1971

A spectral sequence structure for homotopy theory of several subspaces

Richard N. Cain
Carnegie Mellon University

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FOR HOMOTOPY THEORY
OF SEVERAL SUBSPACES

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Report 71-45

September 1971
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Abstract

The structure that we shall describe here is a dual version of the Leray spectral sequence of a covering (Cf. [4], p. 212) but is not extensive enough to satisfy the precise definition of "spectral sequence." Yet, it does enable one to derive a functorial spectral sequence that relates the (generalized) homology sheaf of a space to the homology of the space. (Cf. [2] for an earlier attempt to obtain such a spectral sequence.)
Let X be a space with base point * e X. For some finite set c9 let I = (X |s c J) be a family of subspaces of X, each containing *. Assume that X is covariant, i.e. that an inclusion s c s' implies an inclusion X <= x t . We then have the notion of an alternating non-degenerate n-cochain $\xi$ (n>0) of X with coefficients in IT (q>2); such a $\xi$ is any family $\{\xi^i| i = (i_1, ..., i_n) e ^{n+1}\}$ such that (i) each $\xi^i e IT (X, )$, (ii) the condition $\xi^i* = 0$ holds if $i = (i_1, ..., i_n)$ contains a repetition, (iii) the condition $\xi^i* = (\text{sign } a)-\xi^a*$ holds if $a$ is any permutation map $\{0, 1, ..., n\} \rightarrow \{0, 1, ..., n\}$ and $i^* = (i^\sigma(0), i^\sigma(1), ..., i^\sigma(n))$. We have also the coboundary $\delta \xi$ of $\xi$, which is defined as the (n+1)-cochain such that

$$(\delta_\xi)i^* = \sum_{t=0}^n (-1)^t \xi^i_1(\xi^i(t)),$$

where $i^x e J^{n+2}$ and $\xi^i_1(\xi^i(t)) = (i_1, ..., i_t-1, i_t, i_{t+1}, ..., i_n+1)$. Denote the function group of all n-cochains as $C_n^\text{X, IT}$. In the linear space $\mathbb{R}^{n+2} = i e$ denote the standard basis somehow, say as $\{e_i| i e <3\}$, and for each subset $s c <\xi$ denote the
convex hull of the corresponding set (e. i e s) as As. Regard Ac9 as a simplicial complex with vertex set (e. i e c9).

Each pair (K,L) of subcomplexes of Ac? provides a function space

\[ 3S(K,L) = \{ \phi : K \rightarrow X \mid \phi L = *; \text{ for each } A_s \in K \} \]

We consider the groups \( T5(K,L) \) (cf.1).

**Proposition 1.** If (K,L,M) is a triple of subcomplexes of \( \Delta J \), then \( 3(K,M) \) is a fibre space over \( 3(L,M) \) with fibre \( T5(K,L) \).

**Proof.** The assertion is that the transformation

\[ 6 : 3 (K,M) \rightarrow 3 (L,M) \]

of each \( \phi \in 3(K,M) \) to its restriction \( \phi_L \) is a map with the homotopy lifting property, while the kernel of \( 9 \) is \( 3(K,L) \).

The latter is obvious, as is continuity of \( 9 \). Now, since \( \Phi \) is finite, \( 9 \) is the composite of the maps

\[ \theta_n : 3 (K_{n-1}UL,M) \rightarrow 3 (K_n UL,M) \]

Each has the homotopy lifting property, according to the following argument: Suppose \( Y \) is any space, together with a map

\[ H_0 : Y \rightarrow 3(K_{n}UL,M) \]

and a homotopy \( h_t : Y \rightarrow 5 (K_{n}^t U'LM) \) \((0 \leq t \leq 1)\) such that \( h_0 = 9 H_0 \). Regard these, respectively, as a map

\[ H^t : Y \times (K_{n}UL) \rightarrow X \] and homotopy \( h^t : Y \times (K_{n-1}UL) \rightarrow X \) \((0 \leq t \leq 1)\).

