Resolution and the consistency of analysis

P. B. (Peter Bruce) Andrews
Carnegie Mellon University

Follow this and additional works at: http://repository.cmu.edu/math
RESOLUTION AND THE CONSISTENCY OF ANALYSIS
by
Peter B. Andrews
Research Report 71-44

September, 1971
RESOLUTION AND THE CONSISTENCY OF ANALYSIS

by

Peter B. Andrews

Abstract

It is shown by a purely syntactic argument how the completeness of resolution in type theory implies the consistency of type theory with axioms of extensionality, descriptions, and infinity. In this system the natural numbers are defined, and Peano's Postulates proved; indeed, classical analysis and much more can be formalized here. Nevertheless, Gödel's results show that the completeness of resolution in type theory cannot be proved in this system.

Mathematical Offprint Service classification numbers:

02B15 Higher order predicate calculus
02G99 Methodology of deductive systems
68A40 Theorem proving

*This research was partially supported by NSF Grant GJ-28457X.*
§1. Introduction.

In [2] we formulated a system ft, called a Resolution system, for refuting finite sets of sentences of type theory, and proved that ft is complete in the (weak) sense that every set of sentences which can be refuted in the system 3 of type theory due to Church [5] can also be refuted in ft. The statement that ft is in this sense complete is a purely syntactic one concerning finite sequences of wffs. However, it is clear that there can be no purely syntactic proof of the completeness of ft, since the completeness of ft is closely related to Takeuti's conjecture [9] (since proved by Takahashi [8] and Pravitz [7]) concerning cut-elimination in type theory. As Takeuti pointed out in [9] and [10], cut-elimination in type theory implies the consistency of analysis. Indeed, Takeuti's conjecture implies the consistency of a formulation of type theory with an axiom of infinity; in such a system classical analysis and much more can be formalized. Hence, to avoid a conflict with Gödel's theorem, any proof of the completeness of resolution in type theory must involve arguments which cannot be formalized in type theory with an axiom of infinity. Indeed, the proof in [2] does involve a semantic argument.

*This research was partially supported by NSF Grant GJ-28457X.
Nevertheless, it must be admitted that anyone who does not find the line of reasoning sketched above completely clear will have difficulty finding a unified and coherent exposition of the entire argument in the published literature. We propose to remedy this situation here.

We presuppose familiarity with §2 (The System 3) and Definitions 4.1 and 5.1 (The Resolution System ft) of [2], and follow the notation used there. In particular, Q stands for the contradictory sentence $Vp p$. To distinguish between formulations of JJ with different sets of parameters, we henceforth assume IT has no parameters, and denote by $3(A_1 \ldots A_n)$ a formulation of the system with parameters $A_1 \ldots A_n$. If $H$ is a set of sentences $B$ shall mean that $B$ is derivable from some finite subset of $W$ in system S. The deduction theorem is proved in §5 of [5].

We shall incorporate into our argument Gandy's results in §3 of [6] with some minor modifications. We also wish to thank Professor Gandy for the basic idea (attributed by him to Turing) used below in showing the relative consistency of the axiom of descriptions. (This idea is mentioned briefly at the top of page 48 of [6].)

We shall have occasion to refer to the following wffs:
The set of axioms of extensionality;

\[ E^0 : \forall P_0 \forall q_0 . \ P_0 \ast q_0 \Rightarrow P_0 = q_0 . \]

\[ E(\alpha\beta) : \forall f_{\alpha\beta} \forall g_{\alpha\beta} . \ \forall x_\beta [f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta] \Rightarrow f_{\alpha\beta} = g_{\alpha\beta} \]

The axiom of descriptions for type \( a \):

\[ D^a : Vf_{oa} . \ \exists x f_{oa} x \ \exists f_{oa}^3 f_{oa}^2 f_{oa}^1 f_{oa}^0 \]

An axiom of infinity for type \( a \):

\[ J^a : \exists r_{ooa} Vx y Vz a . \ Sw r_{ooa} x w a \ A \]

\[ \sim r_{ooa} x a a A - r_{ooa} x y V - r_{ooa} x y z V - r_{ooa} x y z \ a \]

We let \( G \) denote the system obtained when one adds to \( \pi \) the axioms \( E, D^1, \) and \( J^1 \). (Description operators and axioms for higher types are not needed, since Church showed [5] that they can be introduced by definition. This matter is also discussed in [3]).

In §4 we shall show how the natural numbers can be defined, and Peano's Postulates can be proved, in \( G \). The basic ideas here go back to Russell and Whitehead [11], of course, but our simple axiom of infinity is not that of Principia Mathematica, but is due to Bernays and Schönfinkel [4]. The natural numbers can be treated in a variety of ways in type theory (e.g., as in [5]), but we believe that the treatment given here has certain advantages of simplicity and naturalness. The simplicity of the axiom of
infinity \( J^1 \) is essential to our program in \( \S 3 \).

