LINEAR PROGRAMMING RELAXATIONS OF QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMS

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Abstract. We investigate the use of linear programming tools for solving semidefinite programming relaxations of quadratically constrained quadratic problems. Classes of valid linear inequalities are presented, including sparse PSD cuts, and principal minors PSD cuts. Computational results based on instances from the literature are presented.

Key words. Quadratic Programming, Semidefinite Programming, relaxation, Linear Programming.

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1. Introduction. Many combinatorial problems have Linear Programming (LP) relaxations that are commonly used for their solution through branch-and-cut algorithms. Some of them also have stronger relaxations involving positive semidefinite (PSD) constraints. In general, stronger relaxations should be preferred when solving a problem, thus using these PSD relaxations is tempting. However, they come with the drawback of requiring a Semidefinite Programming (SDP) solver, creating practical difficulties for an efficient implementation within a branch-and-cut algorithm. Indeed, a major weakness of current SDP solvers compared to LP solvers is their lack of efficient warm starting mechanisms. Another weakness is solving problems involving a mix of PSD constraints and a large number of linear inequalities, as these linear inequalities put a heavy toll on the linear algebra steps required during the solution process.

In this paper, we investigate LP relaxations of PSD constraints with the aim of capturing most of the strength of the PSD relaxation, while still being able to use an LP solver. The LP relaxation we obtain is an outer-approximation of the PSD cone, with the typical convergence difficulties when aiming to solve problems to optimality. We thus do not cast this work as an efficient way to solve PSD problems, but we aim at finding practical ways to approximate PSD constraints with linear ones.

We restrict our experiments to Quadratically Constrained Quadratic Program ($QCQP$). A QCQP problem with variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ is

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a problem of the form

$$\begin{array}{lll} \max & x^TQ_0x + a_0^Tx + b_0^Ty & \\ \text{s.t.} & & \\ & x^TQ_kx + a_k^Tx + b_k^Ty \quad \le c_k & \text{for } \; k = 1,2,\dots,p \\ & l_{x_i} \le x_i \le u_{x_i} & \text{for } \; i = 1,2,\dots,n \\ & l_{y_j} \le y_j \le u_{y_j} & \text{for } \; j = 1,2,\dots,m \end{array} \qquad (\mathbf{QCQP})$$

where, for $k = 0,1,2,\dots,p$, Q_k is a rational symmetric $n \times n$-matrix, a_k is a rational n-vector, b_k is a rational m-vector, and $c_k \in \mathbb{Q}$. Moreover, the lower and upper bounds l_{x_i}, u_{x_i} for $i = 1,\dots,n$, and l_{y_j}, u_{y_j} for $j = 1,\dots,m$ are all finite. If Q_0 is negative semidefinite and Q_k is positive semidefinite for each $k = 1,2,\dots,p$, problem **QCQP** is convex and thus easy to solve. Otherwise, the problem is NP-hard [6].

An alternative *lifted* formulation for **QCQP** is obtained by replacing each quadratic term x_ix_j with a new variable X_{ij}. Let $X = xx^T$ be the matrix with entry X_{ij} corresponding to the quadratic term x_ix_j. For square matrices A and B of the same dimension, let $A \bullet B$ denote the *Frobenius inner product* of A and B, i.e., the trace of A^TB. Problem **QCQP** is then equivalent to

$$\begin{array}{lll} \max & Q_0 \bullet X + a_0^Tx + b_0^Ty & \\ \text{s.t.} & & \\ & Q_k \bullet X + a_k^Tx + b_k^Ty \quad \le c_k & \text{for } \; k = 1,2,\dots,p \\ & l_{x_i} \le x_i \le u_{x_i} & \text{for } \; i = 1,2,\dots,n \\ & l_{y_j} \le y_j \le u_{y_j} & \text{for } \; j = 1,2,\dots,m \\ & X = xx^T. & \end{array} \qquad (\mathbf{LIFT})$$

The difficulty in solving problem **LIFT** lies in the non-convex constraint $X = xx^T$. A relaxation, dubbed PSD, that is possible to solve relatively efficiently is obtained by relaxing this constraint to the requirement that $X - xx^T$ be positive semidefinite, i.e., $X - xx^T \succeq 0$. An alternative relaxation of **QCQP**, dubbed RLT, is obtained by the Reformulation Linearization Technique [17], using products of pairs of original constraints and bounds and replacing nonlinear terms with new variables.

Anstreicher [2] compares the PSD and RLT relaxations on a set of quadratic problems with box constraints, i.e., **QCQP** problems with $p = 0$ and with all the variables bounded between 0 and 1. He shows that the PSD relaxations of these instances are fairly good and that combining the PSD and RLT relaxations yields significantly tighter relaxations than either of the PSD or RLT relaxations. The drawback of combining the two relaxations is that current SDP solvers have difficulties to handle the large number of linear constraints of the RLT.

Our aim is to solve relaxations of **QCQP** using exclusively linear programming tools. The RLT is readily applicable for our purposes, while the PSD technique requires a cutting plane approach as described in Section 2.

In Section 3 we consider several families of valid cuts. The focus is essentially on capturing the strength of the positive semidefinite condition using standard cuts [18], and some sparse versions of these.

We analyze empirically the strength of the considered cuts on instances taken from GLOBALLib [10] and quadratic programs with box constraints described in more details in the next section. Implementation and computational results are presented in Section 4. Finally, Section 5 summarizes the results and gives possible directions for future research.

2. Relaxations of QCQP problems. A typical approach to get bounds on the optimal value of a **QCQP** is to solve a convex relaxation. Since our aim is to work with linear relaxations, the first step is to linearize **LIFT** by relaxing the last constraint to $X = X^T$. We thus get the Extended formulation

$$\begin{array}{llll} \max & Q_0 \bullet X + a_0^T x + b_0^T y & & \\ \text{s.t.} & & & \\ & Q_k \bullet X + a_k^T x + b_k^T y \quad \leq c_k & \text{for} & k = 1, 2, \ldots, p \\ & l_{x_i} \leq x_i \leq u_{x_i} & \text{for} & i = 1, 2, \ldots, n \\ & l_{y_j} \leq y_j \leq u_{y_j} & \text{for} & j = 1, 2, \ldots, m \\ & X = X^T. & & \end{array} \qquad (\textbf{EXT})$$

EXT is a Linear Program with $n(n + 3)/2 + m$ variables and the same number of constraints as **QCQP**. Note that the optimal value of **EXT** is usually a weak upper bound for **QCQP**, as no constraint links the values of the x and X variables. Two main approaches for doing that have been proposed and are based on relaxations of the last constraint of **LIFT**, namely

$$X - xx^T = 0. \qquad (2.1)$$

They are known as the Positive Semidefinite (PSD) relaxation and the Reformulation Linearization Technique (RLT) relaxation.

