OSCILLATORY PROPERTIES OF
COMPLEX DIFFERENTIAL SYSTEMS

by

Zeev Nehari*

Report 71-43

September 1971

* Work supported by the National Science Foundation under
Grant GP 23112.
OSCILLATORY PROPERTIES OF COMPLEX DIFFERENTIAL SYSTEMS

by

Zeev Nehari

Abstract

Let $A(z)$ be an $n \times n$ matrix whose elements are analytic functions of a complex variable $z$ in a simply-connected domain $D$. The differential equation

$$\frac{dw}{dz} = A(z)w$$

is said to be nonoscillatory in $D$ if it does not possess a non-trivial solution vector $w = (w_1, \ldots, w_n)$ such that $w_k(z_k) = 0$, $k = 1, \ldots, n$, where the $z_k$ are points of $D$. The equation is called suborthogonal in $D$ if, for any $z, \zeta \in D$ and any non-trivial solution $w$, $\text{Re}\{w(z)w(\zeta)\} > 0$.

The problem studied in this paper is that of establishing connections between these properties (or their absence) and certain simple properties of the coefficient matrix $A$. A number of criteria are obtained.
OSCILLATORY PROPERTIES OF COMPLEX DIFFERENTIAL SYSTEMS

by

Zeev Nehari

1. The systems to be discussed in this paper are of the form

\[ \frac{d}{dz} w = A(z)w, \]

where \( A(z) \) is an \( nxn \) matrix whose elements are analytic functions in a simply-connected domain \( D \) in the complex plane, and \( w \) is a vector whose \( n \) components are analytic in \( D \).

The system (1.1) is said to be nonoscillatory in \( D \) if every nontrivial solution vector \( w \) has at least one component which does not vanish in \( D \). Evidently, this property may also be defined by the requirement that, for arbitrary complex constants \( a_1, \ldots, a_n \) and arbitrary \( z_1, \ldots, z_n \) in \( D \), (1.1) should have exactly one solution \( w = (w_1, \ldots, w_n) \) for which \( w_j(z_j) = a_j \)

A related property, which is likewise useful in the study of the oscillatory behavior of the solutions of (1.1), is that of suborthogonality. The equation is said to be suborthogonal in \( D \) if, for any nontrivial solution \( w \) and for any two points \( \xi, \eta \) in \( D \), \( \text{Re}(w(\xi)w(\eta)) > 0 \) (where the expression in braces is the scalar product of the two vectors in question). For \( n = 2 \), suborthogonality evidently implies nonoscillation; for \( n > 2 \), the relation between the two concepts is less obvious.

* Work supported by the National Science Foundation under Grant GP 23112.
In formulating the nonoscillation condition of this section we shall assume that the domain $D$ is bounded by a contour $\partial D$ and that the coefficient matrix $A$ is analytic in $D + SD$. This is done in the interest of more concise formulation and is not an essential restriction. Our condition will be stated in terms of the Hölder norm $\|A\|_p (p>1)$ of the coefficient matrix $A$, induced by the norm $\|p\|_p = [\sum |P_1|^p + \ldots + |P_n|^p]^{1/p}$ of the vector $\beta = (\beta_1, \ldots, \beta_n)$.

**Theorem J3-JL** Let $D$ be a simply-connected domain in the complex plane, bounded by a contour $\partial D$. If the matrix $A$ is analytic in $D + \partial D$ and

$$\int_{\partial D} \|A\|_p \, |dz| < 2\pi (p-1) [\pi \sin \alpha]^{-1} = c_p, \quad (p>1),$$

then the system (1.1) is nonoscillatory in $D$.

For $p = 2$, (1.2) reduces to the condition

$$\int \|A\|_2 |dz| < \pi,$$

recently obtained by B. Schwarz [9], who also showed that the constant $\pi$ is the best possible (the fact that the system is nonoscillatory if the left-hand side of (1.3) is bounded by 1 had previously been proved by W. J. Kim [3]). Whether condition (1.2) is sharp for other values of $p$ is an open question. It is easily confirmed that $c_p = c_q$ if $p^{-1} + q^{-1} = 1$, and that $c_p$ decreases from $\pi$ to 2 if $p$ increases from 2 to $\infty$. Thus, $c_p > 2$ except for $p = 1, \infty$. In the two limiting cases, condition (1.2) is contained in a recent result of Schwarz [8] according to which the inequality
where $||A||$ is the matrix norm induced by an arbitrary absolute vector norm, is sufficient to guarantee the nonoscillation of equation (1.1) in $D$. A short proof of this result will be found at the end of this section.