Once one has represented the natural numbers in \( G \), one can easily represent the primitive recursive functions. (With minor changes in type symbols, the details can be found in Chapter 3 of [1].) Syntactic statements about \( \text{wffs} \) can be represented in the usual way by \( \text{wffs} \) of \( G \) via the device of Gödel numbering. Thus there is a \( \text{wff} \) \( \text{Consis} \) of \( G \) whose interpretation is that \( G \) is consistent, and by Gödel's theorem it is not the case that \( h \downarrow \text{Consis} \). Nevertheless, much of mathematics can be formalized in \( G \).

The completeness theorem for \( \text{ft} \) (Theorem 5.3 of [2]) is also a purely syntactic statement, and hence can be represented by a \( \text{wff} \) \( R \) of \( G \). After preparing the ground in \( \S 2 \) with some preliminary results, in \( \S 3 \) we shall show that by using the completeness of \( \text{ft} \) we can prove the consistency of \( G \). This argument will be purely syntactic, and could be formalized in \( G \), so \( h \uparrow \downarrow [R \text{ Consis}^3] \). Thus it is not the case that \( h \uparrow R^\downarrow \) so any proof of the completeness of resolution in type theory must transcend the rather considerable means of proof available in \( G \). Of course such a proof can be formalized in transfinite type theory or in Zermelo set theory.

\( \S 2 \). Preliminary Definitions and Lemmas.

We first establish some preliminary results which will be useful in \( \S 3 \). The reader may wish to postpone the proofs of
this section and proceed rapidly to §3.

In presenting proofs of theorems of 3 (and extensions of 3), we shall make extensive use of proofs from hypotheses and the deduction theorem. Each line of a proof will have a number, which will appear at the left hand margin in parentheses. For the sake of brevity, this number will be used as an abbreviation for the wff which is asserted in that line. At the right hand margin we shall list the number(s) of the line(s) from which the given line is inferred (unless it is simply inferred from the preceding line). We use "hyp" to indicate that the wff is inferred with the aid of one or more of the hypotheses of the given line. Thus in

\begin{align*}
(\cdot1) & \quad \text{\footnotesize \textbf{TA}} \\
(\cdot2) & \quad B \quad \text{\footnotesize \textbf{hyp}} \\
(\cdot3) & \quad B \quad \text{\footnotesize \textbf{h} C} \\
(\cdot4) & \quad D \quad \text{\footnotesize \textbf{h} C} \\
\end{align*}

the hypothesis \( B \) is introduced in line \( \cdot2 \), and \( \xi \) is inferred from \( B \) and the theorem \( A \) in line \( \cdot3 \); \( \zeta \) is also inferred from \( A \) and a different hypothesis \( D \) in line \( \cdot4 \). However, if the wffs \( B \) and \( C \) are long, we may write this proof instead as follows:
A generally useful derived rule of inference is that if \( ft \) is a set of hypotheses such that \( \vdash 3x A \) and \( \# A \vdash JBj \), where \( x \) does not occur free in \( \varepsilon \) or any wff of it, then \( M \vdash B \).

We shall indicate applications of this rule in the following fashion:

\[
\begin{align*}
(1) & \quad \h A \\
(2) & \quad .2 \h P \quad \text{hyp} \\
(3) & \quad .2 \h C \\
(4) & \quad D \h.3 \quad .1, .2 \quad -1, \text{hyp} \\
\end{align*}
\]

If the wff \( A \) is long, we might write step (.17) as follows:

\[
(17) \quad Jt \vdash 3x.20
\]

We shall present only abstracts of proofs, omitting many steps and using familiar laws of quantification theory, equality, and \( \text{7\vdash}\)-conversion quite freely. We shall usually omit type symbols on occurrences of variables after the first.

DEFINITION. For each wff \( A \) of \( 3T(t, , , ,) \), let \( A \)

be the wff of \( U \) which is the result of replacing the primitive
constant $t_{01(0(01))^*}$ everywhere by the wff

$$[Af_{o(0(01))}^A z_i, 3x_{01}, f_{0(01)^x}^A x_{01} z_i].$$

**Lemma 1.** $E^o, E^o V_f \# D^{o_1}$.

Proof: First note that $# D^{o_1} \text{ conv } V_f f x = f [Az, 3x_{o_1} x A x z]$

\[(.1) \h 1 h \text{ 3-x f, } x o t o(ot) o i \quad \text{hyp} \]

\[(.2) \h 2 h f o(ot)^x x o i A V_u o i, f u 3 u = x \quad \text{choose } x(\cdot 1) \]

\[(.3) \h \text{choose } x(\cdot 2) \]

\[(.4) E^o_h 1, 2 \text{ I- } V f. x o t z = 3 x o t, o(ot)^x x A x z \quad .3, E^o \]

\[(.5) E^o, E^o, 2 h x Q_i = [Az, 3x_{o_1}, f_{o(ot)^x} A x z] \quad .4, E^{o_1} \]

\[(.6) E^o, E^o, 1, 2 h f_{Q(ot)} [Az, a x Q_i, f x A x z] \quad .2, .5 \]

\[(.7) E^o, E^o, 1 H \quad .6 \]

\[(.8) E^o, E^o h \# D^{o_1} \quad .7 \]