2.1. PSD Relaxation. As $X - xx^T = 0$ implies $X - xx^T \succcurlyeq 0$, using this last constraint yields a convex relaxation of **QCQP**. This is the approach used in [18, 20, 21, 23], among others.

Moreover, using Schur's complement

$$X - xx^T \succcurlyeq 0 \qquad \Leftrightarrow \qquad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succcurlyeq 0,$$

and defining

$$\tilde{Q}_k = \begin{pmatrix} -c_k & a_k^T/2 \\ a_k/2 & Q_k \end{pmatrix}, \qquad \tilde{X} = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix},$$

we can write the PSD relaxation of **QCQP** in the compact form

$$\begin{array}{lll} \max & \tilde{Q}_0 \bullet \tilde{X} + b_0^T y & \\ \text{s.t.} & & \\ & \tilde{Q} \bullet \tilde{X} + b_k^T y \le 0 & k = 1, 2, \ldots, p \\ & l_{x_i} \le x_i \le u_{x_i} & i = 1, 2, \ldots, n \\ & l_{y_j} \le y_j \le u_{y_j} & j = 1, 2, \ldots, m \\ & \tilde{X} \succcurlyeq 0. & \end{array} \qquad (\mathbf{PSD})$$

This is a positive semidefinite problem with linear constraints. It can thus be solved in polynomial time using interior point algorithms. **PSD** is tighter than usual linear relaxations for problems such as the Maximum Cut, Stable Set, and Quadratic Assignment problems [25]. All these problems can be formulated as **QCQP**s.

2.2. RLT Relaxation. The Reformulation Linearization Technique [17] can be used to produce a relaxation of **QCQP**. It adds linear inequalities to **EXT**. These inequalities are derived from the variable bounds and constraints of the original problem as follows: multiply together two original constraints or bounds and replace each product term $x_i x_j$ with the variable X_{ij}. For instance, let $x_i, x_j,\ i, j \in \{1, 2, \ldots, n\}$ be two variables from **QCQP**. By taking into account only the four original bounds $x_i - l_{x_i} \ge 0$, $x_i - u_{x_i} \le 0$, $x_j - l_{x_j} \ge 0$, $x_j - u_{x_j} \le 0$, we get the RLT inequalities

$$\begin{array}{ll} X_{ij} - l_{x_i} x_j - l_{x_j} x_i & \ge -l_{x_i} l_{x_j}, \\ X_{ij} - u_{x_i} x_j - u_{x_j} x_i & \ge -u_{x_i} u_{x_j}, \\ X_{ij} - l_{x_i} x_j - u_{x_j} x_i & \le -l_{x_i} u_{x_j}, \\ X_{ij} - u_{x_i} x_j - l_{x_j} x_i & \le -u_{x_i} l_{x_j}\ . \end{array} \qquad (2.2)$$

Anstreicher [2] observes that, for Quadratic Programs with box constraints, the PSD and RLT constraints together yield much better bounds than those obtained from the **PSD** or **RLT** relaxations. In this work, we want to capture the strength of both techniques and generate a Linear Programming relaxation of **QCQP**.

Notice that the four inequalities above, introduced by McCormick [12], constitute the convex envelope of the set $\{(x_i, x_j, X_{ij}) \in \mathbb{R}^3 : l_{x_i} \le x_i \le u_{x_i}, l_{x_j} \le x_j \le u_{x_j}, X_{ij} = x_i x_j\}$ as proven by Al-Khayyal and Falk [1], i.e., they are the tightest relaxation for the single term X_{ij}.

3. Our Framework. While the RLT constraints are linear in the variables in the **EXT** formulation and therefore can be added directly to **EXT**, this is not the case for the PSD constraint. We use a linear outer-approximation of the **PSD** relaxation and a cutting plane framework, adding a linear inequality separating the current solution from the PSD cone.

The initial relaxation we use and the various cuts generated by our separation procedure are described in more details in the next sections.

3.1. Initial Relaxation. Our initial relaxation is the **EXT** formulation together with the $O(n^2)$ RLT constraints derived from the bounds on the variables $x_i,\ i = 1, 2, \ldots, n$. We did not include the RLT constraints derived from the problem constraints due to their large number and the fact that we want to avoid the introduction of extra variables for the multivariate terms that occur when quadratic constraints are multiplied together.

The bounds $[L_{ij}, U_{ij}]$ for the extended variables X_{ij} are computed as follows:

$$\begin{aligned} L_{ij} &= \min\{l_{x_i}l_{x_j};\, l_{x_i}u_{x_j};\, u_{x_i}l_{x_j};\, u_{x_i}u_{x_j}\}, \quad \forall i = 1, 2, \ldots, n;\ j = i, \ldots, n \\ U_{ij} &= \max\{l_{x_i}l_{x_j};\, l_{x_i}u_{x_j};\, u_{x_i}l_{x_j};\, u_{x_i}u_{x_j}\}, \quad \forall i = 1, 2, \ldots, n;\ j = i, \ldots, n. \end{aligned}$$

In addition, equality (2.1) implies $X_{ii} \geq x_i^2$. We therefore also make sure that $L_{ii} \geq 0$. In the remainder of the paper, this initial relaxation is identified as **EXT+RLT**.

3.2. PSD Cuts. We use the equivalence that a matrix is positive semidefinite if and only if

$$v^T \tilde{X} v \geq 0 \quad \text{for all} \quad v \in \mathbb{R}^{n+1} \,. \tag{3.1}$$

We can reformulate **PSD** as the semi-infinite Linear Program

$$\begin{array}{llll} \max & \tilde{Q}_0 \bullet \tilde{X} + b_0^T y & & \\ \text{s.t.} & & & \\ & \tilde{Q} \bullet \tilde{X} + b_k^T y & \leq c_k & \text{for } \ k = 1, 2, \ldots, p \\ & l_{x_i} \leq x_i \leq u_{x_i} & & \text{for } \ i = 1, 2, \ldots, n \\ & l_{y_j} \leq y_j \leq u_{y_j} & & \text{for } \ j = 1, 2, \ldots, m \\ & v^T \tilde{X} v & \geq 0 & \text{for all } \ v \in \mathbb{R}^{n+1}. \end{array} \tag{\textbf{PSDLP}}$$

A practical way to use **PSDLP** is to adopt a cutting plane approach to separate constraints (3.1) as done in [18].

