RESOLUTION AND THE CONSISTENCY
OF ANALYSIS
by
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Abstract

It is shown by a purely syntactic argument how the completeness of resolution in type theory implies the consistency of type theory with axioms of extensionality, descriptions, and infinity. In this system the natural numbers are defined, and Peano's Postulates proved; indeed, classical analysis and much more can be formalized here. Nevertheless, Gödel's results show that the completeness of resolution in type theory cannot be proved in this system.

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Resolution and the Consistency of Analysis

Peter B. Andrews

§1. Introduction.

In [2] we formulated a system ft, called a Resolution system, for refuting finite sets of sentences of type theory, and proved that ft is complete in the (weak) sense that every set of sentences which can be refuted in the system 3 of type theory due to Church [5] can also be refuted in ft. The statement that ft is in this sense complete is a purely syntactic one concerning finite sequences of wffs. However, it is clear that there can be no purely syntactic proof of the completeness of ft, since the completeness of ft is closely related to Takeuti's conjecture [9] (since proved by Takahashi [8] and Pravitz [7]) concerning cut-elimination in type theory. As Takeuti pointed out in [9] and [10], cut-elimination in type theory implies the consistency of analysis. Indeed, Takeuti's conjecture implies the consistency of a formulation of type theory with an axiom of infinity; in such a system classical analysis and much more can be formalized. Hence, to avoid a conflict with Gödel's theorem, any proof of the completeness of resolution in type theory must involve arguments which cannot be formalized in type theory with an axiom of infinity. Indeed, the proof in [2] does involve a semantic argument.

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Nevertheless, it must be admitted that anyone who does not find the line of reasoning sketched above completely clear will have difficulty finding a unified and coherent exposition of the entire argument in the published literature. We propose to remedy this situation here.

We presuppose familiarity with §2 (The System 3) and Definitions 4.1 and 5.1 (The Resolution System ft) of [2], and follow the notation used there. In particular, Q stands for the contradictory sentence $\neg p \vee p$. To distinguish between formulations of JJ with different sets of parameters, we henceforth assume IT has no parameters, and denote by $\Gamma(\alpha_1, \ldots, \alpha_n)$ a formulation of the system with parameters $\alpha_1, \ldots, \alpha_n$. If $H$ is a set of sentences, $\vdash B$ shall mean that $B$ is derivable from some finite subset of $W$ in system $S$. The deduction theorem is proved in §5 of [5].

We shall incorporate into our argument Gandy's results in §3 of [6] with some minor modifications. We also wish to thank Professor Gandy for the basic idea (attributed by him to Turing) used below in showing the relative consistency of the axiom of descriptions. (This idea is mentioned briefly at the top of page 48 of [6].)

We shall have occasion to refer to the following wffs:
The set 8 of axioms of extensionality:

\[ E^0: \quad \forall \varphi \sigma \varphi \sigma . \quad \varphi \sigma \rightarrow \varphi \sigma . \]  

\[ E^{(\alpha \beta)}: \quad \forall f_{\alpha \beta} g_{\alpha \beta} . \quad \forall x_\beta [f_{\alpha \beta} x_\beta = g_{\alpha \beta} x_\beta] \Rightarrow f_{\alpha \beta} = g_{\alpha \beta} \]

The axiom of descriptions for type \( a \):

\[ D^a: \quad \forall f_{oa} . \quad \exists x f_o x \exists f_{oa} [f_{oa} x_{oa} \Rightarrow f_{oa} x_{oa}] \]

An axiom of infinity for type \( a \):

\[ J^a: \quad \exists r_{oa} \forall x \forall y \forall z a . \quad \exists w a \exists w a \exists w a \]  

\[ \neg r_{oa} x a a A \rightarrow r_{oa} x y V \rightarrow r_{oa} y z V \rightarrow r_{oa} x z V \rightarrow r_{oa} x z V \]

We let \( G \) denote the system obtained when one adds to \( \{ \alpha / \sigma \} \) the axioms \( E, D^1, \) and \( J^1 \). (Description operators and axioms for higher types are not needed, since Church showed [5] that they can be introduced by definition. This matter is also discussed in [3]).

In §4 we shall show how the natural numbers can be defined, and Peano's Postulates can be proved, in \( G \). The basic ideas here go back to Russell and Whitehead [11], of course, but our simple axiom of infinity is not that of Principia Mathematica, but is due to Bernays and Schönfinkel [4]. The natural numbers can be treated in a variety of ways in type theory (e.g., as in [5]), but we believe that the treatment given here has certain advantages of simplicity and naturalness. The simplicity of the axiom of
infinity \( J^1 \) is essential to our program in §3.

Once one has represented the natural numbers in \( G \), one can easily represent the primitive recursive functions. (With minor changes in type symbols, the details can be found in Chapter 3 of [1].) Syntactic statements about wffs can be represented in the usual way by wffs of \( G \) via the device of Gödel numbering. Thus there is a wff \begin{math} \text{Consis} \end{math} of \( G \) whose interpretation is that \( G \) is consistent, and by Gödel's theorem it is not the case that \( \text{h- Consis} \). Nevertheless, much of mathematics can be formalized in \( G \).