**Lemma 2.** $J^1 h J^{o_1}$

Proof: We assume $J^x$.

\[(.1) \h 1 h Y x V y V z, 8 w r x w A \sim r x x A, \sim r x y V \sim r y z V r x z \quad \text{choose } r \quad \text{ott o t t} \]
Let \( K_{o(ot)(ot)} \) be \[ A^n o / o t - a V o t S A, \sim 3s_t o t s_t V \]

\[ ^3V \ u_{o t t} ^A V V v_{o t t} ^3 r o t t s_t t \]

We shall establish in lines (.11), (.16) and (.31) that \( K \) has the properties necessary to establish \( J^{oi} \). To attack (.11) we consider two cases, (.2) and (.5).

(.2) .2 \( h \sim 3s_{t, o t t} \) hyp (case 1)

(.3) .2 \( H K_{x_{o t}, T A t t} \) \( t = t \) \[ .2, \text{def. of } K \]

(.4) .2 \( l - 3s_{t, o t w} \) \[ .3 \]

(.5) .5 \( (- 3s_{t, o t s_t} \) hyp (case 2)

(.6) .5, .6 \( h x_{o t s_t} \)

(.7) .1, .5, .6, .7 \( h r - s w \)

(.8) .1, .5, .6, .7 \( h K_{x_t} [A_t t, w = t] \)

(.9) .1, .5, .6, .7 \sim 3w_{t, Kx_{t, W}} \[ .8 \]

(.10) .1, .5 \( H .9 \)

(ii) .1 \( V 3w_{t} K_{x_t} w \)

Next we attack (.16). The proof is by contradiction.

(.12) .12 \( h K_{x_t} x \) hyp

(.13) .12 \( l - 3s_{t, x, s} A \) \( V_t, x t 3r_s t \)

(.14) .12 \( h 3s_{t, r, s} s s \)

(.13) \( \text{instantiate } t \) with \( s \).
Finally we attack (.31).

In (.20) and (.21) we consider the two possibilities set forth in (.19).

choose s (.21)
We next repeat Gandy's definitions in [6] with some minor modifications.

**DEFINITION.** By induction on \( y \), we define wffs \( \text{Mod}_{\gamma} \) and \( M_{\gamma} \) for each type symbol \( y \).

\[ A_B \] stands for \( M_{\gamma} A_B \).

\[ \sim y \] stands for \( [\text{Ax}, x] \) for \( K = o,i \).

\[ M \] stands for \( [\text{?Ay}, x = y] \).

\[ \text{Mod}_{\gamma} \] stands for \( [\lambda f_{\alpha \beta} \cdot \forall x_{\beta} \cdot \forall y_{\beta} : \text{Mod}_{\alpha \beta} x_{\beta} \land \text{Mod}_{\alpha \beta} y_{\beta} \land x_{\beta} \equiv y_{\beta} \rightarrow \] \( \text{Mod}_{\alpha \beta} [f_{\alpha \beta} x_{\beta}] \land f_{\alpha \beta} x_{\beta} \equiv f_{\alpha \beta} y_{\beta}^\gamma] \).

\[ \text{Mod}_{\gamma} \] stands for \( [\lambda f_{\alpha \beta} \cdot \forall x_{\beta} \cdot \forall y_{\beta} : \text{Mod}_{\alpha \beta} x_{\beta} \land \text{Mod}_{\alpha \beta} y_{\beta} \land x_{\beta} \equiv y_{\beta} \rightarrow \] \( \text{Mod}_{\alpha \beta} [f_{\alpha \beta} x_{\beta}] \land f_{\alpha \beta} x_{\beta} \equiv f_{\alpha \beta} y_{\beta}^\gamma] \).

**LEMMA 3.** \( \Gamma \vdash x_{\alpha} \equiv a \land x_{\alpha} = a = x_{\alpha} = \gamma a = z_{a} = x_{a} \equiv a = z_{a} = a = M \).

Proof: by induction on \( a \).
DEFINITION. For each wff \( A \) of \( \mathcal{L} \), \( A^T \) is the result of replacing \( II, x \) by \([Af \cdot Vx \cdot \text{Mod } x \cdot 3 f x\] every where in \( A \).

LEMMA 4. If \( A, \ldots, A^n \) and \( B \) are sentences of \( \mathcal{L} \) such that \( A^i, \ldots, A^n \models B \), then \( (A^i)^T, \ldots, (A^n)^T \models B^T \).