Let $\tilde{X}^*$ be an arbitrary point in the space of the $\tilde{X}$ variables. The spectral decomposition of $\tilde{X}^*$ is used to decide if $\tilde{X}^*$ is in the PSD cone or not. Let the eigenvalues and corresponding orthonormal eigenvectors of $\tilde{X}^*$ be λ_k and v_k for $k = 1, 2, \ldots, n$, and assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and let $t \in \{0, \ldots, n\}$ such that $\lambda_t < 0 \leq \lambda_{t+1}$. If $t = 0$, then all the eigenvalues are non negative and $\tilde{X}^*$ is positive semidefinite. Otherwise, $v_k^T \tilde{X}^* v_k = v_k^T \lambda_k v_k = \lambda_k < 0$ for $k = 1, \ldots, t$. Hence, the valid cut

$$v_k^T \tilde{X} v_k \geq 0 \tag{3.2}$$

is violated by $\tilde{X}^*$. Cuts of the form (3.2) are called PSDCUTs in the remainder of the paper.

The above procedure has two major weaknesses: First, only one cut is obtained from eigenvector v_k for $k = 1, \ldots, t$, while computing the spectral

Sparsify$(v, \tilde{X}, pct_{NZ}, pct_{VIOL})$

1 $min_{VIOL} \leftarrow -v^T \tilde{X} v \cdot pct_{VIOL}$
2 $max_{NZ} \leftarrow \lfloor length[v] \cdot pct_{NZ} \rfloor$
3 $w \leftarrow v$
4 $perm \leftarrow$ random permutation of 1 **to** $length[w]$
5 **for** $j \leftarrow 1$ **to** $length[w]$
6   **do**
7         $z \leftarrow w$, $z[perm[j]] \leftarrow 0$
8   **if** $-z^T \tilde{X} z > min_{VIOL}$
9   **then** $w \leftarrow z$
10 **if** number of non-zeroes in $w < max_{NZ}$
11   **then** output w

Fig. 1. *Sparsification procedure for PSD cuts*

decomposition requires a non trivial investment in cpu time, and second, the cuts are usually very dense, i.e. almost all entries in vv^T are nonzero. Dense cuts are frowned upon when used in a cutting plane approach, as they might slow down considerably the reoptimization of the linear relaxation.

To address these weaknesses, we describe in the next section a heuristic to generate several sparser cuts from each of the vectors v_k for $k = 1, \ldots, t$.

3.3. Sparsification of PSD cuts. A simple idea to get sparse cuts is to start with vector $w = v_k$, for $k = 1, \ldots, t$, and iteratively set to zero some component of w, provided that $w^T \tilde{X}^* w$ remains sufficiently negative. If the entries are considered in random order, several cuts can be obtained from a single eigenvector v_k. For example, consider the *Sparsify* procedure in Figure 1, taking as parameters an initial vector v, a matrix $\tilde{X}$, and two numbers between 0 and 1, pct_{NZ} and pct_{VIOL}, that control the maximum percentage of nonzero entries in the final vector and the minimum violation requested for the corresponding cut, respectively. In the procedure, parameter $length[v]$ identifies the size of vector v.

It is possible to implement this procedure to run in $O(n^2)$ if $length[v] = n + 1$: Compute and update a vector m such that

$$m_j = \sum_{i=1}^{n+1} w_j w_i \tilde{X}_{ij} \text{ for } \; j = 1, \ldots, n+1 \; .$$

Its initial computation takes $O(n^2)$ and its update, after a single entry of w is set to 0, takes $O(n)$. The vector m can be used to compute the left hand side of the test in step 8 in constant time given the value of the violation d for the inequality generated by the current vector w: Setting the entry $\ell = perm[j]$ of w to zero reduces the violation by $2m_\ell - w_\ell^2 \tilde{X}_{\ell\ell}$ and thus the violation of the resulting vector is $(d - 2m_\ell + w_\ell^2 \tilde{X}_{\ell\ell})$.

A slight modification of the procedure is used to obtain several cuts from the same eigenvector: Change the loop condition in step 5 to consider the entries in $perm$ in cyclical order, from all possible starting points s in $\{1, 2 \ldots, length[w]\}$, with the additional condition that entry $s-1$ is not set to 0 when starting from s to guarantee that we do not generate always the same cut. From our experiments, this simple idea produces collections of sparse and well-diversified cuts. This is referred to as SPARSE1 in the remainder of the paper.

We also consider the following variant of the procedure given in Figure 1. Given a vector w, let $\tilde{X}_{[w]}$ be the principal minor of $\tilde{X}$ induced by the indices of the nonzero entries in w. Replace step 7 with

> 7. $z \leftarrow \bar{w}$ where $\bar{w}$ is an eigenvector corresponding to the most negative eigenvalue of a spectral decomposition of $\tilde{X}_{[w]}$, $z[perm[j]] \leftarrow 0$.

This is referred to as SPARSE2 in the remainder, and we call the cuts generated by SPARSE1 or SPARSE2 described above *Sparse PSD cuts*.

Once sparse PSD cuts are generated, for each vector w generated, we can also add all PSD cuts given by the eigenvectors corresponding to negative eigenvalues of a spectral decomposition of $\tilde{X}_{[w]}$. These cuts are valid and sparse. They are called *Minor PSD cuts* and denoted by MINOR in the following.

An experiment to determine good values for the parameters pct_{NZ} and pct_{VIOL} was performed on the 38 GLOBALLIB instances and 51 BoxQP instances described in Section 4.1. It is run by selecting two sets of three values in $[0, 1]$, $\{V_{LOW}, V_{MID}, V_{UP}\}$ for pct_{VIOL} and $\{N_{LOW}, N_{MID}, N_{UP}\}$ for pct_{NZ}. The nine possible combinations of these parameter values are used and the best of the nine (V_{best}, N_{best}) is selected. We then center and reduce the possible ranges around V_{best} and N_{best}, respectively, and repeat the operation. The procedure is stopped when the best candidate parameters are (V_{MID}, N_{MID}) and the size of the ranges satisfy $|V_{UP} - V_{LOW}| \leq 0.2$ and $|N_{UP} - N_{LOW}| \leq 0.1$.

In order to select the best value of the parameters, we compare the bounds obtained by both algorithms after $1, 2, 5, 10, 20$, and 30 seconds of computation. At each of these times, we count the number of times each algorithm outperforms the other by at least 1% and the winner is the algorithm with the largest number of wins over the 6 clocked times. It is worth noting that typically the majority of the comparisons end up as ties, implying that the results are not extremely sensitive to the selected values for the parameters.

For SPARSE1, the best parameter values are $pct_{VIOL} = 0.6$ and $pct_{NZ} = 0.2$. For SPARSE2, they are $pct_{VIOL} = 0.6$ and $pct_{NZ} = 0.4$. These values are used in all experiments using either SPARSE1 or SPARSE2 in the remainder of the paper.