For each \( n \)-simplex \( A_s \) of \( K \) not in \( L \) there are \( H^t \) and \( h^t \), which take values in at most \( X \), so by the Homotopy Extension Property for \((YXAS, YX5AS)\) they extend to form a homotopy

\[ H^t : Y \times A_s - \rightarrow X \] \((0 \leq t \leq 1)\).
3.

Doing this for each \( n \)-simplex \( \triangle s \) in \( K \) and not in \( L \), and extending to agree with \( \{ h'_t \} \), we define a homotopy

\[
H'_t : Y \times (K \cup L) \to X \quad (0 \leq t \leq 1),
\]

which is the same as the required

\[
H_t : Y \to \mathcal{F}(K \cup L, M) \quad (0 \leq t \leq 1).
\]

So, \( \theta_n \), and therefore \( \theta \), has the homotopy lifting property. \( \square \)

**Corollary 1.1.** For each such triple \((K, L, M)\) there is an exact sequence of base-pointed sets

\[
\cdots \to \pi_{q+1} \mathcal{F}(L, M) \xrightarrow{\partial} \pi_q \mathcal{F}(K, L) \to \pi_q \mathcal{F}(K, M) \to \cdots \to \pi_0 \mathcal{F}(L, M).
\]

**Corollary 1.2.** There is essentially a spectral sequence

\[
[E^n_{r,q} : d^n_{r,q} : E^n_r \to E^{n+r}_{r,q}]
\]

with \( E^n_r,q \) defined for all \( n \in \mathbb{Z} \), all \( r \geq 1 \), all \( q \geq 2 \), and with \( d^n_{r,q} \) defined for all \( n \in \mathbb{Z} \), all \( r \geq 1 \), all \( q \geq 3 \). The formulas are:

(i) \( E^n_r,q \) = the homology subquotient of the half-exact sequence

\[
\pi_{q+1} \mathcal{F}(\Delta \delta_{n-1}, \Delta \delta_{n-r}) \to \pi_q \mathcal{F}(\Delta \delta_n, \Delta \delta_{n-1}) \to \pi_{q-1} \mathcal{F}(\Delta \delta_{n+r-1}, \Delta \delta_n);
\]

(ii) \( d^n_{r,q} \) is induced by the additive relation

\[
\pi_q \mathcal{F}(\Delta \delta_n, \Delta \delta_{n-1}) \xrightarrow{\partial} \pi_{q-1} \mathcal{F}(\Delta \delta_{n+r}, \Delta \delta_n)
\]

\( \cdots \to \pi_{q-1} \mathcal{F}(\Delta \delta_{n+r}, \Delta \delta_{n+r-1}). \)
Corollary 1.3. The spectral sequence converges to the set of homology subquotients $E^n_{\infty,q}$ of the half-exact sequences

$$\pi_{q+1}^q \mathfrak{F}(\Delta^j_{n-1}, \emptyset) \xrightarrow{\partial} \pi_q^q \mathfrak{F}(\Delta^j_n, \Delta^j_{n-1}) \xrightarrow{\partial} \pi_{q-1}^q \mathfrak{F}(\Delta^j, \Delta^j_n),$$

and these in turn are isomorphic to the subquotients

$$F^n_q \mathfrak{F}(\Delta^j, \emptyset)$$

$$F^{n+1}_q \mathfrak{F}(\Delta^j, \emptyset)$$

of $\mathfrak{F}(\mathcal{A}, 0)$, where

$$F^n_q \mathfrak{F}(\mathcal{A}, 0) = \text{Im}[\text{Tr}_q^3(\mathcal{A}, \mathcal{A}, n, \emptyset) \to \pi_q^q \mathfrak{F}(\Delta^j, \emptyset)].$$

The isomorphism in question is induced by the additive relation

$$\pi_q^q \mathfrak{F}(\Delta^j_n, \Delta^j_{n-1}) \xrightarrow{\partial} \pi_q^q \mathfrak{F}(\Delta^j, \Delta^j_n) \to \pi_q^q \mathfrak{F}(\Delta^j, \emptyset).$$

(The Corollaries 1.2-1.3 assume familiarity with the often-referred-to section in [3], p.333.)