The completeness theorem for \( ft \) (Theorem 5.3 of [2]) is also a purely syntactic statement, and hence can be represented by a wff \( R \) of \( G \). After preparing the ground in §2 with some preliminary results, in §3 we shall show that by using the completeness of \( ft \) we can prove the consistency of \( G \). This argument will be purely syntactic, and could be formalized in \( G \), so \( \text{h-R^3D Consisi} \). Thus it is not the case that \( \text{h-R}^3 \) so any proof of the completeness of resolution in type theory must transcend the rather considerable means of proof available in \( G \). Of course such a proof can be formalized in transfinite type theory or in Zermelo set theory.

§2. Preliminary Definitions and Lemmas.

We first establish some preliminary results which will be useful in §3. The reader may wish to postpone the proofs of
this section and proceed rapidly to §3.

In presenting proofs of theorems of 3 (and extensions of 3), we shall make extensive use of proofs from hypotheses and the deduction theorem. Each line of a proof will have a number, which will appear at the left hand margin in parentheses. For the sake of brevity, this number will be used as an abbreviation for the wff which is asserted in that line. At the right hand margin we shall list the number(s) of the line(s) from which the given line is inferred (unless it is simply inferred from the preceding line). We use "hyp" to indicate that the wff is inferred with the aid of one or more of the hypotheses of the given line. Thus in

\begin{align*}
(.1) \quad & \top A \\
(.2) \quad & B \rightarrow B \quad \text{hyp} \\
(.3) \quad & B \rightarrow C \quad \text{.1,.2} \\
(.4) \quad & D \rightarrow C \quad \text{.1,hyp}
\end{align*}

the hypothesis \( B \) is introduced in line .2, and \( \top \) is inferred from \( B \) and the theorem \( A \) in line .3; \( C \) is also inferred from \( A \) and a different hypothesis \( D \) in line .4. However, if the wffs \( B \) and \( C \) are long, we may write this proof instead as follows:
A generally useful derived rule of inference is that if \( \forall \) \( \exists \) is a set of hypotheses such that \( \forall \exists \) and \( \forall \exists \), where \( \exists \) does not occur free in \( \forall \) or any wff of it, then \( \forall \exists \). We shall indicate applications of this rule in the following fashion:

1. \( \forall \exists \) \( \exists \)
2. \( \exists \) \( \exists \) \( \exists \)
3. \( \exists \) \( \exists \)
4. \( \exists \) \( \exists \)

If the wff \( \exists \) is long, we might write step (1) as follows:

1. \( \exists \) \( \exists \) \( \exists \)

We shall present only abstracts of proofs, omitting many steps and using familiar laws of quantification theory, equality, and \( \otimes \)-conversion quite freely. We shall usually omit type symbols on occurrences of variables after the first.

**DEFINITION.** For each wff \( \exists \) of \( \exists \) \( \exists \), let \( \# \) \( \exists \) be the wff of \( \exists \) which is the result of replacing the primitive...
constant t \( o_0 (o_{01}) \) everywhere by the wff

\[ \left[ A f_{0 (0_{01})} A z_i \cdot 3 x_{01} \cdot f_{0 (0_{11}) 01} A x_{01} z_i \right] \]

**Lemma 1.** \( E^0, E^0 V f f \# D^{01} \).

Proof: First note that \( # D^{01} \) conv \( V f / x . 3 x f x = \) \( f [A z_{01}, 3 x_{01}, fx A xz] \)

\((.1)\) \( 1 \ h \ 3 x f . \) \( \text{hyp} \)

\((.2)\) \( 1., 2 h f_{o (0t)} x_0 A V u_{0i} \cdot f u 3 u = x \) \( \text{choose} \ x (.1) \)

\((.3)\) \( 1., 2 h x_{ot} z_i = 3 x_{ot} . o_{o (0t)} x A xz \) \( \text{choose} \ x (.1) \)

\((.4)\) \( E^0.1, 2 I- V f A xz \cdot x_{ot} z_i = s x_{ot} . f_{o (0t)} x A xz \) \( E^0 \)

\((.5)\) \( E^0, E^{01}, 1, 2 h x_{qi} = [A z \ldots 3 x_{ot} . f_{o (0t)} x A xz] \) \( E^{01} \)

\((.6)\) \( E^0, E^{01}, 1, 2 h f_{o (0t)} [A z \ldots a x_{qi} . fx A xz] \) \( .2, 5 \)

\((.7)\) \( E^0, E^{01}, 1 H 6 \)

\((.8)\) \( E^0, E^{01} h \# D^{01} \)

**Lemma 2.** \( J^1 h J^{01} \)

Proof: We assume \( J^x \).