4. Computational Results. In the implementation, we have used the Open Solver Interface (Osi-0.97.1) from COIN-OR [8] to create and modify the LPs and to interface with the LP solvers ILOG Cplex-11.1. To compute eigenvalues and eigenvectors, we use the `dsyevx` function provided by the LAPACK library version 3.1.1. We also include a cut management procedure to reduce the number of constraints in the outer approximation LP. This procedure, applied at the end of each iteration, removes the cuts that are not satisfied with equality by the optimal solution. Note however that the constraints from the **EXT+RLT** formulation are never removed, only constraints from added cutting planes are possibly removed.

The machine used for the tests is a 64 bit 2.66GHz AMD processor, 64GB of RAM memory, and Linux kernel 2.6.29. Tolerances on the accuracy of the primal and dual solutions of the LP solver and LAPACK calls are set to 10^{-8}.

The set of instances used for most experiments consists of 51 BoxQP instances with at most 50 variables and the 38 GLOBALLib instances as described in Section 4.1.

For an instance $\mathcal{I}$ and a given relaxation of it, we define the *gap closed* by the relaxation as

$$100 \cdot \frac{RLT - BND}{RLT - OPT}, \tag{4.1}$$

where BND and RLT are the optimal value for the given relaxation and the **EXT+RLT** relaxation respectively, and OPT is either the optimal value of $\mathcal{I}$ or the best known value for a feasible solution. The OPT values are taken from [14].

4.1. Instances. Tests are performed on a subset of instances from GLOBALLib [10] and on Box Constrained Quadratic Programs (BoxQPs) [24]. GLOBALLib contains 413 continuous global optimization problems of various sizes and types, such as BoxQPs, problems with complementarity constraints, and general QCQPs. Following [14], we select 160 instances from GLOBALLib having at most 50 variables and that can easily be formulated as **QCQP**. The conversion of a non-linear expression into a quadratic expression, when possible, is performed by adding new variables and constraints to the problem. Additionally, bounds on the variables are derived using linear programming techniques and these bound are included in the formulation. From these 160 instances in AMPL format, we substitute each bilinear term $x_i x_j$ by the new variable X_{ij} as described for the **LIFT** formulation. We build two collections of linearized instances in MPS format, one with the original precision on the coefficients and right hand side, and the second with 8-digit precision. In our experiments we used the latter.

As observed in [14], using together the **SDP** and **RLT** relaxations yields stronger bounds than those given by the **RLT** relaxation only for 38 out of 160 GLOBALLib instances. Hence, we focus on these 38 instances to test the effectiveness of the PSD Cuts and their sparse versions.

The BoxQP collection contains 90 instances with a number of variables ranging from 20 to 100. Due to time limit constraints and the number of experiments to run, we consider only instances with a number of variables between 20 to 50, for a total of 51 BoxQP problems.

The converted GLOBALLib and BoxQP instances are available in MPS format from [13].

4.2. Effectiveness of each class of cuts. We first compare the effectiveness of the various classes of cuts when used in combination with the standard PSDCUTs. For these tests, at most 1,000 cutting iterations are performed, at most 600 seconds are used, and operations are stopped if tailing off is detected. More precisely, let z_t be the optimal value of the linear relaxation at iteration t. The operations are halted if $t \geq 50$ and $z_t \geq (1 - 0.0001) \cdot z_{t-50}$. A cut purging procedure is used to remove cuts that are not tight at iteration t if the condition $z_t \geq (1 - 0.0001) \cdot z_{t-1}$ is satisfied. On average in each iteration the algorithm generates $\frac{n^2}{2}$ cuts, of which only $\frac{n}{2}$ are are kept by the cut purging procedure and the rest are discarded.

In order to compare two different cutting plane algorithms, we compare the closed gaps values first after a fixed number of iterations, and second at several given times, for all QCQP instances at avail. Comparisons at fixed iterations indicate the quality of the cuts, irrespective of the time used to generate them. Comparisons at given times are useful if only limited time is available for running the cutting plane algorithms and a good approximation of the PSD cone is sought. The closed gaps obtained at a given point are deemed different only if their difference is at least $g\%$ of the initial gap. We report comparisons for $g = 1$ and $g = 5$. Comparisons at one point is possible only if both algorithms reach that point. The number of problems for which this does not happen – because, at a given time, either result was not available or one of the two algorithms had already stopped, or because either algorithm had terminated in fewer iterations – is listed in the “inc.” (incomparable) columns in the tables. For the remaining problems, we report the percentage of problems for which one algorithm is better than the other and the percentage of problems were they are tied. Finally, we also report the average improvement in gap closed for the second algorithm over the first algorithm in the column labeled “impr.”.

Tests are first performed to decide which combination of the SPARSE1, SPARSE2 and MINOR cuts perform best on average. Based on Tables 1 and 2 below, we conclude that using MINOR is useful both in terms of iteration and time, and that the algorithm using PSDCUT+SPARSE2+MINOR (abbreviated *S2M* in the remainder) dominates the algorithm using PSDCUT+SPARSE1+MINOR (abbreviated *S1M*) both in terms of iteration and time. Table 1 gives the comparison between S1M and S2M at different iterations. S2M dominates clearly S1M in the very first iteration and after 200 iterations, while after the first few iterations S1M also manages to obtain good bounds. Table 2 gives the comparison between these two algorithms at different times. For comparisons with $g = 1$, S1M is better than S2M only in at most 2.25% of the problems, while the converse varies between roughly 50% (at early times) and 8% (for late times). For $g = 5$, S2M still dominates S1M in most cases.

TABLE 1
Comparison of S1M with S2M at several iterations.

	g = 1			g = 5				
Iteration	S1M	S2M	Tie	S1M	S2M	Tie	inc.	impr.
1	7.87	39.33	52.80	1.12	19.1	79.78	0.00	3.21
2	17.98	28.09	53.93	0.00	10.11	89.89	0.00	2.05
3	17.98	19.10	62.92	1.12	7.87	91.01	0.00	1.50
5	12.36	14.61	73.03	3.37	5.62	91.01	0.00	1.77
10	10.11	13.48	76.41	0.00	5.62	94.38	0.00	1.42
15	4.49	13.48	82.03	1.12	6.74	92.14	0.00	1.12
20	1.12	10.11	78.66	1.12	6.74	82.02	10.11	1.02
30	1.12	8.99	79.78	1.12	5.62	83.15	10.11	0.79
50	2.25	6.74	80.90	1.12	4.49	84.28	10.11	0.47
100	0.00	4.49	28.09	0.00	2.25	30.33	67.42	1.88
200	0.00	3.37	15.73	0.00	2.25	16.85	80.90	2.51
300	0.00	2.25	12.36	0.00	2.25	12.36	85.39	3.30
500	0.00	2.25	7.87	0.00	2.25	7.87	89.88	3.85
1000	0.00	2.25	3.37	0.00	2.25	3.37	94.38	7.43

Sparse cuts yield better bounds than using solely the standard PSD cuts. The observed improvement is around 3% and 5% respectively for SPARSE1 and SPARSE2. When we are using the MINOR cuts, this value gets to 6% and 8% respectively for each type of sparsification algorithm used. Table 3 compares PSDCUT (abbreviated by S) with S2M. The table shows that the sparse cuts generated by the sparsification procedures and minor PSD cuts yield better bounds than the standard cutting plane algorithm at fixed iterations. Comparisons performed at fixed times, on the other hand, show that considering the whole set of instances we do not get any improvement in the first 60 to 120 seconds of computation (see Table 4). Indeed S2M initially performs worse than the standard cutting plane algorithm, but after 60 to 120 seconds, it produces better bounds on average.