Proposition 2. There is a natural isomorphism

$$C^n(X) \cong \mathfrak{F}(A^\wedge, J^n) \quad \text{such that} \quad d^n \cong \delta^n(q, j).$$

Proof. First, $E^n = IT J(A^\wedge, J^n)$, by definition. But

$$\mathfrak{F}(\Delta^j_n, A^\wedge, n, n) \cong x \text{ 'x' (As, dAs) by restriction of the members of $$

$$\mathfrak{F}(\Delta^j_n, A^\wedge, n, n) \to each \text{n-simplex. This means } E^n = \text{IT } \mathfrak{F}(\mathcal{A}, \Delta^j \mathcal{A}).$$

(naturally). Sticking momentarily to one $n$-simplex $\mathcal{A}$, and putting its indices into a sequence $i = (i, \ldots, i)$, we form next an

isomorphism
\( \phi^* : \pi_q \mathcal{F}(\Delta s, \partial \Delta s) \xrightarrow{\sim} \pi_{q+n}(X_s) \): Its formula entails the set
\[ s(t) = \{ i_t, i_{t+1}, \ldots, i_n \} \quad (0 \leq t \leq n) \]
via the diagram

\[
\begin{array}{c}
\pi_q \mathcal{F}(\Delta s, \partial \Delta s) \\
\uparrow \\
\pi_q \text{Top}_0 (\Delta s(0), \partial \Delta s(0); X_s) \\
\uparrow \\
\pi_{q+k} \text{Top}_0 (\Delta s(k), \partial \Delta s(k); X_s) \\
\uparrow \\
\pi_{q+k+1} \text{Top}_0 (\partial \Delta s(k), \partial \Delta s(k) - \text{int} \Delta s(k+1); X_s) \\
\downarrow \\
\pi_{q+k+1} \text{Top}_0 (\Delta s(k+1), \partial \Delta s(k+1); X_s) \\
\vdots \\
\pi_{q+n} \text{Top}_0 (\Delta s(n), \emptyset; X_s) \\
\uparrow \\
\pi_{q+n}(X_s).
\end{array}
\]

(Top\(_0\) stands for function space, with base point of \( X_s \) understood.)
The rest of the proof is exactly as presented in [1], Lemma 2.5. \( \square \)

Thus, the groups \( \pi_q \mathcal{F}(\Delta J, \emptyset) \) for \( q \geq 2 \) are related to the groups \( \pi_j(X_s) \) for \( j \geq 2 \) and \( s \subset J \); we have

\[
\frac{\pi^n q \mathcal{F}(\Delta J, \emptyset)}{\pi^{n+1} q \mathcal{F}(\Delta J, \emptyset)} \cong \text{a subquotient of } \prod \pi_{q+n}(X_s) \quad \text{dim} \Delta s = n
\]
(where this isomorphism entails both Corollary 1.3 and Proposition 2).

To approximate this subquotient, one forms subquotients \( E^n_{r,q} \) \((r \geq 2)\)
by reference to differentials, with the exception of the case \( q = 2 \).
(The \( d^n_{r,2} \)'s are missing, since the corresponding \( E^n_{r,1} \)'s are not
available to serve as their codomains).

Application to general homology. Before going any further we
note that there is a map \( b : \bigotimes\limits_{s \subseteq \Delta} X_s \rightarrow \mathcal{F}(\Delta^d, \emptyset) \) which sends each
\( x \in \bigotimes\limits_{s \subseteq \Delta} X_s \) into the constant map which has value \( x \). The map \( b \) is
obviously natural.

Now let \( h = \{ h_q, \partial_q \mid q \in \mathbb{Z} \} \) be any general homology theory
(Cf. [5]). If \( h^* \) is its polar general cohomology theory, there is
an \( S \)-module (syn:spectrum) \( \mathcal{W} = \{ W(k) \mid k = 0, 1, \ldots \} \) of base-pointed
spaces that classifies \( h^* \). For compact polyhedral pairs \((X, A)\) we
must then have \( h_q(X, A) \simeq \lim_{k \to \infty} \pi_{q+k}(W(k)^X, W(k)^A) \) for all \( q \in \mathbb{Z} \)
(where for any base-pointed space \( M \) one defines \( M^X \) to be
\((M \times X)/\{x \times X\})\).