\((.1)\) \( 1 \ h Y x V x V z \ldots 8 w r x w A \ldots r x x A \ldots r x y V \ldots r y z V \ldots r x z \)

\( \text{choose} \ r \) \( ot t \)
Let \( K_{o\otimes}(o\otimes) \) be \( \{A_{o\otimes}/V_{o\otimes} - aV_{o\otimes}S_{A\otimes} \sim s_{t\otimes}S_{t\otimes}V \}
\]
\[
3V^{u_{o\otimes}t\otimes}A_{o\otimes}V^{x_{o\otimes}t\otimes}V^{o\otimes t\otimes}3s_{t\otimes}S_{t\otimes}t_{t\otimes}1
\]

We shall establish in lines (.11), (.16) and (.31) that \( K \) has the properties necessary to establish \( J^{o\otimes} \). To attack (.11) we consider two cases, (2) and (5).

(2) \( .2 \ h \sim 3s_{x_{o\otimes}t\otimes} \) hyp (case 1)
(3) \( .2 \ H \ K_{x_{o\otimes}t\otimes}T_{A\otimes}x_{t\otimes}t_{t\otimes} = t \) \( .2 \), def. of \( K \)
(4) \( .2 \ 1- 3w_{t\otimes}K_{x_{o\otimes}t\otimes}w \) \( .3 \)
(5) \( .5 \ (- 3s_{x_{o\otimes}t\otimes}S_{t\otimes}) \) hyp (case 2)
(6) \( .5, .6 h \ x_{o\otimes}S_{t\otimes} \) choose \( s \) (5)
(7) \( .1, .5, .6, .7 h \ r \sim s_{w_{t\otimes}} \) ch9ose \( w \) (.1)
(8) \( .1, .5, .6, .7 h \ K_{x_{o\otimes}t\otimes}T_{A\otimes}w_{.t\otimes} = t \) \( .6, .1 \), def. of \( K \)
(9) \( .1, .5, .6, .7 \ 1- 3w_{t\otimes}K_{x_{o\otimes}t\otimes}w \) \( .8 \)
(10) \( .1, .5 H .9 \) \( .9, .1, .5 \)

( ii) \( .1 \ V \ 3w \ K_{x_{t\otimes}}w \)

Next we attack (.16). The proof is by contradiction.

(12) \( .12 h K_{x_{t\otimes}}x \) hyp
(13) \( .12 F \ 3s_{x_{t\otimes}}S_{A\otimes}T_{x_{t\otimes}}. xt \ 3r_{x_{t\otimes}}S_{t\otimes} \) \( .12 \), def. of \( K \)
(14) \( .12 h 3s_{r\otimes}S_{s_{t\otimes}}S_{t\otimes} \) \( .13 \) (instantiate \( t \) with \( s \))
Finally we attack (.31).

In (.20) and (.21) we consider the two possibilities set forth in (.19).

Choose s (.21)

Choose q (-23)

In (.20) and (.21) we consider the two possibilities set forth in (.19).

Choose s (.21)
We next repeat Gandy's definitions in [6] with some minor modifications.

**DEFINITION.** By induction on $y$, we define wffs $\text{Mod}_{\gamma}^{\gamma}$ and $\text{M}_{\gamma\gamma}$ for each type symbol $y$.

$A \gamma \equiv B$ stands for $\text{Mod}_{\gamma\gamma}^{\gamma\gamma} A B$.

$\text{Mod}_{\gamma}^{\gamma}$ stands for $[A_{\gamma}, p p ]$ for $K = o, i$.

$\text{M}_{\gamma\gamma}$ stands for $[\neg p, s q ]$.

$\text{M}_{\gamma\gamma}$ stands for $[\neg w, A y . x = y ]$.

$\text{Mod}_{\gamma}^{\gamma}(a,0)$ stands for $[\lambda f_{\alpha}^\gamma \beta, \forall x_{\gamma}^\beta \forall y_{\beta} \gamma, \text{Mod}_{\gamma}^{\gamma} x_{\beta}^\gamma \wedge \text{Mod}_{\gamma}^{\gamma} y_{\beta} \gamma \wedge x_{\beta}^\gamma \equiv y_{\beta} \gamma, \text{Mod}_{\gamma}^{\gamma} (f_{\alpha}^\gamma x_{\beta}^\gamma) A \beta, f_{\alpha}^\gamma x_{\beta}^\gamma \equiv f_{\alpha}^\gamma y_{\beta} \gamma]$.

$\text{M}_{\gamma}(a, 3, 3)$ stands for $[A f^\gamma A g^\gamma, \forall x^\gamma, \text{Mod}^\gamma 3, f_{\alpha}^\gamma x_{\beta}^\gamma \equiv g_{\alpha} x_{\beta}^\gamma]$.

**Lemma 3.** $\vdash x_{\alpha}^\gamma = x_{\alpha}^\alpha, x_{\alpha}^\alpha = ^\alpha = !, z_{\alpha}^\gamma = z_{\alpha}^\alpha = !, z_{\alpha}^\gamma = z_{\alpha}^\alpha = !, z_{\alpha}^\gamma = y_{\alpha}^\gamma$.

