$. These incidence matrices provide our greatest insight into the structure of particular BIBDs. We use these $0,1$-matrices to find sets of solutions for the k-clique problem that have desirable properties such as evenly covering the edges of a graph.

A *large set* is a set of BIBDs such that every possible block appears in exactly one of the designs. That is, if a large set exists for the parameters (v, k, λ), then there exists a set of (v, k, λ)-BIBDs that cover each k subset of the v elements exactly once.

Theorem 1 *(Lu-Teirlinck [5]) A large set of $(v, 3, 1)$-BIBDs exists if and only if $v \equiv 1, 3 \pmod 6$ and $v \neq 7$.*

We later make use of large sets extensively in proving Theorem 3.

There are three important necessary conditions for the existence of a (v, b, r, k, λ)-BIBD. The first two listed here are trivial and the third is known as Fisher's Inequality:

(a) $vr = bk$.

(b) $r(k-1) = \lambda(v-1)$.

(c) $b \geq v$.

The following theorem, Wilson's Asymptotic Existence Theorem, tells us that given any k and λ, there exists an infinite number of v such that the above conditions are sufficient for the existence of a (v, k, λ)-design.

Theorem 2 *(Wilson [14]) Let k and λ be positive integers. There exists a v_0 such that for every $v \geq v_0$ where b and r as defined in (1) are integral and the necessary conditions (a)-(c) described above are satisfied, there exists a (v, k, λ)-BIBD.*

Note that Lu-Teirlinck's Theorem 1 is a stronger statement than Theorem 2 for the case $k = 3, \lambda = 1$ and $v_0 = 9$. Wilson's Theorem 2 is the foundation of the generalization of Theorem 3 presented in Section 4.

3 Minimum Number of Below Average Triangles

As the domination number is a function relating to rank, we can make use of maps that send the weight function of one instance to a weight function of another instance such that the relative ranks of the feasible solutions remains unchanged. To this effect, we define an *order-preserving operation* σ to be a map between weight functions c and $\sigma(c)$ such that for any two feasible solutions S_i and S_j, both of the following hold:

(1). $c(S_i) \leq c(S_j)$ if and only if $\sigma(c)(S_i) \leq \sigma(c)(S_j)$,

(2). $c(S_i) \leq A_{\text{clique}}(k, G)$ if and only if $\sigma(c)(S_i) \leq A_{\text{clique}}(k, \sigma(G))$, where $\sigma(G) = (K_n, \sigma(c))$.

Hence, if $\mathcal{A}$ returns the solution S_i when run on both of the instances $(\mathcal{S}, c)$ and $(\mathcal{S}, \sigma(c))$, then $\mathcal{A}$ has the same domination number on both instances.

One technique for calculating a lower bound for the domination number of an average dominating algorithm is to find the minimum number of below average k-cliques that can be subgraphs of a weighted complete graph on n nodes; we denote this value $D(n, k)$. In order to simplify the language, a k-clique whose weight equals the average weight of all the k-cliques is said to have *below average weight.* This partitions the k-cliques into two sets, below average k-cliques and *above average k-cliques.* Note that 3-cliques are also called *triangles.*

In this paper, we are mostly interested in the case $k = 3$. Note that we have trivially that $D(3, 3) = 1$. We also have that $D(4, 3) = 1$, as shown by the cost function assigning a cost of -1 on the three edges of a triangle and a weight of 1 on the remaining edges. Similarly, $D(5, 3) = 1$, as shown by the cost function assigning a cost of -1 on the three edges of a triangle and on the edge linking the two vertices not in the triangle, and a weight of $2/3$ on the remaining edges. For $n = 6$, we have that $D(n, 6) = 2$ as shown in [3]. The remaining cases are covered by the following theorem, which is the main result of this paper:

Theorem 3 *For* $n = 7, 8, \ldots,$

$$D(n, 3) = n - 2 \ .$$

That is, the minimum number of below average triangles in any weighted complete graph on $n \geq 7$ *nodes is* $(n - 2)$.

In the following sections, we prove Theorem 3 for the cases where $n \geq 7$ and $n \equiv 1, 2, 3, 4 \pmod{6}$. We do this by using results from design theory (discussed in the previous section), graph theory, as well as determining the exact structure of the below average triangles when a complete graph on n nodes contains exactly $D(n, 3)$ below average triangles.

We use the notation $G = (K_n, c)$, to mean that G is a weighted complete graph on n nodes, where c is the weight function. Let $T(G)$ denote the number of below average triangles in G. For a set of edges F, we denote the sum of the weights of those edges by $c(F)$, that is

$$c(F) = \sum_{e \in F} c(e) \ .$$

To simplify the notation, we let $c(G) = c(E(G))$. If N is a set of nodes, let $c(N) = c(G[N])$, that is the sum of the weights of the edges in the graph induced by the nodes in N. For simplicity, we write $c(u, v, w)$ instead of $c(\{u, v, w\})$. In all else, we follow the conventions set forth in [15] for graph theory.

Above we calculated $A_{\text{clique}}(k, G)$. Applying this formula to triangles, or 3-cliques, we get:

$$A_{\text{clique}}(3, G) = \frac{6}{n(n-1)} c(G) \ . \tag{2}$$

Since $A_{\text{clique}}(3, G)$ is a constant multiple of $c(G)$ and adding $-\dfrac{c(G)}{|E(G)|}$ to each edge of G is an order preserving operation, we assume without loss of generality

$$c(G) = A_{\text{clique}}(3, G) = 0 \ .$$

In the following two subsections we prove Theorem 3. We begin with the special cases of n equal to 7 and 8. In establishing these results, we develop much of the theory we need to prove the remainder of the theorem. The second section extends Theorem 3 to cover the cases where $n \geq 9$ and $n \equiv 1, 2, 3, 4 \pmod{6}$. The end of the proof, $n \equiv 5, 6 \pmod{6}$, has been omitted and can be found in [3].