In Section 6 detailed computational results are given in Tables 5 and 6 where for each instance we compare the duality gap closed by S and S2M at several iterations and times. The initial duality gap is obtained as in (4.1) as $RLT - OPT$. We then let S2M run with no time limit until the

Table 2
Comparison of S1M with S2M at several times.

	g = 1			g = 5				
Time	S1M	S2M	Tie	S1M	S2M	Tie	inc.	impr.
0.5	3.37	52.81	12.36	0.00	43.82	24.72	31.46	2.77
1	0.00	51.68	14.61	0.00	40.45	25.84	33.71	4.35
2	0.00	47.19	15.73	0.00	39.33	23.59	37.08	5.89
3	1.12	44.94	14.61	0.00	34.83	25.84	39.33	5.11
5	1.12	43.82	15.73	0.00	38.20	22.47	39.33	6.07
10	1.12	41.58	16.85	0.00	24.72	34.83	40.45	4.97
15	2.25	37.08	16.85	1.12	21.35	33.71	43.82	3.64
20	1.12	35.96	16.85	1.12	17.98	34.83	46.07	3.49
30	1.12	28.09	22.48	1.12	16.86	33.71	48.31	2.99
60	1.12	20.23	28.09	0.00	12.36	37.08	50.56	2.62
120	0.00	15.73	32.58	0.00	10.11	38.20	51.69	1.73
180	0.00	13.49	32.58	0.00	5.62	40.45	53.93	1.19
300	0.00	11.24	31.46	0.00	3.37	39.33	57.30	0.92
600	0.00	7.86	24.72	0.00	0.00	32.58	67.42	0.72

Table 3
Comparison of S with S2M at several iterations.

	g = 1			g = 5				
Iteration	S	S2M	Tie	S	S2M	Tie	inc.	impr.
1	0.00	76.40	23.60	0.00	61.80	38.20	0.00	10.47
2	0.00	84.27	15.73	0.00	55.06	44.94	0.00	10.26
3	0.00	83.15	16.85	0.00	48.31	51.69	0.00	10.38
5	0.00	80.90	19.10	0.00	40.45	59.55	0.00	10.09
10	1.12	71.91	26.97	0.00	41.57	58.43	0.00	8.87
15	1.12	60.67	38.21	1.12	35.96	62.92	0.00	7.49
20	1.12	53.93	40.45	1.12	29.21	65.17	4.50	6.22
30	1.12	34.83	53.93	0.00	16.85	73.03	10.12	5.04
50	1.12	25.84	62.92	0.00	13.48	76.40	10.12	3.75
100	1.12	8.99	21.35	0.00	5.62	25.84	68.54	5.57
200	0.00	5.62	8.99	0.00	3.37	11.24	85.39	7.66
300	0.00	3.37	7.87	0.00	3.37	7.87	88.76	8.86
500	0.00	3.37	5.62	0.00	3.37	5.62	91.01	8.72
1000	0.00	2.25	0.00	0.00	2.25	0.00	97.75	26.00

value s obtained does not improve by at least 0.01% over ten consecutive iterations. This value s is an upper bound on the value of the **PSD+RLT** relaxation. The column "bound" in the tables gives the value of $RLT - s$ as a percentage of the gap $RLT - OPT$, i.e. an approximation of the percentage of the gap closed by the **PSD+RLT** relaxation. The columns

TABLE 4
Comparison of S with S2M at several times.

	g = 1			g = 5				
Time	S	S2M	Tie	S	S2M	Tie	inc.	impr.
0.5	41.57	17.98	5.62	41.57	17.98	5.62	34.83	-9.42
1	41.57	14.61	5.62	39.33	13.48	8.99	38.20	-8.66
2	42.70	10.11	6.74	29.21	8.99	21.35	40.45	-8.73
3	41.57	8.99	8.99	31.46	6.74	21.35	40.45	-8.78
5	35.96	7.87	15.72	33.71	5.62	20.22	40.45	-7.87
10	34.84	7.87	13.48	30.34	4.50	21.35	43.81	-5.95
15	37.07	5.62	11.24	22.47	2.25	29.21	46.07	-5.48
20	37.07	5.62	8.99	17.98	1.12	32.58	48.32	-4.99
30	30.34	5.62	15.72	11.24	1.12	39.32	48.32	-3.9
60	11.24	12.36	25.84	11.24	2.25	35.95	50.56	-1.15
120	8.99	12.36	24.72	2.25	2.25	41.57	53.93	0.48
180	2.25	14.61	29.21	0.00	4.50	41.57	53.93	1.09
300	0.00	15.73	26.97	0.00	6.74	35.96	57.30	1.60
600	0.00	14.61	13.48	0.00	5.62	22.47	71.91	2.73

labeled S and S2M in the tables give the gap closed by the corresponding algorithms at different iterations.

Note that although S2M relies on numerous spectral decomposition computations, most of its running time is spent in generating cuts and reoptimization of the LP. For example, on the BoxQP instances with a time limit of 300 seconds, the average percentage of CPU time spent for obtaining spectral decompositions is below 21 for instances of size 30, below 15 for instances of size 40 and below 7 for instances of size 50.

5. Conclusions. This paper studies linearizations of the PSD cone based on spectral decompositions. Sparsification of eigenvectors corresponding to negative eigenvalues is shown to produce useful cuts in practice, in particular when the minor cuts are used. The goal of capturing most of the strength of an PSD relaxation through linear inequalities is achieved, although tailing off occurs relatively quickly. As an illustration of typical behavior of a PSD solver and our linear outer-approximation scheme, consider the two instances, spar020-100-1 and spar030-060-1, with respectively 20 and 30 variables. We use the SDP solver `SeDuMi` and S2M, keeping track at each iteration of the bound achieved and the time spent. Figure 2 and Figure 3 compare the bounds obtained by the two solvers at a given time. For the small size instance spar020-100-1, we note that S2M converges to the bound value more than twenty times faster than `SeDuMi`. In the medium size instance spar030-060-1 we note that S2M closes a large gap in the first ten to twenty iterations, and then tailing off occurs. To compute the exact bound, `SeDuMi` requires 408 seconds while S2M requires 2,442 seconds to reach the same precision. Nevertheless, for our purposes, most of the benefits of the PSD constraints are captured in the early iterations.