Proof: by induction on $a$. 
DEFINITION. For each wff $A$ of $\mathcal{L}$, $A^T$ is the result of replacing $\forall x$ by $[\forall f \cdot \forall x \cdot \text{Mod } x \exists f x]$ everywhere in $A$.

LEMMA 4. If $A_0, \ldots, A_n$ and $B$ are sentences of $\mathcal{L}$ such that $A_0^T, \ldots, A_n^T \models B$, then $(A_0^T)^T, \ldots, (A_n^T)^T \models B^T$.

Proof: This is an immediate consequence of Theorem 3.26 of [6], since Gandy's full translation $Q$ of $jQ$ is $C$ when $C$ is a sentence. Our modifications of Gandy's definitions do not injure the proof.

LEMMA 5. $h \models \text{Mod}[M z 1]$. 

Proof: $\text{Mod}[M z 1]$ is equivalent to 

$$\forall x \forall y (\text{Mod } x \exists y \forall z \forall f \text{Mod } z \exists f z).$$

This is readily proved using the definition of Mod and Lemma 3.

LEMMA 6. $\models_{\mathcal{E}} (E')^T$ for each $E'$ in 6.

Proof: $(E')$ is equivalent to 

$$\forall p_0 [\text{Mod } p_0 \circ \forall q \circ \text{Mod } q \exists f_0 \circ \circ f_0 \circ q_0].$$
which is easily proved using the definition of $\text{Mod } f_{\infty}$.

\[ (E^{0}/3) T \quad \text{is equivalent to} \]

\[ Vf \quad \text{[Mod } f \Rightarrow Vg_{Q/3}. \text{Mod } g = Vx_{/3} \text{[Mod } x \Rightarrow Yh_{Q a}. \text{Mod } h 3. h[f] z > h.gx] } \]

\[ 3 \quad Vk_{o(a/3)} \cdot \text{Mod } k \Rightarrow k f => k g , \]

which we prove as follows:

\[(.1) \quad \text{It- [Mod } f A \text{Mod } g_{Q/3} \quad \text{hyp} \]

\[(.2) \quad 2(- Vx \quad [\text{Mod } x \Rightarrow Vh_{Q a}. \text{Mod } h \Rightarrow h[f] 3. h.gx \quad \text{hyp} \]

\[(.3) \quad 3. h \text{Mod } k_{Q(a/3)} \quad \text{hyp} \]

\[(.4) \quad h \text{Mod } o(a) - k_{o a} - [f_{a0}^p \quad \text{Lenunasa} \]

\[(.5) \quad 2. \text{Mod } x_{/3} V- [M_{o a a} \cdot f_{a0}^p \cdot x_{/3} ] [f_{a0}^p \cdot x_{/3}] \Rightarrow [M_{o a a} \cdot f_{a0}^p \cdot x_{/3}] . g_{a0}^p \cdot x_{/3} \]

\[ 2.4 \quad \text{(instantiate } h_{o a} \text{ with } M[f]) \]

\[(.6) \quad \vdash M_{o a a} [f_{a0}^p \cdot x_{/3} ] [f_{a0}^p \cdot x_{/3}] \quad \text{Lemma 3} \]

\[(.7) \quad 2. \text{Mod } x_{/3} H f_{Q/3} x_{/3} \quad g_{a0}^p / x_{/3} \quad 5.6 \]

\[ <^h \quad 2 \quad f_{a3}^c \quad g_{a0}^p / x_{/3} \quad \text{B} \]

\[ 7. \text{def. of } M_{o(cO)(a/S)} \]

\[(.9) \quad 1, 2, 3. h k_{o(a/3)} f_{a0}^p / k_{o(a/0)} g_{a0}^p \quad 3, \text{def. of } \text{Mod } k_{o(a/0)} \quad 1, 8 \]

\[(.10) \quad h(E^{a3}) T \quad 9 \]

**Lemma 7.** $h \sim \text{Mod } D_{\infty}$
Proof: Mod $z$ is equivalent to

$$\forall x \forall y \forall z \forall A \forall x = y \supset \left[ \text{Mod}(z_0 x) \land A \land x = y \right]$$

so $h \forall z \forall z \forall z \land \left[ \text{Mod}(z) \right]$.

Mod $r$ is equivalent to

$$\forall x \forall y \forall z \forall A \forall x = y \supset \left[ \text{Mod}(r x) \right]$$

which is easily proved.

**Lemma 8.** $J^1 H^1 (J^x)^T$.

Proof: $(J^t)^T$ is equivalent to

$$\forall x \forall y \forall z \forall A \forall x = y \supset \left[ \text{Mod}(r x) \right]$$

This is easily derived from $J^1$ with the aid of Lemma 1.