3.1 $D(n, 3) = n - 2$ for $n = 7, 8$

The first step of the proof for Theorem 3 is establishing an upper bound on $D(n, 3)$. The easiest way to do this is by giving an example of a cost function that achieves the bound.

Lemma 4 *For $n = 3, 4, \ldots,$ $D(n, 3) \leq (n - 2)$.*

Proof: Let c be the cost function that weights one edge with $(1 - |E(K_n)|)$ and the remaining edges with 1. We know $c(G) = 0$, so only the triangles that use the negative weight edge are below average. □

Having established an upper bound for all n, it remains only to show that for $n \geq 7$, $(n - 2)$ is also a lower bound. The proof is divided into several cases. The remainder of this section is devoted to proving the special cases of $n = 7$ and $n = 8$. While proving these cases, we develop several results necessary to prove the remainder of the theorem. In fact, the steps followed in proving the theorem for $n = 8$ are similar to the steps for proving the theorem in general. The heart of Theorem 6 which establishes the result for $n = 7$ is in the following lemma.

Lemma 5 *If an $(n, 3, 1)$-BIBD B exists, then*

$$D(n, 3) \geq (n - 2) \ .$$

Moreover, if G has exactly $(n - 2)$ below average triangles and an $(n, 3, 1)$-BIBD B exists, then every isomorphism between the elements of B and the nodes of G maps exactly one block of B to one below average triangle of G.

Proof: An isomorphism h exists between the elements of B and the nodes of G because both are sets of size n. That is, under h, we can consider the elements of B to be the nodes of G. Every block of B contains three elements, and so under h, each block of B is isomorphic to three vertices of G. By abuse of language, we say that each block of B is isomorphic to a triangle in G. Since $\lambda = 1$, each edge is in exactly one triangle that is isomorphic to a block of B. Hence, if we sum the weights of the edges of the triangles isomorphic to blocks of B, we get 0, since $c(G) = 0$. This implies that at least one of the blocks in B is isomorphic to a below average triangle of G.

There are $n!$ isomorphisms between the elements of B and the nodes of G. Call this set of isomorphisms $\mathcal{D}$. We want to count how many $h \in \mathcal{D}$ send some block of B to a particular triangle T. A particular block of B, say the m^{th}, has three 1's and $(n - 3)$ 0's. An isomorphism h sends the m^{th} block to T if and only if the elements in the m^{th} block are sent to the nodes that are in T. There are 3! ways to this, and $(n - 3)!$ ways to send the remaining four elements of B to the remaining four nodes of G. We then get a total of $6 \cdot (n - 3)!$ isomorphisms between the m^{th} block of B and T. There are

$\dfrac{n \cdot (n-1)}{6}$ blocks, so $n \cdot (n-1) \cdot (n-3)!$ different isomorphisms send a block of B to T. Hence, G must have at least

$$\frac{n!}{n \cdot (n-1) \cdot (n-3)!} \;=\; (n-2)$$

below average trianglesto ensure that each isomorphism sends at least one block on a below average triangle. It follows that $D(n, 3) \geq (n-2)$. $\square$

Theorem 6 $D(7, 3) = 5$.

Proof: A BIBD B exists with parameters $(7, 3, 1)$ [5]. Lemma 4 and Lemma 5 prove the result. $\square$

The BIBD used in the proof of Theorem 6 does more than help establish a lower bound on $D(7, 3)$; it also helps to define what the subgraph of G induced by the edges of the below average triangles can contain. Let G^T be the set of below average triangles of $G = (K_n, c)$. We say that a subgraph of G is *induced by* G^T if that subgraph contains exactly the edges that are in below average triangles of G. If $|G^T| = D(n, 3)$, we say that G is *minimally triangled*.

We use Lemma 5 to establish that if $G = (K_7, c)$ is minimally triangled (that is, G has exactly 5 below average triangles), then G^T can only be one of two graphs. A *crown* (see Figure 1) is a graph where all triangles share a single edge e, which is called the *base* of the crown. Notice that the weight function in Lemma 4 (one edge very negative, the rest positive) yields a graph whose set of below average triangles induce a crown.

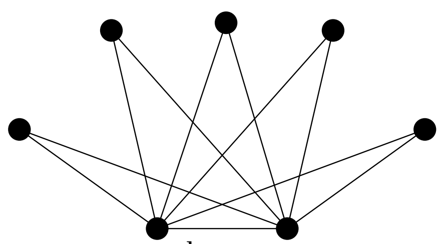

Figure 1: Example of a crown

Lemma 7 *If* $G = (K_7, c)$ *is minimally triangled, then* G^T *induces either* $K_4 \cup K_3$ *or a crown.*

Proof: Let B be a BIBD with parameters (7,3,1). We know that B partitions the edges of G into triangles, and since each node has degree 6 in G, B must contain two triangles that share exactly one node. Hence, no two below average triangles of G can share exactly one node; otherwise we could label the nodes so that these two below average triangles corresponded to blocks of the BIBD, which contradicts Lemma 5.