Proof: This is an immediate consequence of Theorem 3.26 of [6], since Gandy's full translation \( J \) of \( jQ \) is \( C \) when \( C \) is a sentence. Our modifications of Gandy's definitions do not injure the proof.

LEMMA 5. \( h \models \text{Mod}[M z_1, 1] \).

Proof: \( \text{Mod}[M z_1, 1] \) is equivalent to

\[
\forall x \forall y [\text{Mod } x \cdot 3 y \cdot a(x, y)] \cdot \forall x \forall y \text{Mod } x \cdot 3 y \cdot a(x, y)
\]

This is readily proved using the definition of \( \text{Mod}_0 \) and Lemma 3.

LEMMA 6. \( \llbracket (E') \rrbracket^T \) for each \( E' \) in 6.

Proof: \( (E') \) is equivalent to

\[
\forall p_0 [\text{Mod } p_0 \cdot 3 q_0 \cdot \text{Mod } q_0 \cdot 3 p_0 = q_0 \Rightarrow Vf_0, \text{ Mod } f_0 \cdot p_0 \cdot f_0 \cdot q_0 \cdot]
\]
which is easily proved using the definition of Mod $f_{oo}$.

$$(E^{O/3}T \text{ i s e q u i v a l e n t to}$$

$${\cal V}f_{\alpha\beta}, \text{ Mod } f \Rightarrow {\cal V}g_{Q/3}, \text{ Mod } g = {\cal V}x_{3/3} \text{ [Mod } x \Rightarrow {\cal Y}h_{Q.a}, \text{ Mod } h 3, \text{ h}[fx] z > h.gx]$$

$${\cal V}k_{o(a/3)} \Rightarrow \text{ Mod } k \Rightarrow \text{ kf } \Rightarrow kg],$$

which we prove as follows:

(.1) $\Rightarrow \text{ Mod } f \Rightarrow \text{ A Mod } g_{Q/3}$ hyp

(.2) $\Rightarrow 2(- \text{ Vx } [\text{ Mod } x \Rightarrow \text{ Vh }_{0}, \text{ Mod } h \Rightarrow \text{ h}[fx] 3 h.gx)$ hyp

(.3) $\Rightarrow 3 h \text{ Mod } k_{Q(a/3)}$ hyp

(.4) $h \text{ Mod } o(oa) \Rightarrow k_{o(a/3)} \Rightarrow f_{a/3}$ p LenunaaS

(.5) $\Rightarrow 2, \text{ Mod } x_{3/3} \Rightarrow [M_{oaa} \Rightarrow f_{a/3}, f_{a/3}] \Rightarrow [M_{oaa} \Rightarrow f_{a/3}, g_{a/3} \Rightarrow \Rightarrow 2, 4$ (instantiate $h_{o/3}$ with $M[fx]$)

(.6) $\Rightarrow M_{oaa} \Rightarrow [f_{a/3}, f_{a/3} \Rightarrow \Rightarrow \text{ Lemma 3}$

(.7) $\Rightarrow 2, \text{ Mod } x_{3/3} \Rightarrow f_{Q/3} x_{3} \Rightarrow \Rightarrow g_{a/3} x_{0}$

(.8) $\Rightarrow 2 \Rightarrow f_{a/3} M_{a/3} \Rightarrow B_{1}$

(.9) $\Rightarrow 1, 2, 3 h \Rightarrow k_{o(a/3)} f_{a/3} - k_{o(a/0)} g_{a/3}$

(.10) $h \Rightarrow (E^{a/3})^{T}$

LEMMA 7. $h \Rightarrow \text{ Mod } r$
Proof: \( \text{Mod } z \) is equivalent to

\[
\forall x \forall y \left( \text{Mod } x \land \text{Mod } y \land x^2 = y^2 \Rightarrow \text{Mod} [z_0 x] \land A \land x \land s \land z \land y \right)
\]

so \( h Vz \text{ Mod } z \).

\( \text{Mod } r \) is equivalent to

\[
\forall x \forall y \left( \text{Mod } x \land \text{Mod } y \land x = y \Rightarrow \text{Mod} [r] \land A \land \text{fr } x \right)
\]

which is easily proved.

**LEMMA 8.** \( J^1 H \land (J^x)^T \).

Proof: \( (J^x)^T \) is equivalent to

\[
3r \left[ \text{Mod } r \land A \forall x . \text{Mod } x \Rightarrow \forall y . \text{Mod } y \Rightarrow \right.
\]

\[
Vz . \text{Mod } z \land 3 . 3w \left[ \text{Mod } w \land A \right] . 1 \land A
\]

This is easily derived from \( J^1 \) with the aid of Lemma 1.