**Definition.** Let $\theta$ be the substitution

$$x^1 \rightarrow y^n, \text{ i.e. } \text{the simultaneous substitution of } A_A \text{ for all } A$$
free occurrences of \( x^i \) for \( 1 \leq i \leq n \), where \( x^1, \ldots, x^n \) are distinct variables and \( A^i \) has the same type as \( x^i \) for \( 1 \leq i \leq n \). If \( B \) is any wff, we let \( \theta * B \) denote \( \eta[\lambda x^1 \ldots \lambda x^n B] A^1 \ldots A^n \). If \( \theta \) is the null substitution (i.e., \( n = 0 \)), then \( \theta * B \) denotes \( \eta B \).

Note that if \( x_\alpha \) and \( y_\beta \) are distinct variables, 
\[
[[\lambda x_\alpha \lambda y_\beta B] A_\alpha C_\beta \text{ conv } [[\lambda y_\beta \lambda x_\alpha B] C_\beta A_\alpha ],
\]
so the definition above is unambiguous. Clearly, if there are no conflicts of bound variables, \( \theta * B \) is simply \( \eta \theta B \), the \( \eta \)-normal form of the result of applying the substitution \( \theta \) to \( B \).

From the definition it is evident that if \( B \) conv \( C \), then \( \theta * B = \theta * C \).

§3. The Consistency of \( G \).

**THEOREM.** \( G \) is consistent.

**Proof:** The proof is by contradiction, so we suppose \( G \) is inconsistent. Thus

(1) \( J^t, \varepsilon, D^t \vdash J^{(t_1(o_1))} \).

(2) \( J^{o_1}, \varepsilon, D^{o_1} \vdash J^{(o_1(o(o_1)))} \).
Proof: Replace the type symbol \(t\) by the type symbol \((ot)\) everywhere in the sequence of wffs which constitutes a proof of \(Q\) whose existence is asserted in step 1. By checking the axioms and rules of inference of \(JT\) one easily sees that a proof of \(Q\) satisfying the requirements of step 2 is obtained.

\[J^{ot}, \epsilon, \#D^{ot} \vdash U.\]

Proof: The replacement of \(A\) by \(#A\) everywhere in the proof whose existence is asserted in step 2 yields a proof satisfying step 3, possibly after the insertion of a few applications of the rule of alphabetic change of bound variables.

\[J^0 \parallel \epsilon \vdash \Box\] by Lemma 1.

\[J \parallel \epsilon \vdash \Box\] by Lemma 2.

\[(J^1)^T, \{(E^T \epsilon \epsilon \epsilon \epsilon \epsilon \epsilon) H_3D\]

Proof: by Lemma 4, since \(\vdash \Box \vdash \Box\).

\[(J^1)^T 1-jD\] by Lemma 6.

\[J^1 h_3 D\] by Lemma 8.

We next introduce parameters \(\overline{r}_{ot t}\) and \(\overline{g}_{11}\).
Let $\bar{x} = \{\bar{V}x \bar{r} x \in \bar{g} x, \bar{V}x \cdot \bar{r} \bar{x} x \in \bar{t}, \bar{V}x \bar{V}y \bar{V}z \cdot \bar{r} \bar{x} y \in \bar{t} \}.$

\[ \sim \bar{r} \bar{x} \bar{y} \bar{z} \sim \bar{r} \bar{y} \bar{z} \bar{v} \bar{r} \bar{x} \bar{z} \} . \]

(9) \quad \text{Ph 3}_{\bar{t}} \bar{g} > \cdot \cdot \cdot

Proof: \( J^1 H \bar{g}(x, g) \bar{g} \bar{y} \bar{z} \bar{r} \bar{x} \bar{y} \bar{z} \bar{v} \bar{r} \bar{x} \bar{z} \) by (8), and \( \bar{H} \bar{g}(x, g) \bar{g} \bar{y} \bar{z} \bar{r} \bar{x} \bar{y} \bar{z} \bar{v} \bar{r} \bar{x} \bar{z} \).

\textbf{do)} \quad \bar{I} - R \bar{D}

Proof: This follows from (9) by the completeness of resolution in type theory, i.e. Theorem 5.3 of [2]. The proof of this theorem is the one non-syntactic step in our present proof of the consistency of \( G. \)

(11) \quad \text{It is not the case that} \quad \bar{g} \bar{h} \bar{F} \bar{g} \bar{t} \bar{w} \bar{v} \bar{r} \bar{x} .

Proof: An 77-wff of the form \( \bar{r} \bar{A} \bar{B} \) will be called \textbf{positive} if the number of occurrences of \( \bar{g} \) in \( \bar{A} \) is strictly less than the number of occurrences of \( \bar{g} \) in \( \bar{B} \), and otherwise \textbf{negative}. An 77-wff of the form \( \sim \bar{r} \bar{A} \bar{B} \) will be called \textbf{positive} iff \( \bar{r} \bar{A} \bar{B} \) is negative, and negative iff \( \bar{r} \bar{A} \bar{B} \) is positive.