Assume that G^T contains two node disjoint triangles, say T_1 on nodes v_1, v_2, v_3 and T_2 on nodes v_4, v_5, v_6. By hypothesis $T(G) = 5$ and $|V(G)| = 7$. Thus there are three other below average triangles, say T_3, T_4 and T_5 as well as a node v_0 not incident to T_1 or T_2. Since the node v_0 is the only node not incident to T_1 or T_2, we know that T_3 must share at least one node with T_1 or T_2, without loss of generality, we assume T_3 contains the node v_1. We know that T_3 and T_1 cannot share exactly one node and so they must share an edge, say (v_1, v_2). Further, the node of T_3 not in T_1 must be v_0; otherwise it is a node of T_2, and then T_2 and T_3 would be two below average triangles sharing exactly one node. Hence, T_3 is on nodes v_0, v_1, v_2. This reasoning also holds for T_4 and T_5, and so say T_4 is on nodes v_0, v_2, v_3 and T_5 on nodes v_0, v_1, v_3. Note that since T_3 shares an edge with T_1, the below average triangles T_4 and T_5 must also share an edge with T_1; otherwise, T_4 or T_5 would have to share an edge with T_2, and we would have two below average triangles (T_3 and T_4 or T_3 and T_5) sharing exactly one node, v_0. Hence the below average triangles T_1, T_3, T_4 and T_5 induce a K_4 subgraph of G and T_2 is a disjoint K_3 subgraph of G.

Assume that G^T does not contain two node disjoint triangles. Then any two below average triangles must share an edge. Since there are 5 below average triangles, these below average triangles must induce a crown. (Note that if there were only 4 below average triangles, they could induce a K_4 subgraph of G.) $\square$

Knowing that the set of below average triangles of a graph induces a crown is our primary tool for proving Theorem 3. The main use of the crown is summed up in Theorem 13, the Crown Extension Theorem. This theorem states that a minimally triangled graph whose below average triangles form a crown cannot be extended (that is, cannot be a proper subgraph of another graph) without adding at least one more below average triangle. Lemma 8 is the heart of the Crown Extension Theorem.

Lemma 8 *Let $G^* = (K_{n-1}, c^*)$ such that: $n \geq 7$, $T(G^*) = (n-3)$, and G^{*T} induces a crown. Then there exists no weighted complete graph $G = (K_n, c)$ with $G^* \subset G$ such that $T(G) = T(G^*)$.*

$$
\begin{array}{lrl}
\max\ t & & \\
\text{s.t.} & \displaystyle\sum_{e \in E(K_n)} x_e & = 0 \\
& c(\overline{T}) & \leq 0, \ \ \forall\, \overline{T} \in \overline{\mathcal{T}} \\
& -c(T) \ + \ t & \leq 0, \ \ \forall\, T \in \mathcal{T} \\
& t & \geq 0\ .
\end{array}
$$

Figure 2: The linear program (**P**).

Before proving Lemma 8 rigorously, we look at a brief sketch. The proof proceeds by contradiction: assume that there exists a G such that G^T induces a crown and that there is exactly one node not in a below average triangle. Let $\overline{\mathcal{T}} = \{\overline{T}\}$ be the set of below average triangles in G, and let $\mathcal{T} = \{T\}$ be the set of above average triangles.

We then note that the edges of G yield a feasible solution to a particular linear program which we denote (**P**). This program has a variable x_e for each edge e in G as well as a variable t that measures the distance between the least weight above average triangle and zero. The constraints are formed from the conditions that the cost of each below average triangle is nonpositive, and the cost of each above average triangle minus t is nonnegative. We also constrain the sum of the edges to be zero since $c(G) = 0$. The dual program to (**P**) is (**D**) (see [4] for an introduction to Linear Programming). The two linear programs are in Figures 2 and 3 respectively. We denote the dual variables corresponding to below average triangles in G by $u_{\overline{\mathcal{T}}}$. The dual variables v_T correspond to above average triangles. The dual variable w corresponds to the constraint that the sum of all the edges is zero.

Note that G is a feasible solution to (**P**) with $t > 0$. Hence, (**D**) must be infeasible, because if it were feasible then its optimal value must be zero, and so, by Strong Duality [4], the optimal value of (**P**) must be zero. We reach a contradiction by finding a feasible solution to (**D**).

Proof of Lemma 8. Assume that there exists a $G = (K_n, c)$ such that the graph of the below average triangles of G induces a crown and that there is exactly one node not in a below average triangle. Denote this node v_0, and denote the edge that is the base of the crown $f = (f_1, f_2)$.

$$
\begin{array}{lrll}
\min 0 & & & \\
\text{s.t.} & \sum_{T \in \mathcal{T}} v_T & \geq 1 & \\
& \sum_{\overline{T} \in \overline{\mathcal{T}} : e \in \overline{T}} u_{\overline{T}} - \sum_{T \in \mathcal{T} : e \in T} v_T \ + w & = 0, & \forall\ e \in E(K_n) \\
& u_{\overline{T}} & \geq 0, & \forall\ \overline{T} \in \overline{\mathcal{T}} \\
& v_T & \geq 0, & \forall\ T \in \mathcal{T}
\end{array}
$$

Figure 3: (**D**), the dual linear program to (**P**).

Let $\overline{\mathcal{T}} = \{\overline{T}\}$ be the set of below average triangles in G, and let $\mathcal{T} = \{T\}$ be the set of above average triangles.

The weights on the edges of G yield a feasible solution to the linear program (**P**) (see Figure 2). Moreover, the weights on the edges of G allow for a solution where $t > 0$. This is because $c(G) = 0$, so the average of the triangle's weights is 0, and thus there is some slack between 0 and the weight of all above average triangles.