**DEFINITION.** Let \( S \) be the substitution

\[
x^1 \land x^n \land S \land \text{, i.e. the simultaneous substitution of } A^x \text{ for all } A
\]

\[
\ldots A
\]
free occurrences of $x^i$ for $1 \leq i \leq n$, where $x^1, \ldots, x^n$ are distinct variables and $A^i$ has the same type as $x^i$ for $1 \leq i \leq n$. If $B$ is any wff, we let $\theta \ast B$ denote $\eta[[\lambda x^1 \ldots \lambda x^n B] A^1 \ldots A^n]$. If $\theta$ is the null substitution (i.e., $n = 0$), then $\theta \ast B$ denotes $\eta B$.

Note that if $x_\alpha$ and $y_\beta$ are distinct variables, $[[\lambda x_\alpha y_\beta B] A_\alpha C_\beta] \text{ conv } [[\lambda y_\beta x_\alpha B] C_\beta A_\alpha]$, so the definition above is unambiguous. Clearly, if there are no conflicts of bound variables, $\theta \ast B$ is simply $\eta \theta B$, the $\eta$-normal form of the result of applying the substitution $\theta$ to $B$.

From the definition it is evident that if $B \text{ conv } C$, then $\theta \ast B = \theta \ast C$.

§3. The Consistency of $G$.

**THEOREM.** $G$ is consistent.

**Proof:** The proof is by contradiction, so we suppose $G$ is inconsistent. Thus

(1) $J^t, \xi, D^t \vdash J(t_1(0_1)) \square$.

(2) $J^o_1, \xi, D^o_1 \vdash J(o_1(o(0_1))) \square$. 

Proof: Replace the type symbol $t$ by the type symbol $(ot)$ everywhere in the sequence of wffs which constitutes a proof of $Q$ whose existence is asserted in step 1. By checking the axioms and rules of inference of $JT$ one easily sees that a proof of $Q$ satisfying the requirements of step 2 is obtained.

(3) $J^{o_1}, e, \# D^{o_1} \vdash \overline{U}$.

Proof: The replacement of $A$ by $\# A$ everywhere in the proof whose existence is asserted in step 2 yields a proof satisfying step 3, possibly after the insertion of a few applications of the rule of alphabetic change of bound variables.

(4) $J^{o_1} \vdash \overline{e} \overline{g}$ by Lemma 1.

(5) $J \vdash \overline{e} \overline{g}$ by Lemma 2.

(6) $(J^1)^T, \{(e^{\gamma})^T \mid e^{\gamma} \in \Xi\} \vdash H_3 D$

Proof: by Lemma 4, since $\vdash \overline{f}^{T} \Rightarrow \overline{e}$.

(7) $(J^1)^T 1-jD$ by Lemma 6.

(8) $J^1 h_s D$ by Lemma 8.

We next introduce parameters $\overline{f}$ and $\overline{g}$.
Let $\mathcal{S} = \{ \forall x \, \overline{r} \circ x \, [\overline{g} \cdot x \}, \, \forall x \, \overline{r} \circ x \, x \, , \, \forall x \, \forall y \, \forall z \, . \, \overline{t} \cdot x \, y \, v \cdot \overline{r} \circ y \cdot z \, V \overline{r} \circ x \, z \}$. 

(9) \[ \text{Ph } 3 \{ \text{7}_{\text{ott}}, \overline{g} \} > \cdot \cdot \cdot \]

Proof: \[ J^j H \overline{\circ} (x, \overline{g}) \overline{\circ} j \text{ by } (8), \text{ and } \wedge H \overline{r} \cdot \circ x, j^j. \]

do: $^1 - R \, D$

Proof: This follows from (9) by the completeness of resolution in type theory, i.e. Theorem 5.3 of [2]. The proof of this theorem is the one non-syntactic step in our present proof of the consistency of $G$.

(11) It is not the case that $\mathcal{S} \, h, F]$. 

Proof: An 77-wff of the form $\overline{r} \circ A \, B$ will be called positive if the number of occurrences of $\overline{q}$ in $\circ A$ is strictly less than the number of occurrences of $\overline{q}$ in $B$, and otherwise negative. An 77-wff of the form $\overline{r} \circ A \, B$ will be called positive iff $\overline{r} \circ A \, B$ is negative, and negative iff $\overline{r} \circ A \, B$ is positive.

Let $\#$ be the set of wffs $\mathcal{G}$ having one of the following six forms:
(a) $\forall x \, \overline{r} \, x[\overline{g} \, x]$

(b) $\forall x \, \neg \, \overline{r} \, x \, x$

(c) $\forall x \forall y \forall z \, [\overline{\overline{r} \, x \, y} \land \neg \, \overline{\overline{r} \, y \, z} \land \overline{\overline{r} \, x \, z}]$ where $x$, $y$, and $z$ are distinct variables.