Let \( \bar{G} \) be the set of wffs \( \bar{G} \) having one of the following six forms:
(a) $\forall x^r x[g x]$

(b) $\forall x \sim r x x$

(c) $\forall x \forall y \forall z [\sim r x y v \sim r y z v r x z]$ where $x$, $y$, and $z$ are distinct variables.

(d) $\forall y \forall z_i [\sim r A y V \sim r y z V r A z_i]$ where $y_i$ and $z_i$ are distinct from one another and from the free variables of $A_i$.

(e) $\forall z_i [\sim r A z_i B V \sim r B z V r A z_i]$ where $z_i$ is distinct from the free variables of $A$ and of $B$.

(f) $G$ is a disjunction of wffs, each of the form $r j \backslash B$, or $\sim r A B$, at least one of which is positive.

Let $C$ be the set of wffs $C$ such that for each substitution $6$, $0 * c$ is in $3$.

We assert that if $p \vdash r C$, then $C \in C$. Clearly $p \in C$, so it suffices to show that $C$ is closed under the rules of inference of SI. For each rule of inference of $SI$ and any substitution $0$, we show that $0 * E \in 5$ for any wff $E$ derived from wff(s) of $C$ by that rule.

Suppose $M V A$ and $N V ^\sim A$ are in $C$, and $M V N$ is obtained from them by cut. Then $0 * [M V A]$ and $0 * [N V ^\sim A]$ must each have form (f). (For $0 * [N V ^\sim A] = [(0 * N) V (0 * A)]$;
even if $N$ is null, this cannot have any of the forms (a)-(e), so if $\theta \ast A$ must have the form $\overline{F B C}$. Then $\theta \ast [M \lor A] = [(\theta \ast M) \lor \theta \ast A]$.

If $\theta \ast A$ is negative, $\theta \ast M$ must contain a positive wff (so $M$ cannot be null), so $\theta \ast [M \lor N]$ does also. If $\theta \ast A$ is positive, then $\theta \ast [\neg A]$ is negative, so $\theta \ast N$ must contain a positive wff, so $\theta \ast [M \lor N]$ does also, and hence has form (f).

Suppose $D$ is in $\mathcal{C}$, and $[\lambda x \alpha D]_{B_\alpha}$ is obtained from $D$ by substitution. Let $\rho$ be the substitution $S_{B_\alpha}$, and let $\theta \ast \rho$ be the substitution which is the composition of $\theta$ with $\rho$ (i.e., $(\theta \ast \rho) \ast C = \theta \ast (\rho \ast C)$ for each wff $C$). Then $\theta \ast ([\lambda x \alpha D]_{B_\alpha}) = \theta \ast [\eta([\lambda x \alpha D]_{B_\alpha})] = \theta \ast (\rho \ast D) = (\theta \ast \rho) \ast D \in \mathcal{F}$ since $D \in \mathcal{C}$, so $[[\lambda x \alpha D]_{B_\alpha}] \in \mathcal{C}$.

Suppose $D \in \mathcal{C}$ and $E$ is derived from $D$ by universal instantiation. Thus $D$ has the form $M \lor \Pi_{\alpha}(\alpha \alpha)\alpha_\alpha$, where $M$ may be null. By considering the null substitution we see that $\eta D \in \mathcal{F}$, so $D$ has the form $\Pi_{\alpha}(\alpha \alpha)\alpha_\alpha$ and $E$ has the form $\alpha_\alpha X_1$. It is easily checked by examining forms (a)-(e) that if $H$ is any wff obtained from a wff of $\mathcal{F}$ by universal instantiation, then $(\theta \ast H) \in \mathcal{F}$. But $(\eta \alpha_\alpha X_1)X_1$ is obtained from $\eta D$ by universal instantiation, so $\theta \ast E = \theta \ast [(\eta \alpha_\alpha X_1)X_1]$ is in $\mathcal{F}$.

The verification that $\mathcal{C}$ is closed under the remaining rules of inference of $\mathcal{F}$ is trivial, so our assertion is proved.
Now [U is not in C, so it is not the case that \(|-^D:\)

(12) The contradiction between (10) and (11) proves our theorem.

§4. The Natural Numbers in G.

We shall define the natural numbers to be equivalence classes of sets of individuals having the same finite cardinality. We let \(o\) denote the type symbol \((o(ot))\). \(cr\) is the type of natural numbers.

DEFINITIONS.

0 stands for \([Ap \ Vx \sim p x]\).

S stands for \([An , \forall Ap . 3x . p x A \\&\& \o(ot) o1 \ I \ \o(ot) ot t \ n o(ot) [At . t ^ x A p o1 t] \].

N stands for \([An \ Vp . [p 0 A Vx . p x D p S x] 3 p n]\).

Vx A stands for \(Vx [ N x ZDA]\).

kx A stands for \(3x [ N x A A]\).