Let $Z = V(G) - \{f_1, f_2, v_0\}$, and consider the dual linear program (**D**) (see Figure 3). This program has a variable for every triangle in G. These triangles fall into six categories. For examples, see Figure 4.

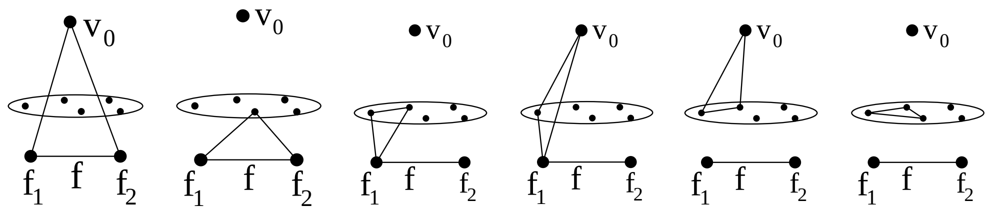

Figure 4: Examples of $T^1, T^2, T^3, T^4, T^5, T^6$

$T^1 = (f_1, f_2, v_0)$
$T^2 = (a, f_1, f_2),\ a \in Z$
$T^3 = (a, b, f_i),\ a, b \in Z,\ i = 1 \text{ or } 2$
$T^4 = (a, f_i, v_0),\ a \in Z,\ i = 1 \text{ or } 2$
$T^5 = (a, b, v_0),\ a, b \in Z$
$T^6 = (a, b, c), a, b, c \in Z$

The edges of G fall into five categories. For examples, see Figure 5.

$e^1 = (f_i, v_0),\ i = 1$ or 2

$e^2 = (f_i, a),\ a \in Z,\ i = 1$ or 2
$e^3 = (a, v_0),\ a \in Z$
$e^4 = (a, b),\ a, b \in Z$
$e^5 = f$

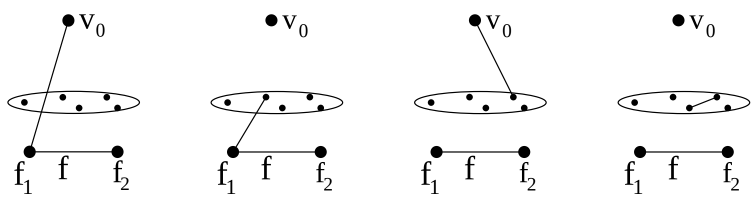

Figure 5: Examples of e^1, e^2, e^3, e^4

Given a feasible solution (u^*, v^*, w^*) of the dual problem, we can get another feasible solution by permuting the nodes in $V - \{v_0, f_1, f_2\}$ arbitrarily and possibly permuting f_1 with f_2. Taking the average of all solutions obtained from (u^*, v^*, w^*) by such permutations, we get a solution where all triangles in the same category have dual variables with identical values. Therefore the linear program $(\overline{\mathbf{D}})$ in Figure 6 is equivalent to $(\mathbf{D})$.

Each constraint in this linear program corresponds to a constraint in $(\mathbf{D})$. The first five constraints of $(\overline{\mathbf{D}})$ are found by fixing an edge e in an edge category ($e^1, \ldots, e^6$ respectively) and counting how many triangles in each triangle category contain that edge. The final constraint is found by counting how many above average triangles there are in each category.

Figure 7 gives a feasible solution for $(\overline{\mathbf{D}})$ when $n \geq 7$ found using Maple.

Hence $(\mathbf{D})$ is feasible. Since $(\mathbf{D})$ is feasible, it has optimal objective value equal to 0. Therefore the optimal objective value of the primal program $(\mathbf{P})$ should be 0. But G yields a solution where $t > 0$, a contradiction. $\square$

Alone, Lemma 8 only tells us that if the below average triangles of G induce a crown, then any graph containing G as a proper subgraph cannot have the same number of below average triangles as G. In order to prove the Crown Extension Theorem 13, we need to be able to compare $D(n_1, 3)$ to $D(n_2, 3)$, where $n_1 < n_2$. Lemma 10 does this with the help of the very useful

$$
\begin{array}{llllll}
\min & 0 & & & & \\
\text{s.t.} & -T^1 & -(n-3)T^4 & & & +w=0 \\
 & T^2 & -(n-4)T^3 & -T^4 & & +w=0 \\
 & -2T^4 & -(n-4)T^5 & & & +w=0 \\
 & -T^1 & -2T^3 & -T^5 & -(n-5)T^6 & +w=0 \\
 & -T^1 & +(n-3)T^2 & & & +w=0 \\
 & & & & & \\
 & T^1 & +2\binom{n-3}{2}T^3 & +2(n-3)T^4+ & & \\
 & & +\binom{n-3}{2}T^5 & +\binom{n-3}{3}T^6 & & \geq 1 \\
 & & & T^i \geq 0, & i=1,\ldots,6 &
\end{array}
$$

Figure 6: $(\overline{\mathbf{D}})$ for the case of the crown

Lemma 9.

For a collection of nodes N, let $q(N)$ equal the sum of the weights of the edges with at least one endpoint in N. Hence, for a single node v, $q(v)$ is simply the sum of the edges with v as an endpoint; this is the *weight of a node.*

Lemma 9 *If $c(G)=0$, then there exists a node $v \in V(G)$ such that $q(v) \geq 0$.*

Proof:

$$
\sum_{v \in V(G)} q(v) = 2c(G)
$$

which is zero if $c(G)=0$. Hence, every node cannot have negative weight. $\square$

Lemma 10 $D(n,3) \geq D(n-1,3)$.