(d) $\forall y \forall z \, \overline{\overline{\neg \, A \, y} \land \overline{\overline{r} \, y \, z} \land \overline{\overline{r} \, A \, z}}$ where $y$ and $z$ are distinct from one another and from the free variables of $A$.

(e) $\forall z \, \overline{\overline{\neg \, A \, \neg \, B} \land \overline{\overline{r} \, B \, z} \land \overline{\overline{r} \, A \, z}}$ where $z$ is distinct from the free variables of $A$ and of $B$.

(f) $G$ is a disjunction of wffs, each of the form $\overline{\overline{r} \, j} \setminus B$, or $\neg \, \overline{\overline{r} \, A \, B}$, at least one of which is positive.

Let $C$ be the set of wffs $C$ such that for each substitution $\theta$, $\theta \star C$ is in $\mathcal{F}$.

We assert that if $p \vdash_r C$, then $C \in C$. Clearly $p \in C$, so it suffices to show that $C$ is closed under the rules of inference of $SI$. For each rule of inference of $SI$ and any substitution $\theta$, we show that $\theta \star E \in C$ for any wff $E$ derived from wff(s) of $C$ by that rule.

Suppose $\overline{M \, V \, A}$ and $\overline{N \, V \, \overline{\overline{A}}}$ are in $C$, and $\overline{M \, V \, \overline{N \, V \, \overline{A}}}$ is obtained from them by cut. Then $\theta \star \overline{[M \, V \, A]}$ and $\theta \star \overline{[N \, V \, A]}$ must each have form $(f)$. (For $\theta \star \overline{[N \, V \, A]} = [(0 \star N) \land \neg \, (0 \star A)J]$;
even if \( N \) is null, this cannot have any of the forms (a)-(e), so \( \theta * A \) must have the form \( \overline{\text{r}} \overline{\text{p}} \overline{\text{c}}_1 \). \( \theta * [M \lor A] = ([\theta * M] \lor \theta * A) \); if \( \theta * A \) is negative, \( \theta * M \) must contain a positive wff (so \( M \) cannot be null), so \( \theta * [M \lor N] \) does also. If \( \theta * A \) is positive, then \( \theta * [\neg A] \) is negative, so \( \theta * N \) must contain a positive wff, so \( \theta * [M \lor N] \) does also, and hence has form (f).

Suppose \( D \) is in \( C \), and \( [\lambda x_\alpha D]B_\alpha \) is obtained from \( D \) by substitution. Let \( \rho \) be the substitution \( s_{B_\alpha} \), and let \( \theta \circ \rho \) be the substitution which is the composition of \( \theta \) with \( \rho \) (i.e., \( (\theta \circ \rho) \circ C = \theta \circ (\rho \circ C) \) for each wff \( C \)). Then
\[
\theta \circ [\lambda x_\alpha D]^B_\alpha = \theta \circ \eta[\lambda x_\alpha D]^B_\alpha = \theta \circ (\rho \circ D) = (\theta \circ \rho) \circ D \in \mathcal{F}
\]
since \( D \in C \), so \( [[\lambda x_\alpha D]^B_\alpha] \in C \).

Suppose \( D \in C \) and \( E \) is derived from \( D \) by universal instantiation. Thus \( D \) has the form \( M \lor \Pi_0(\alpha)A_{\alpha} \), where \( M \) may be null. By considering the null substitution we see that \( \eta D \in \mathcal{F} \), so \( D \) has the form \( \Pi_0(\alpha)A_{\alpha} \) and \( E \) has the form \( A_{\alpha_0}x_1 \). It is easily checked by examining forms (a)-(e) that if \( H \) is any wff obtained from a wff of \( \mathcal{F} \) by universal instantiation, then \( (\theta \circ H) \in \mathcal{F} \). But \( (\eta A_{\alpha_0})x_1 \) is obtained from \( \eta D \) by universal instantiation, so \( \theta \circ E = \theta \circ [(\eta A_{\alpha_0})x_1] \) is in \( \mathcal{F} \).

The verification that \( C \) is closed under the remaining rules of inference of \( \theta \) is trivial, so our assertion is proved.
Now \[U\] is not in \(C\), so it is not the case that \(^\uparrow\neg\neg^\downarrow\). 

(12) The contradiction between (10) and (11) proves our theorem.

§4. The Natural Numbers in \(G\).

We shall define the natural numbers to be equivalence classes of sets of individuals having the same finite cardinality. We let \(o\) denote the type symbol \((o(ot))\). \(cr\) is the type of natural numbers.

**DEFINITIONS.**

\(0\) stands for \([Ap \ Vx \ \neg p x]\).

\(S\) stands for \([An, \ pAp \ 3x \ p x A\)

\(N\) stands for \([An \ Vp \ [p 0 A Vx \ p x D p S x] 3 p n\].

\(Vx A\) stands for \(Vx [N x ZDA]\).