Let \( X \) be any compact polyhedron without base point and
\( \{(X_i, A_i) \mid i \in \mathcal{I}\} \) a finite family of pairs of subpolyhedra of \( X \).
For each \( k = 0, 1, \ldots \) define \( Y(k-1) = \Omega(W(k)^X, W(k)^X) \),
\( Y_s(k-1) = \Omega(W(k)^X_s, W(k)^A_s) \) \((s \subseteq \Delta)\), where \((X_s, A_s) = (\bigcup_{i \in \mathcal{I}} X_i, \bigcup_{i \in \mathcal{I}} A_i)\).
It follows that \( Y = \{ Y(k) \} \) and \( Y_s = \{ Y_s(k) \} \) are \( S \)-modules, with
\( \lim_{k \to \infty} \pi_{q+k}(Y_s(k)) \) (henceforth denoted \( \pi_{q}(Y_s) \)) = \( h_q(X_s, A_s) \). The
function space \( \mathcal{F}(K,L) \) for \( (Y(k); \mathcal{J}; \{ Y_s(k) \mid s \subseteq \mathcal{J} \}) \) will be denoted \( \mathcal{F}(K,L)(k) \). The structure of \( S \)-module for \( Y \) carries over, so that there is an \( S \)-module \( \mathcal{F}(K,L) = \{ \mathcal{F}(K,L)(k) \mid k = 0,1,\ldots \} \). Returning now to \( b \), we see that we have a map

\[
b(k) : \bigcap_{s \subseteq \mathcal{J}} Y_s(k) \rightarrow \mathcal{F}(\Delta \mathcal{J}, \emptyset)(k)
\]

which commutes with the \( S \)-module structure, so that there is an \( S \)-module transformation \( b : \bigcap_{s \subseteq \mathcal{J}} Y_s \rightarrow \mathcal{F}(\Delta \mathcal{J}, \emptyset) \).

**Proposition 3.** \( \pi_q(b) \) is an isomorphism

\[
h_q(\bigcap_{i \in \mathcal{J}} Y_i, \bigcap_{i \in \mathcal{J}} A_i) = \pi_q(\bigcap_{s \subseteq \mathcal{J}} Y_s) \cong \pi_q(\mathcal{F}(\Delta \mathcal{J}, \emptyset)).
\]

**Proof.** Assume first that \( A_i = \{\ast\} \) for all \( i \in \mathcal{J} \), because the general case will follow from this case (see below). Write \( Z_s(k) \) for \( W(k) \wedge X_s \), to define an \( S \)-module \( \tilde{Z}_s \) such that

\[
\pi_q Z_s \cong \pi_q Y_s \cong h_q(X_s). \quad \text{We shall denote} \quad \bigcap_{i \in s} X_i \quad \text{as} \quad X^s.
\]

Consider the following commutative diagram for some 1-simplex \( \Delta ij \) of \( \Delta \mathcal{J} \):

\[
\cdots \rightarrow \pi_q(\Omega_{i,j}^{Z_{ij}}) \rightarrow \pi_q(Z_{ij}^{ij}) \rightarrow \pi_q(Z_i) \rightarrow \cdots
\]
The indicated isomorphisms are simply the renaming of a parameter, while $e$ is induced by inclusion and $?\sim$ is given by the same formula as $b$. But $e$ is actually the map $h_{\Delta^j} (X_i, X_i^{ij}) \rightarrow h_{\Delta^j} (X_i, X_j)$, which is an isomorphism by the general homology excision property for compact polyhedra. By the diagrammatic 5-lemma it follows that $?\sim$ is an isomorphism.