Proof: Let $G=(K_n,c)$ be minimally triangled. By Lemma 9, there is some node v such that $q(v) \geq 0$. Hence, since $c(G-v)=c(G)-q(v)$, we know $0 \geq c(G-v)$. Recall that the average weight of a triangle in a weighted complete graph is proportional to the sum of all the weights in the graph. Thus, $A_{\text{clique}}(3,G)=0$ and $A_{\text{clique}}(3,G-v) \leq 0$. Since $G-v$ is a subgraph of G, all the triangles in $G-v$ are in G. Further, a below average triangle

$$\begin{bmatrix} T^1 \\ T^2 \\ T^3 \\ T^4 \\ T^5 \\ T^6 \\ w \end{bmatrix} = \begin{bmatrix} \dfrac{6}{n(n-1)} \\ 0 \\ \dfrac{6}{n(n-1)(n-4)} \\ 0 \\ \dfrac{6}{n(n-1)(n-4)} \\ \dfrac{6(n-7)}{n(n-1)(n-4)(n-5)} \\ \dfrac{6}{n(n-1)} \end{bmatrix}$$

Figure 7: Solution to ($\overline{\mathbf{D}}$) for the case of the crown

in $G - v$ has nonpositive weight, and so this triangle must be below average in G. Therefore $T(G) \geq T(G - v)$. By assumption $T(G) = D(n, 3)$, and by definition $T(G - v) \geq D(n - 1, 3)$, which yields the result. □

The technique of deletion used to prove Lemma 10 is a useful one. The Deletion Lemma 11 formalizes this idea.

Lemma 11 *(Deletion Lemma) If $c(G) = 0$, then there exists a nonempty subset of the nodes N such that $q(N) \geq 0$ and*

$$T(G - N) \leq T(G) \ .$$

Moreover, if we let $T(G, N)$ denote the number of below average triangles using at least one node of N, then

$$T(G, N) \leq T(G) - D(n - |N|, 3) \ .$$

Proof: The existence of the set N with $q(N) \geq 0$ is proved by Lemma 9. The graph $G - N$ is a subgraph of G, so any triangle in $G - N$ has the same weight that it has in G. Since $c(G) = 0$ and $q(N) \geq 0$, $c(G - N) \leq 0$. Hence, because the average weight of a triangle is proportional to the weight of the graph,

$$A_{\text{clique}}(3, G - N) \leq A_{\text{clique}}(3, G)$$

which implies that any below average triangle in $G - N$ is also below average in G.

The statement that $T(G, N) \leq T(G) - D(n - |N|, 3)$ follows trivially from the definitions. □

Lemma 12 follows easily from Lemma 8 and Lemma 10.

Lemma 12 *Let $G^* = (K_{n-1}, c)$ be a minimally triangled graph such that $n \geq 7$, $|G^{*T}| = (n-3)$, and G^{*T} induces a crown. Then every graph containing G^* as a proper subgraph has at least one more below average triangle than G^*.*

Proof: Let G be a graph which contains G^* as a proper subgraph. Then from Lemma 8, $T(G^*) \neq T(G)$. By hypothesis, G^* is minimally triangled, so $T(G^*) = D(n-1, 3)$, and by Lemma 10, $D(n-1, 3) \leq D(|V(G)|, 3)$. Hence, $T(G^*) < T(G)$. □

The Crown Extension Theorem 13 is a simple application of Lemma 12.

Theorem 13 *(Crown Extension Theorem) If every minimally triangled graph on $(n-1)$ nodes $G^* = (K_{n-1}, c)$ contains exactly $(n-3)$ below average triangles and these below average triangles induce a crown, then $D(n, 3) = (n-2)$.*

Proof: By hypothesis and Lemmas 4 and 10, we know that $(n-3) = D(n-1, 3) \leq D(n, 3) \leq (n-2)$.

Assume that $D(n, 3) = (n-3)$, and let $G = (K_n, c)$ be minimally triangled. By the Deletion Lemma 11, there exists a node $v_0 \in V(G)$ such that $q(v_0) \geq 0$ and all the below average triangles in $G - v_0$ are below average in G. The graph $G - v_0$ must have at least $(n-3)$ below average triangles. By assumption, the graph G has exactly $(n-3)$ below average triangles, and hence v_0 must be adjacent to no below average triangles and $G - v_0$ is minimally triangled.

Since $G - v_0$ is minimally triangled, by hypothesis, $(G - v_0)^T$ induces a crown. But if $(G - v_0)^T$ induced a crown, then G would be an extension of a minimally triangled graph where no new below average triangles were added. This is a contradiction with Lemma 12. □

Recall that Lemma 7 states that if $G = (K_7, c)$, then G^T induces either $K_4 \cup K_3$ or a crown. The Crown Extension Theorem 13 allows us to examine the latter case. The following lemmas examine the former.