Two additional improvements are possible. The first one is to use a cut separation procedure for the RLT inequalities, avoiding their inclusion in the initial LP and managing them as other cutting planes. This could potentially speed up the reoptimization of the LP. Another possibility is to use a mix of the S and S2M algorithms, using the former in the early iterations and then switching to the latter.

FIG. 2. *Instance spar020-100-1*

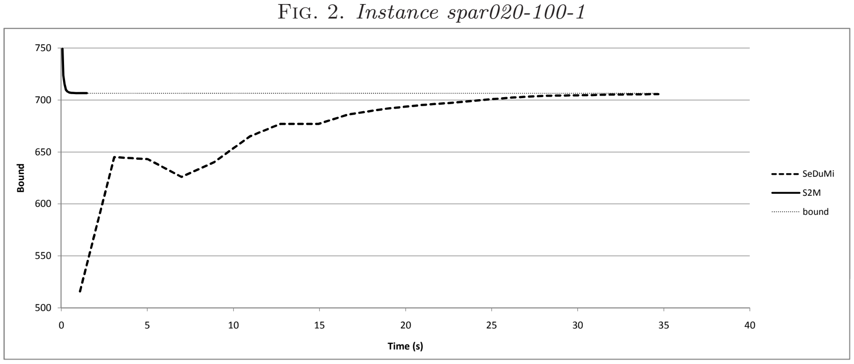
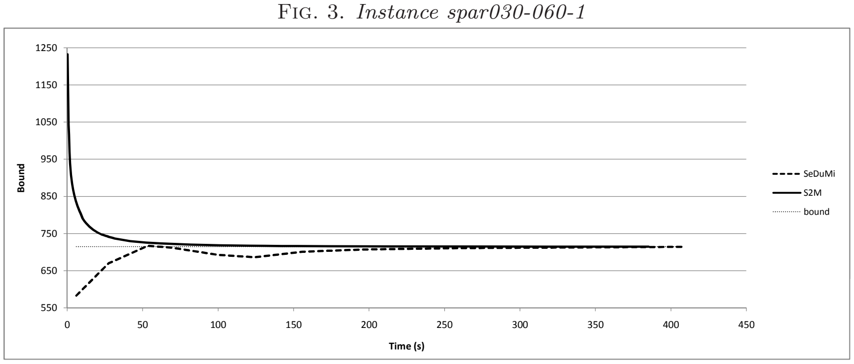


FIG. 3. *Instance spar030-060-1*



Acknowledgments. The authors warmly thank Anureet Saxena for the useful discussions that led to the results obtained in this work.

6. Appendix.

TABLE 5
Duality gap closed at several iterations for each instance.

Instance	$\|x\|$	$\|y\|$	bound	iter. 2 S	iter. 2 S2M	iter. 10 S	iter. 10 S2M	iter. 50 S	iter. 50 S2M
circle	3	0	45.79	0.00	0.00	10.97	41.31	45.77	45.79
dispatch	3	1	100.00	25.59	27.92	37.25	35.76	95.90	92.17
ex2_1_10	20	0	22.05	3.93	8.65	15.93	21.05	22.05	22.05
ex3_1_2	5	0	49.75	49.75	49.75	49.75	49.75	49.75	49.75
ex4_1_1	3	0	100.00	99.81	99.84	100.00	100.00	100.00	100.00
ex4_1_3	3	0	56.40	0.00	0.00	51.19	51.19	56.40	56.40
ex4_1_4	3	0	100.00	22.33	42.78	98.98	99.98	100.00	100.00
ex4_1_6	3	0	100.00	69.44	69.87	92.62	99.94	100.00	100.00
ex4_1_7	3	0	100.00	18.00	48.17	96.86	99.90	100.00	100.00
ex4_1_8	3	0	100.00	56.90	81.93	99.76	99.93	100.00	100.00
ex8_1_4	4	0	100.00	94.91	95.19	99.98	100.00	100.00	100.00
ex8_1_5	5	0	68.26	32.32	39.17	59.01	66.76	68.00	68.25
ex8_1_7	9	0	77.43	3.04	33.75	33.13	53.44	64.03	75.38
ex8_4_1	21	1	91.81	4.45	21.80	18.60	45.08	38.07	69.83
ex9_2_1	10	0	54.52	0.00	42.55	0.01	50.13	0.01	51.90
ex9_2_2	10	0	70.37	0.00	14.08	2.34	51.97	7.12	69.41
ex9_2_4	6	2	99.87	0.00	24.84	25.24	99.85	86.37	99.87
ex9_2_6	16	0	99.88	3.50	99.42	23.09	99.86	62.32	99.88
ex9_2_7	10	0	42.30	0.00	4.59	0.00	27.34	3.14	34.91
himmel11	5	4	49.75	49.75	49.75	49.75	49.75	49.75	49.75
hydro	12	19	52.06	0.00	20.87	21.95	29.03	26.04	31.39
mathopt1	4	0	100.00	95.76	100.00	99.96	100.00	100.00	100.00
mathopt2	3	0	100.00	99.84	99.93	100.00	100.00	100.00	100.00
meanvar	7	1	100.00	0.00	0.00	78.35	95.84	100.00	100.00
nemhaus	5	0	53.97	26.00	26.41	48.49	50.16	53.87	53.96
prob06	2	0	100.00	90.61	92.39	98.39	98.39	98.39	98.39
prob09	3	1	100.00	0.00	99.00	61.14	99.96	99.64	100.00
process	9	3	8.00	0.00	4.25	0.00	4.98	0.00	5.73