Now let $\Delta s$ be any simplex of $\Delta^j$ and $e_i$ some vertex not in $\Delta s$. Assume that $\pi_q(Z_s) \sim \pi_q(?\sim(\Delta s))$ holds. Form the commutative diagram

$$
\begin{array}{c}
\cdots \pi \Omega(Z_i, Z_i^{S_s}) \to \pi (Z_i^{S_s}) \to \pi (Z_i) \\
\downarrow e \quad \quad \downarrow ?\sim \quad \quad \downarrow \?\sim
\end{array}
$$

where $?^*(\Delta s)$ is the same as $?^*(\Delta s)$ but for the system $(X; J; \{X_i \cup X_s \mid s \in J\})$. (Thus, $?^*(\Delta s)$ are subspaces of $?^*(\Delta s).$)

The map $e$ has again the form of an excision isomorphism $h_{\Delta^j} (X_i, X_i^{S_s}) \rightarrow h_{\Delta^j} (X_i \cup X_s, X_s)$. To prove that $?\sim$ is an isomorphism, we apply the diagrammatic 5-lemma to the commutative diagram
where the isomorphism at the left is of the same type as the one at the right. Now that "" is seen to be an isomorphism, the earlier diagram similarly shows that "" is an isomorphism. This completes the inductive proof in case $A_i = \{\star\}$ for all $i \in \mathcal{J}$.

For the general case recall that

$$Y_s(k-1) = \Omega(Z_s^0(k), Z_s^0(k)),$$

where $Z_s^1(k) = \mathbb{W}(k) \wedge X_s$ and $Z_s^0(k) = \mathbb{W}(k) \wedge A_s$. Write

$$\mathcal{F}(K,L)(k) = \{\varphi : K \to \mathbb{Z}(k) | \varphi(L) = \star, \text{ and for each } \Delta s \subseteq K \varphi(\Delta s) \subseteq \mathbb{Z}_s \} \cup \{\varphi : K \to \mathbb{Z}(k) \mid \varphi(L) = \star, \text{ and for each } \Delta s \subseteq K \varphi(\Delta s) \subseteq \mathbb{Z}_s \},$$

($\forall = 0,1$). From the definition of $\mathcal{F}(K,L)$ it is evident that

$$\mathcal{F}(K,L)(k-1) \cong \Omega(\mathcal{F}_1(K,L)(k), \mathcal{F}_0(K,L)(k))$$

by rearranging the priority of variables. So, we have the commutative diagram

$$\cdots \pi_q \Omega(Z_s^0(k), Z_s^0(k)) \to \pi_q (Z_s^0(k), Z_s^0(k)) \to \pi_q (Z_s^0(k)) \cdots$$

and

$$\cdots \pi_q \mathcal{F}_1(\Delta^s) \to \pi_q \mathcal{F}(\Delta^s) \to \pi_q \mathcal{F}_0(\Delta^s) \cdots,$$

in which the left and right isomorphisms have just been established, so that the map $\pi_q(b)$ is an isomorphism by the diagrammatic 5-lemma. □
Corollary 3.1. For any general homology theory $h$ and finite family $\{X_i, A_i\}_{i \in I}$ of pairs of subpolyhedra of a compact polyhedron $X$, there is a corresponding spectral sequence

$$\{f^n_i; d^n_i: E^n_i \to E^{n+1}_j; j \in \mathbb{Z}, r \geq 1\}$$

such that $E^* \cong c^*(X; h)$. Under this isomorphism $d^n_i$ corresponds to the cochain coboundary operator, and $E^n$ is isomorphic to

$$F^n h_{g \mathcal{X} \mathcal{A}} \mathcal{A}$$

for some filtration

$$\ldots \longrightarrow F^{n+1} h(X, A) \longrightarrow \ldots$$

of $h(X, A)$ (containing both $h(X, A)$ and $\{0\}$), where

$$(X, A) = (\mathcal{X}, \mathcal{A}) \in I.$$  Moreover, the entire structure is functorial in $(X, c9; I)$.

Corollary 3.2. If $X$ is a compact polyhedron and $h$ a general homology theory there is a spectral sequence

$$\{E^n r; q d^n r; q: E^n \to E^{n+r}; r \geq 1, q \in \mathbb{Z}, r \geq 2\}$$

with $E^* \cong H^*(X; S)$, where $S$ stands for the induced sheaf of the general homology presheaf $U \to h(X, X-U)$. The group $E^n$ is isomorphic to

$$\mathcal{F}^n h_{q}(X)$$

for a suitable filtration

$$\mathcal{F}^{n+1} h_{q}(X) \to \cdots$$

of $h(X)$ (containing both $h(X)$ and $\{0\}$).
Bibliography