\(kx A\) stands for \(3x [N x A A]\).

Thus zero is the collection of all sets with zero members, i.e., the collection containing just the empty set \([Ax \emptyset]\). \(S\) represents the successor function. If \(n, \ xx\) is a finite cardinal \((o(oi))\) (say 2), then a set \(pot\) (say \([a, b, c]\)) is in \(Sn\) iff there
is an individual (say c) which is in \( P_{O_1} \) and whose deletion from \( P_{O_1} \) leaves a set \( \{a, b\} \) which is in \( n \). \( N_{O_0} \) represents the set of natural numbers, i.e., the intersection of all sets which contain \( 0 \) and are closed under \( S \).

We now prove Peano's Postulates (Theorems 1, 2, 3, 4, and 7 below.) In this section \( \vdash B \) means \( B \) is a theorem of \( G \).

1 \( \vdash N_{O_0} 0 \) by the def. of \( N \).

2 \( \vdash \forall x. \ N_{O_0} x \supset N_{O_0} \cdot S_{O_0} x \)

Proof:

(1) \( \exists x. \ N_{O_0} x \vdash p_{O_0} 0 \land \forall x. \ p x \supset p. S x \) hyp

(2) \( \exists x. \ p_{O_0} x \) hyp, def. of \( N \)

(3) \( \exists x. \ p_{O_0} S x \) hyp, def. of \( N \)

(4) \( \forall x. \ N. S x \)

3 The Induction Theorem

\( \vdash \forall x. [p_{O_0} x \land \forall x. p_{O_0} x \supset p_{O_0} \cdot S_{O_0} x] \supset \forall x. p_{O_0} x \)

Proof: Let \( P_{O_0} \) be \( [\lambda t. N t \land p_{O_0} t] \).
Proof by contradiction:

(1) \(1 \vdash S_n = 0\)  

(2) \(\vdash o_\sigma[\lambda x_i \Box]\)  

(3) \(1 \vdash S_n[\lambda x_i \Box]\)  

(4) \(1 \vdash \exists x_i \Box\)  

(5) \(\vdash S_n \neq 0\)  

(6) \(\vdash \forall n_\sigma. S_n \neq 0\)  

Our first step in proving Theorem 7 is to show that if we remove any element from a set of cardinality \(S_n\) we obtain a set of cardinality \(n\).
5 ⊨ \forall_{\sigma} \forall p_{O_1} . \sim p_{O_1} w_1 \wedge S_{\sigma} n_{\sigma} [\lambda t_1 . \ t = w_1 \vee p_{O_1} t_1] \supset n_{\sigma} p_{O_1}.

The proof is by induction on \( n \). First we treat the case \( n = 0 \).

(1) \( 1 \vdash \sim p_{O_1} w_1 \wedge S O [\lambda t_1 . \ t = w \vee pt] \) \hspace{1cm} \text{hyp}

(2) \( 1 \vdash \exists x_1 . 3 \)

(3) \( 1, 3 \vdash [x_1 = w_1 \vee p_{O_1} x_1] \wedge O[\lambda t_1 . \ t \neq x_1 \wedge \ t = w \vee pt] \)

choose \( x \) (2)

(4) \( 1, 3 \vdash \sim w_1 \neq x_1 \wedge w = w \vee p_{O_1} w_1 \)

(5) \( 1, 3 \vdash w_1 = x_1 \)

(6) \( 1, 3 \vdash \forall t_1 . p_{O_1} t \equiv t \neq x_1 \wedge \ t = w_1 \vee pt \)

(7) \( 1, 3 \vdash p_{O_1} = [\lambda t_1 . \ t \neq x_1 \wedge \ t = w_1 \vee pt] \)

(8) \( 1, 3 \vdash 0 p_{O_1} \)

(9) \( \vdash \forall_{\sigma} \cdot \sim p_{O_1} w_1 \wedge S_{\sigma} n_{\sigma} [\lambda t_1 . \ t = w \vee pt] \supset 0 p \)

Next we treat the induction step

(10) \( 10 \vdash N_{\sigma} \wedge \forall_{\sigma} p_{O_1} . \sim p w_1 \wedge S_{\sigma} n_{\sigma} [\lambda t_1 . \ t = w \vee pt] \supset n p \)

(inductive) hyp
From (.11) we must prove \([Sn]p\). We consider two cases in (.14) and (.17).

In case 2 we shall use the inductive hypothesis.
This completes the induction step. The theorem now follows from .9 and .24 by the Induction Theorem.