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Table 5 – Continued

Instance	$\|x\|$	$\|y\|$	bound	iter. 2 S	iter. 2 S2M	iter. 10 S	iter. 10 S2M	iter. 50 S	iter. 50 S2M
qp1	50	0	100.00	79.59	89.09	93.89	99.77	98.93	100.00
qp2	50	0	100.00	55.94	70.99	82.42	93.92	93.04	99.35
rbrock	3	0	100.00	97.48	100.00	99.96	100.00	100.00	100.00
st_e10	3	1	100.00	56.90	81.93	99.76	99.93	100.00	100.00
st_e18	2	0	100.00	0.00	0.00	98.72	98.72	100.00	100.00
st_e19	3	1	93.51	5.14	15.93	29.97	60.10	93.40	93.50
st_e25	4	0	87.55	55.80	55.80	87.02	87.01	87.23	87.23
st_e28	5	4	49.75	49.75	49.75	49.75	49.75	49.75	49.75
st_iqpbk1	8	0	97.99	71.99	76.69	97.20	97.95	97.99	97.99
st_iqpbk2	8	0	97.93	70.55	75.16	94.93	97.52	97.93	97.93
spar020-100-1	20	0	100.00	91.15	94.64	99.77	99.99	100.00	100.00
spar020-100-2	20	0	99.70	90.12	92.64	98.17	99.32	99.66	99.69
spar020-100-3	20	0	100.00	96.96	98.51	100.00	100.00	100.00	100.00
spar030-060-1	30	0	98.87	43.53	53.64	79.61	87.39	93.90	97.14
spar030-060-2	30	0	100.00	80.74	89.73	99.89	100.00	100.00	100.00
spar030-060-3	30	0	99.40	67.43	71.94	91.48	95.68	98.75	99.26
spar030-070-1	30	0	97.99	49.05	54.94	76.54	86.51	91.15	95.68
spar030-070-2	30	0	100.00	81.19	85.82	99.26	99.99	100.00	100.00
spar030-070-3	30	0	99.98	85.97	87.43	98.44	99.52	99.92	99.97
spar030-080-1	30	0	98.99	64.44	70.99	87.32	92.11	96.23	98.01
spar030-080-2	30	0	100.00	92.78	95.45	100.00	100.00	100.00	100.00
spar030-080-3	30	0	100.00	92.71	94.18	99.99	100.00	100.00	100.00
spar030-090-1	30	0	100.00	80.37	86.35	97.27	99.30	100.00	100.00
spar030-090-2	30	0	100.00	86.09	89.26	98.13	99.65	100.00	100.00
spar030-090-3	30	0	100.00	90.65	91.56	99.97	100.00	100.00	100.00
spar030-100-1	30	0	100.00	77.28	83.25	95.20	98.30	99.85	100.00
spar030-100-2	30	0	99.96	76.78	81.65	93.44	96.84	98.70	99.72
spar030-100-3	30	0	99.85	86.82	88.74	97.45	98.75	99.75	99.83
spar040-030-1	40	0	100.00	25.60	41.96	73.59	84.72	99.13	100.00
spar040-030-2	40	0	100.00	30.93	53.39	79.34	95.62	99.46	100.00
spar040-030-3	40	0	100.00	9.21	31.38	66.46	86.62	98.53	100.00
spar040-040-1	40	0	96.74	23.62	29.03	63.04	75.93	85.93	93.29
spar040-040-2	40	0	100.00	33.17	48.87	89.08	97.94	100.00	100.00
spar040-040-3	40	0	99.18	21.77	30.31	70.44	80.96	91.37	96.69
spar040-050-1	40	0	99.42	35.62	44.87	73.11	84.05	92.81	97.21

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Table 5 – Continued

Instance	$\lvert x\rvert$	$\lvert y\rvert$	bound	iter. 2 S	iter. 2 S2M	iter. 10 S	iter. 10 S2M	iter. 50 S	iter. 50 S2M
spar040-050-2	40	0	99.48	36.79	47.68	82.38	91.27	97.26	98.93
spar040-050-3	40	0	100.00	41.91	51.72	84.04	90.70	96.88	99.34
spar040-060-1	40	0	98.09	46.22	52.89	81.65	87.28	92.39	95.97
spar040-060-2	40	0	100.00	63.02	72.87	94.09	97.66	99.78	100.00
spar040-060-3	40	0	100.00	78.09	87.91	99.30	99.99	100.00	100.00
spar040-070-1	40	0	100.00	64.02	71.33	93.92	97.35	99.77	100.00
spar040-070-2	40	0	100.00	67.49	76.78	95.12	97.97	99.97	100.00
spar040-070-3	40	0	100.00	70.13	79.43	95.65	97.99	99.75	100.00
spar040-080-1	40	0	100.00	63.06	69.40	91.09	95.44	99.00	99.97
spar040-080-2	40	0	100.00	71.42	79.77	94.98	97.62	99.92	100.00
spar040-080-3	40	0	99.99	83.93	88.65	97.76	98.86	99.81	99.95
spar040-090-1	40	0	100.00	75.73	79.96	95.34	97.43	99.46	99.91
spar040-090-2	40	0	99.97	76.39	80.97	95.16	96.72	99.20	99.81
spar040-090-3	40	0	100.00	84.90	87.04	98.33	99.52	100.00	100.00
spar040-100-1	40	0	100.00	87.64	90.43	98.27	99.35	99.98	100.00
spar040-100-2	40	0	99.87	79.78	83.02	94.58	96.76	98.74	99.50
spar040-100-3	40	0	98.70	72.69	78.31	90.83	93.03	95.84	97.36
spar050-030-1	50	0	100.00	3.11	17.60	58.23	79.98	-	-
spar050-030-2	50	0	99.27	1.35	16.67	51.11	70.58	-	-
spar050-030-3	50	0	99.29	0.08	13.63	50.19	67.46	-	-
spar050-040-1	50	0	100.00	23.13	30.86	72.10	81.73	-	-
spar050-040-2	50	0	99.39	21.89	34.45	71.24	81.63	-	-
spar050-040-3	50	0	100.00	27.18	37.42	83.96	91.70	-	-
spar050-050-1	50	0	93.02	25.24	33.77	61.42	68.75	-	-
spar050-050-2	50	0	98.74	32.10	41.26	77.48	83.48	-	-
spar050-050-3	50	0	98.84	38.57	44.67	80.97	85.36	-	-
Average	-	-	-	48.75	59.00	75.53	84.39	85.85	89.60

Table 6

Duality gap closed at several times for each instance. (Instances solved in less than 1 second are not shown)