Thus zero is the collection of all sets with zero members, i.e., the collection containing just the empty set \([A x, o]\). S represents the successor function. If \(n, x x\) is a finite cardinal \((o(o1))\) (say 2), then a set \(p o1\) (say \([a, b, c]\)) is in \(Sn\) iff there
is an individual (say c) which is in \( p_0 \) and whose deletion from \( p_0 \) leaves a set \( \{a, b\} \) which is in n. \( N_0 \) represents the set of natural numbers, i.e., the intersection of all sets which contain 0 and are closed under S.

We now prove Peano's Postulates (Theorems 1, 2, 3, 4, and 7 below.) In this section \( \vdash B \) means B is a theorem of G.

1 \( \vdash N_0 \) 0 by the def. of \( N \).

2 \( \vdash \forall x_{\sigma}. N_0 x_{\sigma} \supset N_0. S_{\sigma} x_{\sigma} \)

**Proof:**

(1) \( N x_{\sigma}. 1 \vdash p_0 0 \wedge \forall x_{\sigma}. px \supset p. Sx \) hyp

(2) \( N x_{\sigma}. 1 \vdash p_0 x_{\sigma} \) .1,hyp,def. of \( N \)

(3) \( N x_{\sigma}. 1 \vdash p_0 . Sx_{\sigma} \) .1,.2

(4) \( N x_{\sigma} \vdash N. Sx_{\sigma} \) .3, def. of \( N \).

3 The Induction Theorem

\( \vdash \forall p_0. [p_0 0 \wedge \forall x_{\sigma}. p_0 x_{\sigma} \supset p_0 . S_{\sigma} x_{\sigma}] \supset \forall x_{\sigma} p_0 x_{\sigma} \)

**Proof:** Let \( p_0 \) be \( [\lambda t_{\sigma}. Nt \wedge p_0 t] \).
Proof by contradiction:

(.1) \( 1 \vdash p_{o} \lor \forall x_{p} \cdot Nx \lor px \lor p \cdot Sx \) \quad hyp

(.2) \( Ny_{p} \vdash [P o \land \forall x_{p} \cdot Px \lor P \cdot Sx] \lor Py_{p} \) \quad hyp, def. of N

(.3) \( 1 \vdash P o \) \quad def. of P, .1, Theorem 1

(.4) \( 1 \vdash \forall x_{p} \cdot Px \lor P \cdot Sx \) \quad def. of P, .1, Theorem 2

(.5) \( 1, Ny_{p} \vdash Py_{p} \) \quad .2, .3, .4

(.6) \( 1 \vdash \forall y_{p} py_{p} \) \quad .5, def. of \( \forall, p \)

\[ 4 \vdash \forall n_{p} \cdot S_{p} n_{p} \neq o_{p} \]

Our first step in proving Theorem 7 is to show that if we remove any element from a set of cardinality \( Sn \) we obtain a set of cardinality \( n \).
The proof is by induction on \( n \). First we treat the case \( n = 0 \).

\[ (.1) \quad \vdash \sim p_{o_1} w_t \land S \sigma n[\lambda t. t = w \lor p_{o_1} t] \supset n p_{o_1} \]

\[ (.2) \quad \vdash \exists x. 3 \quad \text{.1, def. of } S \]

\[ (.3) \quad \vdash [x = w \lor p_{o_1} x] \land 0[\lambda t. t \neq x \land t = w \lor pt] \quad \text{choose } x \text{ (.2)} \]

\[ (.4) \quad \vdash \sim w_t \neq x_t \land w = w \lor p_{o_1} w_t \quad \text{.3, def. of } 0 \]

\[ (.5) \quad \vdash w_t = x_t \quad .4 \]

\[ (.6) \quad \vdash \forall t. p_{o_1} t = .t \neq x_t \land t = w \lor pt \quad \text{.1, .5} \]

\[ (.7) \quad \vdash p_{o_1} = [\lambda t. t \neq x_t \land t = w \lor pt] \quad \text{.6, E}^0, E^{o_1} \]

\[ (.8) \quad \vdash 0 p_{o_1} \quad .3, .7 \]

\[ (.9) \quad \vdash \forall p_{o_1} \sim p_{o_1} w_t \land S \sigma n[\lambda t. t = w \lor pt] \supset n p \quad \text{.2, .8} \]

Next we treat the induction step

\[ (.10) \quad \vdash n n \sigma \land \forall p_{o_1} \sim p w_t \land S n[\lambda t. t = w \lor pt] \supset np \quad \text{(inductive) hyp} \]
From (.11) we must prove \([Sn]p\). We consider two cases in (.14) and (.17).

In case 2 we shall use the inductive hypothesis.
This completes the induction step. The theorem now follows from .9 and .24 by the Induction Theorem.

It will be observed that so far in this section we have not used the axiom of infinity $J$. We shall use it in proving the next theorem, which will also be used to prove Theorem 7.