Lemma 14 *Let $G = (K_n, c)$ such that $c(G) = 0$. Also let N be a set of nodes in $V(G)$ with $|N| \geq 2$ and let C be the subgraph of G induced by the nodes of N. If $c(C) \leq 0$, $(n+2-3|N|) \geq 0$, and $\sum_{v \in N} q(v) \leq 0$, then there exists a below average triangle of G containing exactly two nodes of N.*

Proof: Let $\mathfrak{T}$ be the triangles sharing an edge with C but not contained in C. Each edge of C is in $(n - |N|)$ triangles of $\mathfrak{T}$ (one for each node not in N). Each edge that has exactly one endpoint in C is in $(|N| - 1)$ triangles of $\mathfrak{T}$. Count the weights of the triangles in $\mathfrak{T}$ by summing $q(v)$ over all $v \in N$, multiply it by $(|N| - 1)$ and subtracting $2(|N| - 1)c(C)$. Hence, the sum of the weights of the triangles in $\mathfrak{T}$ is:

$$(n - |N|)c(C) + (|N| - 1)\sum_{v \in N} q(v) \; - (2|N| - 2)c(C) =$$

$$(n - 3|N| + 2)c(C) + (|N| - 1)\sum_{v \in N} q(v) \; .$$

Applying the conditions of the hypothesis, the above is nonpositive. Hence, the sum of the weights of the triangles using exactly two nodes of N is nonpositive, and so at least one of these triangles must have nonpositive weight. Therefore, since $c(G) = 0$, this triangle is below average. □

We apply Lemma 14 to a particular structure of below average triangles. A *clique of below average triangles* is defined as a clique on three or more nodes in which every triangle is below average.

Lemma 15 *Let $G = (K_n, c)$ such that $c(G) = 0$. Also let $\{v_1, \ldots, v_m\}$ be the nodes in a non empty clique of below average triangles C in G. If $(n+2-3m) \geq 0$ and $\sum_{i=1}^{m} q(v_i) \leq 0$, then there exists a below average triangle of G containing an edge of C but not contained in C.*

Proof: In order to apply Lemma 14, we need only conclude that $c(C) \leq 0$. Each edge of C is in $(|C| - 2)$ triangles of C, and so $c(C)$ equals the sum of all the triangles in C divided by $(|C| - 2)$. We know C is a clique of below average triangles, and so every triangle in C has nonpositive weight, which implies $c(C) \leq 0$. □

Note that if $n = 8$, the statement in the hypothesis that $(n + 2 - 3m) \geq 0$ holds for $m = 3$. Also, for $n \geq 10$, then this part of the hypothesis holds for $m = 4$.

We are now ready to establish Theorem 3 for $n = 8$. Note that the proof is similar to that for the Crown Extension Theorem 13.

Theorem 16 $D(8,3) = 6$.

Proof: By Theorem 6 and Lemmas 4 and 10, we know that $5 = D(7,3) \leq D(8,3) \leq 6$.

Assume that $D(8,3) = 5$, and let $G = (K_8, c)$ be minimally triangled.

By the Deletion Lemma 11, there exists a node v_0 in $V(G)$ such that $q(v_0) \geq 0$ which we can delete such that all the below average triangles in $G - v_0$ are below average in G. The graph $G - v_0$ has seven nodes, and hence must have at least 5 below average triangles. By assumption, the graph G has exactly 5 below average triangles, and hence v_0 must be adjacent to no below average triangles and $G - v_0$ is minimally triangled. Moreover, if any node in a below average triangle had positive weight, we could apply Lemma 11 and delete that node. The resulting graph would have one less below average triangle and only 7 nodes. This would contradict Theorem 6. Therefore all the nodes in below average triangles have negative weight.

Since $G - v_0$ is minimally triangled, we can apply Lemma 7, and so $(G - v_0)^T$ induces either $K_4 \cup K_3$ or a crown. We know $(G - v_0)^T$ cannot induce a crown, because we would have a contradiction to the Crown Extension Theorem 13. Therefore $(G - v_0)^T$ induces $K_4 \cup K_3$. In particular, this means that $(G - v_0)^T$ is the disjoint union of two cliques of below average triangles. Above we concluded that any node in a below average triangle has negative weight. Hence, the sum of all the nodes in either clique of below average triangles is nonpositive. The K_3 subgraph is a clique of below average triangles on three nodes. Then, by Lemma 15, the K_3 subgraph has a below average triangle not in the clique but sharing an edge with the clique. This is a contradiction, because no such triangle exists in $(G - v_0)^T$.

Therefore $D(8,3) \neq 5$. The result follows. □

3.2 $D(n,3)$ for $n \equiv 1,2,3,4 \pmod 6$

In this subsection we prove Theorem 3 for $n \geq 9$ and $n \equiv 1,2,3,4 \pmod 6$. We also show that if $G = (K_n, c)$ is minimally triangled and $n \equiv 1,3 \pmod 6$, then G^T induces a crown incident to all the nodes of G. The basic idea of the proof is to use large sets to partition the triangles of G. Then we note that each BIBD in the large set can contain at most 1 below average triangle. More, we can use the nature of the partition the large set creates to decide how the below average triangles in a minimally triangled graph relate to each other.