Instance	bound	1 s		60 s		180 s		300 s		600 s	
		S	S2M	S	S2M	S	S2M	S	S2M	S	S2M
ex4_1_4	100.00	-	100.00	-	-	-	-	-	-	-	-
ex8_1_4	100.00	-	100.00	-	-	-	-	-	-	-	-
ex8_1_7	77.43	77.43	77.37	-	-	-	-	-	-	-	-
ex8_4_1	91.81	28.14	36.24	61.60	90.43	-	-	-	-	-	-
ex9_2_2	70.37	-	70.35	-	-	-	-	-	-	-	-
ex9_2_6	99.88	96.28	-	-	-	-	-	-	-	-	-
hydro	52.06	26.43	31.46	-	-	-	-	-	-	-	-
mathopt2	100.00	-	100.00	-	-	-	-	-	-	-	-
process	8.00	-	7.66	-	-	-	-	-	-	-	-
qp1	100.00	79.99	80.28	98.22	99.52	99.73	99.96	99.92	99.98	99.99	100.00
qp2	100.00	55.82	55.27	91.74	95.56	95.86	98.69	97.41	99.66	98.80	100.00
spar020-100-1	100.00	100.00	100.00	-	-	-	-	-	-	-	-
spar020-100-2	99.70	99.67	99.61	-	-	-	-	-	-	-	-
spar020-100-3	100.00	-	100.00	-	-	-	-	-	-	-	-
spar030-060-1	98.87	69.98	58.72	96.53	97.61	98.45	98.70	98.68	98.82	-	-
spar030-060-2	100.00	96.52	91.05	-	-	-	-	-	-	-	-
spar030-060-3	99.40	82.99	76.15	99.27	99.32	99.38	99.39	99.39	99.40	99.40	99.40
spar030-070-1	97.99	69.81	60.36	94.50	96.38	97.29	97.73	97.70	97.91	-	97.98
spar030-070-2	100.00	96.05	87.93	-	-	-	-	-	-	-	-
spar030-070-3	99.98	96.26	90.42	99.98	99.98	99.98	99.98	-	99.98	-	-
spar030-080-1	98.99	83.36	74.42	97.80	98.11	98.74	98.88	98.89	98.96	-	98.99
spar030-080-2	100.00	99.83	96.70	-	-	-	-	-	-	-	-
spar030-080-3	100.00	99.88	95.87	-	-	-	-	-	-	-	-
spar030-090-1	100.00	92.86	87.69	-	-	-	-	-	-	-	-
spar030-090-2	100.00	93.80	88.46	-	100.00	-	-	-	-	-	-
spar030-090-3	100.00	97.78	91.35	-	-	-	-	-	-	-	-
spar030-100-1	100.00	91.04	84.34	100.00	100.00	-	-	-	-	-	-
spar030-100-2	99.96	90.21	83.14	99.56	99.75	99.91	99.95	99.95	99.96	-	99.96
spar030-100-3	99.85	94.26	89.55	99.84	99.84	99.85	99.85	99.85	99.85	99.85	99.85
spar040-030-1	100.00	28.97	40.51	89.30	84.19	99.06	99.98	99.98	100.00	-	100.00

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Table 6 – Continued

Instance	bound	1 s		60 s		180 s		300 s		600 s	
		S	S2M	S	S2M	S	S2M	S	S2M	S	S2M
spar040-030-2	100.00	31.97	48.01	94.01	96.39	99.58	99.98	99.99	100.00	-	-
spar040-030-3	100.00	9.20	27.59	81.66	85.43	97.25	99.86	99.81	100.00	100.00	-
spar040-040-1	96.74	19.38	22.90	70.35	75.45	80.73	88.63	85.34	92.29	90.79	94.74
spar040-040-2	100.00	24.51	29.87	98.63	98.60	100.00	100.00	-	-	-	-
spar040-040-3	99.18	20.88	21.31	78.28	79.31	86.02	91.22	89.52	95.04	94.07	97.71
spar040-050-1	99.42	28.96	21.27	80.18	84.01	88.70	94.62	92.75	96.71	96.53	98.32
spar040-050-2	99.48	29.52	16.91	91.33	91.42	97.01	97.97	98.26	98.87	-	99.31
spar040-050-3	100.00	28.67	19.81	90.03	90.72	95.68	97.51	97.49	99.08	98.92	99.89
spar040-060-1	98.09	37.16	17.10	86.26	87.13	90.18	93.50	92.25	95.32	95.05	96.84
spar040-060-2	100.00	39.57	22.83	98.09	98.22	99.90	99.96	100.00	100.00	100.00	-
spar040-060-3	100.00	52.41	30.57	100.00	99.99	-	-	-	-	-	-
spar040-070-1	100.00	50.01	21.79	97.74	97.78	99.80	99.87	99.97	99.99	100.00	100.00
spar040-070-2	100.00	47.57	25.19	98.81	98.46	99.99	99.99	100.00	100.00	-	-
spar040-070-3	100.00	47.22	21.95	98.96	98.70	99.88	99.92	99.98	100.00	100.00	100.00
spar040-080-1	100.00	51.66	28.00	95.13	95.38	98.29	99.05	99.09	99.74	99.77	99.99
spar040-080-2	100.00	52.24	25.94	98.71	98.31	99.95	99.97	100.00	100.00	-	-
spar040-080-3	99.99	56.05	26.98	99.54	99.25	99.89	99.88	99.94	99.95	99.97	99.98
spar040-090-1	100.00	59.71	28.17	98.10	97.86	99.43	99.61	99.70	99.86	99.90	99.99
spar040-090-2	99.97	59.14	29.82	97.83	97.70	99.34	99.58	99.68	99.81	99.86	99.93
spar040-090-3	100.00	63.07	34.62	99.94	99.85	100.00	100.00	-	-	-	-
spar040-100-1	100.00	69.47	28.24	99.66	99.47	99.99	99.99	100.00	100.00	-	-
spar040-100-2	99.87	65.27	26.07	97.34	96.87	98.60	98.98	99.02	99.39	99.44	99.69
spar040-100-3	98.70	61.40	29.61	93.01	93.17	94.91	96.02	95.81	97.00	96.84	97.77
spar050-030-1	100.00	0.37	3.63	54.46	37.52	70.10	73.34	76.87	84.75	86.23	96.33
spar050-030-2	99.27	0.08	2.79	44.68	38.62	59.58	64.94	67.79	74.98	77.02	86.58
spar050-030-3	99.29	0.00	2.75	44.32	32.31	57.13	59.07	62.54	68.99	71.18	82.86
spar050-040-1	100.00	3.76	1.77	69.97	56.87	77.15	78.30	80.31	84.30	84.90	91.79
spar050-040-2	99.39	2.08	2.84	68.64	58.47	77.72	77.61	81.54	83.63	86.40	90.94
spar050-040-3	100.00	1.76	2.31	79.44	65.71	89.73	87.74	92.67	93.00	95.99	97.69
spar050-050-1	93.02	4.91	1.84	60.64	53.28	65.52	66.42	66.81	70.38	68.45	74.76
spar050-050-2	98.74	6.18	3.39	76.56	68.33	82.34	82.21	84.94	86.52	-	91.34
spar050-050-3	98.84	6.12	2.82	79.38	69.23	84.95	83.23	86.99	86.98	89.77	91.57
Average	-	51.45	42.96	87.50	86.38	92.14	93.22	93.18	94.77	93.16	95.86

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