$6hVn . np \Rightarrow \exists w t \sim p_{ot} w t$

(.1) $1 I- VxVyVz . 3w r xw a \sim r x x A . \sim r xy V \sim r yz V r x z$

choose r $(J)$

Let $P_{og}$ be $[An Vp . np z > 3z Vw . r . zw 3 \sim pw]$

We may informally interpret $r z w$ as meaning that $z$ is below $w$. Thus $P n$ means that if $p$ is in $n$, then there is an element $z$ which is below no member of $p$. We shall prove $\forall n P n$ by induction on $n$.

(.2) $0p_{ot} h - p_{ot} w t$ def. of 0

(.3) $\models P 0$ .2, def. of $P$
Next we treat the induction step.

(.4) \( .4 \vdash N_n \sigma \land P_n \)  
    (inductive) hyp

(.5) \( .5 \vdash S_n p_{o1} \)  
    hyp

(.6) \( .5 \vdash \exists x \_ .7 \)  
    .5, def. of \( S \)

(.7) \( .5, .7 \vdash p_{o1} x \_ \land n \sigma [\lambda t \_ . t \neq x \land pt] \)  
    choose \( x \) (.6)

(.8) \( .4, .5, .7 \vdash \exists z \_ .9 \)  
    .4, def. of \( P \), .7

(.9) \( .4, .5, .7, .9 \vdash \forall w \_ . r_{o1} z \_ w \supset w = x \_ \lor \sim p_{o1} w \)  
    choose \( z \) (.8)

Thus from the inductive hypothesis we see that there is an 
element \( z \) which is under nothing in \( p - \{x\} \). We must show 
that there is an element which is under nothing in \( p \). We 
consider two cases, (.10) and (.14).

(.10) \( .10 \vdash \sim r_{o1} z \_ x \_ \)  
    hyp (case 1)

(.11) \( .4, .5, .7, .9, .10 \vdash r_{o1} z \_ w \supset w \neq x \_ \)  
    .10

(.12) \( .4, .5, .7, .9, .10 \vdash \forall w \_ . r_{o1} z \_ w \supset \sim p_{o1} w \)  
    .9, .11

(.13) \( .4, .5, .7, .9, .10 \vdash \exists z \_ .12 \)  
    .12

Next we consider case 2, and show that \( x \) is under nothing 
in \( p \).
Having finished the inductive proof, we proceed to prove the main theorem.

\[ (.24) .24 \vdash \text{Nn}_\sigma \land n \text{Po}_t \quad \text{hyp} \]

\[ (.25) .1, .24 \vdash \exists z \forall w_t. \text{r}_{\text{ot}_t} zw \supset \sim \text{Po}_t w \quad .23, .24, \text{def. of P} \]

\[ (.26) .1 \vdash \forall z \exists w_t \text{r}_{\text{ot}_t} zw \quad .1 \]

\[ (.27) .1, .24 \vdash \exists w_t \sim \text{Po}_t w_t \quad .25, .26 \]
(28) \( 1 \vdash \forall \sigma \cdot n_{\sigma} \supset \exists w_t \sim p_{O_1} w_t \)

(29) \( \vdash .28 \)

7 \( \vdash \forall \sigma \forall m \cdot s_{\sigma} n_{\sigma} = s_{\sigma} m_{\sigma} \supset n_{\sigma} = m_{\sigma} \)

Proof:

(1) \( 1 \vdash n_{\sigma} \land n_{m} \land s_{\sigma} = s_{m} \) hyp

(2) \( 2 \vdash n_{\sigma} p_{O_1} \) hyp

(3) \( 1, 2, \vdash \exists w_t \sim p_{O_1} w_t \) \( .1, .2 \), Theorem 6

(4) \( 1, 2, 4 \vdash p_{O_1} t \)

(5) \( 1, 2, 4 \vdash p_{O_1} = [\lambda t_1 \cdot t \neq w_t \land t = w \lor p t] \) \( .4, E^0, E^0_t \)

(6) \( 1, 2, 4 \vdash n_{\sigma} [\lambda t_1 \cdot t \neq w_t \land t = w \lor p_{O_1} t] \) \( .2, .5 \)

(7) \( 1, 2, 4 \vdash s_{\sigma} [\lambda t_1 \cdot t = w_t \lor p_{O_1} t] \) \( .6, \text{def. of } S \)

(8) \( 1, 2, 4 \vdash s_{\sigma} [\lambda t_1 \cdot t = w_t \lor p_{O_1} t] \) \( .1, .7 \)

(9) \( 1, 2, 4 \vdash m_{\sigma} p_{O_1} \) \( .1, .4, .8, \text{Theorem 5} \)

(10) \( 1 \vdash n_{\sigma} p_{O_1} \supset m_{\sigma} p \) \( .3, .9 \)

(11) \( 1 \vdash m_{\sigma} p_{O_1} \supset n_{\sigma} p \) proof as for .10
\[ .1 \vdash \forall p: \sigma. n_\sigma p = m_\sigma p \] \[ .1 \vdash n_\sigma = m_\sigma \]

**Bibliography**