Lemma 17 *Let $G = (K_n, c)$ be a graph where the parameters $(n, 3, 1)$ admit a large set L. Also let no BIBD in L contain more than ℓ rows corresponding to below average triangles in G. If L contains a BIBD B with $\ell + 1$ rows corresponding to some configuration of triangles C, then G cannot contain a set of $\ell + 1$ below average triangles in the configuration of C.*

Proof: Consider a permutation on the nodes of G. This corresponds to a permutation on the elements of each BIBD in the large set L. Thus every permutation in the permutation group S_n, when applied to L, yields a large set with the same partitioning properties as L. Hence, if G contains a set of $\ell + 1$ below average triangles in the configuration of C and has $\ell + 1$ rows corresponding to some configuration of triangles C, then there would exist a permutation that labeled the nodes so that the said $\ell + 1$ below average triangles corresponded to the said $\ell + 1$ rows of B. This is a contradiction with the hypothesis that no BIBD in L contains more than ℓ rows corresponding to below average triangles in G. $\square$

Theorem 18 *If $n \geq 9$ and $n \equiv 1, 3 \pmod{6}$, then $D(n, 3) = (n - 2)$. Moreover, if $G = (K_n, c)$ is minimally triangled, then G^T induces a crown.*

Proof: From Lu-Teirlinck's Theorem 1, we know that if $n \geq 9$ and $n \equiv 1, 3 \pmod{6}$, then a large set L exists for the parameters $(n, 3, 1)$. Each element of L is a BIBD. Consider the elements of these BIBDs as the nodes of K_n. Thus the blocks of the BIBDs, which are three element subsets of the nodes, induce triangles in K_n, and so L is a partitioning of the triangles of K_n into designs. Recall that a (v, k, λ)-BIBD has $b = \dfrac{\lambda v(v-1)}{k(k-1)}$ blocks, so each design in L has $\dfrac{n(n-1)}{6}$ blocks. The complete graph K_n contains $\binom{n}{3}$ triangles. Therefore, $|L| = (n - 2)$.

Choose a design D in L. Since $\lambda = 1$, D partitions the edges of G into blocks which are themselves triangles. Hence the sum of all weights of the edges in a block of D is the same as the sum of all the edges of G, which is zero. Therefore some block of D must induce a below average triangle. This is true for every element of L, so

$$D(n, 3) \geq (n - 2) \ .$$

With Lemma 4, we get the result that $D(n, 3) = (n - 2)$.

Recall that the set of below average triangles in the example given in Lemma 4 induces a crown (one edge is very negative, the rest positive). We want to show that the set of below average triangles in every minimally triangled graph $G = (K_n, c)$ induces a crown. In order to do this, we simply show that every two below average triangles share an edge. Then, because $(n-2) > 4$, the below average triangles are forced into a crown.

The large set L partitions the triangles into BIBDs. At least one below average triangle is in each BIBD of L. The size of L is $(n-2)$. Hence, if G has exactly $(n-2)$ below average triangles, then exactly one below average triangle is in each BIBD of L.

Each design $D \in L$ contains two blocks that induce two triangles that share exactly one node. Hence, by Lemma 17, no two below average triangles can share exactly one node.

Each design in L contains two node disjoint triangles. This is easy to see by a simple counting argument: we know that every edge is in exactly one triangle induced by a block of the BIBD, and so a triangle induced by the BIBD is adjacent to at most $\dfrac{3(n-3)}{2}$ other triangles induced by the BIBD; we know that each BIBD has $\dfrac{n(n-1)}{6}$ blocks; and we know that if $n > 7$, then $\dfrac{3(n-3)}{2} + 1 < \dfrac{n(n-1)}{6}$. Thus, since each BIBD induces two node disjoint triangles, by Lemma 17, no two below average triangles can be disjoint. Hence, every two below average triangles share an edge, and G^T induces a crown. □

The proof of Theorem 3 in the cases of $n \equiv 2, 4 \pmod 6$ is now a direct application of The Crown Extension Theorem 13.

Theorem 19 *If $n \geq 9$ and $n \equiv 2, 4 \pmod 6$, then $D(n, 3) = (n-2)$.*

Proof: By hypothesis, $n \geq 9$ and $n \equiv 2, 4 \pmod 6$. Thus, $(n-1) \equiv 1, 3 \pmod 6$. Theorem 18 yields the results necessary to satisfy the hypothesis of the Crown Extension Theorem 13. □

The remainder of the proof for Theorem 3 concerning the cases where $n \equiv 5, 6 \pmod 6$ can be found in [3]. The case $n \equiv 5 \pmod 6$ is a result of establishing that for $n \equiv 4 \pmod 6$ and G minimally triangled, G^T induces a crown. This result uses the structure of large sets and is not a trivial one. For $n \equiv 6 \pmod 6$, we make use of what is known about the case for $n \equiv 1 \pmod 6$ as well as some structural lemmas regarding cliques of below average triangles.

4 $D(n,k)$ for $k > 3$

Unfortunately, as of this writing, the complete set of values for $D(n,k)$ $k = 4, 5, \ldots$ is unknown. However, attempts to extend the proof techniques used for establishing $D(n,3)$, lead to the following conjecture:

Conjecture 20 *Let n and k be integers such that $k \geq 2$ and $n \geq k$. Then there exists an n_0 such that for $n \geq n_0$*

$$D(n,k) = \binom{n-2}{k-2} .$$

Note that if $k = 2$ the conjecture is trivial to establish for $n_0 = 2$, and for $k = 3$ and $n_0 = 7$, the conjecture is simply Theorem 3.

Our first step in proving Theorem 3 is to establish an upper bound on $D(n,3)$ by demonstrating a class of graphs with the appropriate number of below average triangles. We shall do the same here with k-cliques using the same weighted graphs we used in Lemma 4. Recall that for a graph $G = (K_n, c)$, we denote the average of the weights of the k-clique subgraphs by $A_{\text{clique}}(k, G)$, and

$$A_{\text{clique}}(k, G) = \frac{k(k-1)}{n(n-1)} c(G) .$$

Thus the average weight of a k-clique is proportional to the weight of the graph.