It will be observed that so far in this section we have not used the axiom of infinity $\text{J}$. We shall use it in proving the next theorem, which will also be used to prove Theorem 7.

\[ \forall h V \forall n . \forall p \quad \exists \forall w . \forall t \quad \text{choose } r \quad (\text{J}^i) \]

Let \( P \) be \( \forall n . \forall p . \exists z > 3z . Vw . \forall r . \forall z w . \forall r x z \). We may informally interpret \( \forall r z w \) as meaning that \( z \) is below \( w \). Thus \( Pn \) means that if \( p \) is in \( n \), then there is an element \( z \) which is below no member of \( p \). We shall prove \( \forall n . Pn \) by induction on \( n \).

\[ \forall o \quad (\text{def. of } 0) \]

\[ \exists \forall o t . h - P o t w t \]

\[ \exists \forall o t . h - P o t w t \quad \text{def. of } 0 \]

\[ \exists \forall o t . h - P o t w t \]

\[ \exists \forall o t . h - P o t w t \quad \text{def. of } P \]
Next we treat the induction step.

(.4) .4 $\vdash \sigma_n \land \rho_n$ (inductive) hyp

(.5) .5 $\vdash S_n \rho_{o_t}$ hyp

(.6) .5 $\vdash \exists x \cdot .7$ .5, def. of $S$

(.7) .5, .7 $\vdash \rho_{o_t} x \land n_{\sigma} [\lambda t \cdot t \neq x \land pt]$ choose $x$ (.6)

(.8) .4, .5, .7 $\vdash \exists z \cdot .9$ .4, def. of $P$, .7

(.9) .4, .5, .7, .9 $\vdash \forall w \cdot r_{o_t} z \supset w \supset p = x \lor \sim \rho_{o_t} w$ choose $z$ (.8)

Thus from the inductive hypothesis we see that there is an element $z$ which is under nothing in $p - \{x\}$. We must show that there is an element which is under nothing in $p$. We consider two cases, (.10) and (.14).

(.10) .10 $\vdash \sim r_{o_t} z \cdot x$ hyp (case 1)

(.11) .4, .5, .7, .9, .10 $\vdash r_{o_t} z \cdot w \supset w \neq x$ .10

(.12) .4, .5, .7, .9, .10 $\vdash \forall w \cdot r_{o_t} z \cdot w \supset \sim \rho_{o_t} w$ .9, .11

(.13) .4, .5, .7, .9, .10 $\vdash \exists z \cdot .12$ .12

Next we consider case 2, and show that $x$ is under nothing in $p$. 
Having finished the inductive proof, we proceed to prove the main theorem.

(24) .24 ⊨ \text{hyp} \quad \text{hyp}

(25) .1, .24 ⊨ \exists z_i \forall w. \ r_{\text{ott}} r_{\text{w}} \bowtie \sim P \ r_{\text{ott}} w \quad .23, .24, \text{def. of P}

(26) .1 ⊨ \forall z_i \exists w. \ r_{\text{ott}} r_{\text{w}} \quad .1

(27) .1, .24 ⊨ \exists w. \sim P \ w \quad .25, .26
(.28) $1 \vdash \forall n \sigma \cdot np_{O_1} \supset \exists w \_t \sim p_{O_1} w \_t$

(.29) $\vdash .28$

7 $\vdash \forall n \_m \cdot s \sigma s \_m = s \sigma m \sigma \supset n \_m = m \_m$

Proof:

(.1) $1 \vdash n \sigma \land n \_m \land s n = s m$ hyp

(.2) $2 \vdash n \sigma p_{O_1}$ hyp

(.3) $1, 2, \vdash \exists w \_t \sim p_{O_1} w \_t$ .1, 2, Theorem 6

(.4) $1, 2, 4 \vdash p_{O_1} w \_t$ choose w (.3)

(.5) $1, 2, 4 \vdash p_{O_1} = [\lambda t \_t \cdot t \neq w \_t \land t = w \lor p t]$ .4, $E^0$, $E^{O_1}$

(.6) $1, 2, 4 \vdash n \sigma [\lambda t \_t \cdot t \neq w \_t \land t = w \lor p_{O_1} t]$ .2, 5

(.7) $1, 2, 4 \vdash s n \sigma [\lambda t \_t \cdot t = w \_t \lor p_{O_1} t]$ .6, def. of $S$

(.8) $1, 2, 4 \vdash s m \sigma [\lambda t \_t \cdot t = w \_t \lor p_{O_1} t]$ .1, 7

(.9) $1, 2, 4 \vdash m \sigma p_{O_1}$ .1, 4, 8, Theorem 5

(.10) $1 \vdash n \sigma p_{O_1} \supset m \sigma p$ .3, 9

(.11) $1 \vdash m \sigma p_{O_1} \supset n \sigma p$ proof as for .10
\[
\begin{align*}
(.12) & \quad 1 \vdash \forall \sigma \in \text{O} \cdot \sigma \in \text{p} \equiv \sigma \in \text{m} & .10, .11 \\
(.13) & \quad 1 \vdash \sigma \in \text{m} & .12, E^0, E^{O^1}
\end{align*}
\]

Bibliography