Lemma 21 *For $n \geq k$,*

$$D(n,k) \leq \binom{n-2}{k-2} .$$

Proof: Let c be the cost function that weights one edge with $(1 - |E(K_n)|)$ and the remaining edges with 1. We know $c(G) = 0$ which implies that $A_{\text{clique}}(k, G) = 0$. Hence, only the k-cliques that use the negative weight edge are below average. □

Our next step in proving Theorem 3, aside from showing the special cases of $n = 7, 8$, is to use Lu-Teirlinck's Theorem 1 to establish a base case for the result: namely $n \geq 9$ and $n \equiv 1, 3 \pmod 6$. We can do the same here for larger values of k using Wilson's Theorem 2 to establish base cases. In order to do this, we need the following definitions and Lemma 22.

As we did with below average triangles, we say that a k-clique is below average if it has weight less than or equal to the average weight of all k-cliques in the graph. Also, a graph $G = (K_n, c)$ is called *minimally k-cliqued* if it contains exactly $D(n,k)$ below average k-cliques.

Lemma 22 *If there exists an $(n,k,1)$-BIBD, then* $D(n,k) \geq \binom{n-2}{k-2}$.

Proof: This proof is a generalization of the technique used to prove Lemma 6.

Let $G = (K_n, c)$ be minimally k-cliqued and B be an $(n,k,1)$-BIBD. An isomorphism h exists between the elements of B and the nodes of G. Every block of B is isomorphic to a k-clique in G, and because $\lambda = 1$, each edge is in exactly one k-clique that is isomorphic to a block of B. Without loss of generality, we can assume $c(G) = 0$ which implies $A_{\text{clique}}(k, G) = 0$. Further, $c(G) = 0$ also implies that the sum of the weights of the k-cliques that are isomorphic to blocks of B equals zero, and hence the average weight of the k-cliques that are isomorphic to blocks of B is zero. Hence, at least one block of B is isomorphic to a below average k-clique.

There are $n!$ isomorphisms between the elements of B and the nodes of G. Call this set of isomorphisms $\mathcal{D}$. We want to count how many $h \in \mathcal{D}$ send some block of B to a particular k-clique C. A particular block of B, say the m^{th}, has k 1's and $(n-k)$ 0's. An isomorphism h sends the m^{th} block to C if and only if the elements in the m^{th} block are sent to the nodes that are in C. There are $k!$ ways to this, and $(n-k)!$ ways to send the remaining elements of B to the remaining nodes of G. We then get a total of $k!(n-k)!$ isomorphisms between the m^{th} block of B and C. There are $\dfrac{n(n-1)}{k(k-1)}$ blocks, so

$$\frac{n(n-1)}{k(k-1)} k!(n-k)!$$

different isomorphisms send a block of B to C. Hence, since each isomorphism must send some block of B to some below average k-clique, we have

$$\begin{aligned} D(n,k) &\geq \frac{n!}{\frac{n(n-1)}{k(k-1)} k!(n-k)!} \\ &= \frac{n!k(k-1)}{(n-k)!k!n(n-1)} \\ &= \frac{(n-2)!}{(k-2)!(n-k)!} \\ &= \binom{n-2}{k-2}. \end{aligned} \tag{3}$$

□

Lemma 22 together with Wilson's Theorem 2 gives us:

Theorem 23 *Let n and k be integers such that $k \geq 2$ and $n \geq k$. Then there exists an n_0 such that for $n \geq n_0$ and satisfying the conditions*

(1). $b = \dfrac{n(n-1)}{k(k-1)}$ *is integer*

(2). $r = \dfrac{n-1}{k-1}$ *is integer*

(3). $nr = bk$

(4). $r(k-1) = \lambda(n-1)$

(5). $b \geq n$

we have

$$D(n,k) = \binom{n-2}{k-2} .$$

If we continue to parallel the proof technique for Theorem 3, we now need some methods for establishing Conjecture 20 for values of n greater than n_0 but not satisfying the above conditions. The Deletion Lemma 11, the Crown Extension Theorem 13, and results showing that the below average triangles of some minimally triangled graphs had to induce a crown were the most important ideas establishing Theorem 3 for $n \not\equiv 1, 3 \pmod 6$. For the case where $k > 3$, we do still have a deletion lemma. Unfortunately, we have no theorem analogous to the Crown Extension Theorem 13 or any results restricting the structure of the below average k-cliques in a minimally k-cliqued graph.

This General Deletion Lemma is not enough to establish our conjecture. Though we can use it to establish that $D(n-1,k) \leq D(n,k)$ (just as we did in Lemma 10), this is not as tight a bound for a general k as it is when $k = 3$. When $k = 3$, we easily can show for $n \geq 7$, $D(n,3) - D(n-1,3) \leq 1$;hence, after showing $D(n-1,3) = (n-3)$, we have only two choices for $D(n,3)$, $(n-3)$ or $(n-2)$. As k increases however, the conjectured difference $D(n,k) - D(n-1,k)$ increases as well, giving us more possibilities to check.

This difficulty extends itself further. Imagine we have some sort of Crown Extension Theorem for k-cliques; that is, assume we know that given a particular structure of below average k-cliques in G, say a k-crown, then every extension of G contains more below average k-cliques than G. In the case of

$k = 3$, this is enough because adding one below average triangle takes us to our upper bound on $D(n, 3)$. For larger k, the upper bound is farther away, and since we are not sure how many below average k-cliques are in the extension, more work is needed to establish the result.

This point, however, is moot as we don't as of yet have a result paralleling the Crown Extension Theorem 13. It seems likely that, for a fixed k and n large enough, all the below average k-cliques in a minimally k-cliqued graph share an edge, however it has not been proven.

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