If $D$ is transformed into another domain $D'$ by a conformal mapping $z \to z'$, (1.1) is transformed into an equation of the same form and the integral on the left-hand side of (1.4) remains invariant [7]. It is therefore sufficient to prove Theorem 1.1 for the case in which $D$ is the unit disk. The situation is further simplified by a result of W. J. Kim [3] who showed that if (1.1) is oscillatory in $|z| \leq 1$, then there exists a solution $w$ of (1.1) such that every component of $w$ has a zero on a circle $|z| = r$ ($0 < r < 1$).

By (1.1), we have

$$\|w'\|_p = \|Aw\|_p \leq \|A\|_p \|w\|_p$$

and therefore, if $\rho$ is a positive continuous function on $|z| = r$,

$$\int_{|z|=r} \rho(\|w'\|_p)^p |dz| \leq \int_{|z|=r} \rho(\|A\|_p)^p (\|w\|_p)^p |dz|. $$

In particular, if we set $\rho = (\|A\|_p)^{1-p}$ and introduce a new (real) variable $s$ by setting $ds = \|A\|_p |dz|$, we obtain

$$\int_0^L (|\frac{dw}{ds}|_p)^p ds \leq \int_0^L (|W|_p)^p ds,$$

where $W(s) = w(z(s))$,

$$L = \int_{|z|=r} \|A\|_p |dz|.$$
and, since the analytic vector \( w \) is single-valued in the unit disk, \( W(s+L) = W(s) \).

If \( W_1, \ldots, W_n \) are the components of \( W \) and if \( a_k \) denotes the zero of \( W_k \) on \( |z| = r \), the periodicity of \( W \) enables us to write (1.5) in the form

\[
\sum_{k=1}^{n} \int_{a_k}^{a_k+L} |s| \leq L \sum_{k=1}^{n} \left| \int_{a_k}^{a_k+L} w \right| ds,
\]

where the prime now denotes differentiation with respect to \( s \).

Setting \( R_k = |w_k| \) and observing that \( |R_k| \leq |W| \), we find that the real functions \( R_k \) satisfy the inequality

\[
\sum_{k=1}^{n} \int_{a_k}^{a_k+L} |R_k|^p ds \leq \sum_{k=1}^{n} \int_{a_k}^{a_k+L} |R_k|^p ds.
\]

and the conditions \( R-fou) = R-fcu+L) = 0 \).

We now use an inequality of Hardy, Littlewood and Pólya [2, Chap. VII], according to which

\[
\int_{0}^{a} f^p ds \leq \left[ \int_{0}^{a} \left( f^p - f' (t) \right) ds \right]^{1/p} \int_{0}^{a} (f')^p ds - \left( 2a \right) \int_{0}^{a} f' ds,
\]

if \( f(0) = 0 \), \( f \) is continuous and nondecreasing in \([0,a]\) and \( f' \in L_1[0,a] \). (In [2], \( p \) is an even integer, but the proof given there is valid for any \( p > 1 \)). The monotonicity assumption for \( f' \) is not essential. If this assumption is dropped and \( f' \) is replaced by \( |f'| \) on the right-hand side of (1.8), the inequality remains true, since \( f \) may be replaced by the function

\[
F(s) = \int_{0}^{s} \left[ f(t) + |f'(t)| \right] dt,
\]

for which \( f(s) \leq F(s) \), \( 0 \leq F \) \( F(s) \leq f'(s) \), and which satisfies the conditions under which (1.8) holds.
Applying this generalized version of (1.8) (with \( a = \frac{L}{2} \)) to the functions \( f(s) = R_k(\alpha_k + s) \) and \( f(s) = R_k(L + \alpha_k - s) \), respectively, and adding the results, we obtain

\[
\int_{a_k}^{a_k+L} R_k^p ds \leq (Lc_p^{-1})^p \int_{a_k}^{a_k+L} |R_k| ds, \quad k = 1, \ldots, n,
\]

and a comparison with (1.7) shows that \( Lc_p^{-1} \geq 1 \), i.e., by (1.6),

\[
\int_{|z|=r} \|A\|_p |dz| \geq c_p.
\]

But the norm \( \|A\|_p \) is a subharmonic function in the closed unit disk [7], and for a subharmonic function \( S \) the mean value

\[
\frac{1}{2\pi r} \int_{|z|=r} S|dz|
\]

is a nondecreasing function of \( r \). Hence, the assumption that (1.1) possesses an oscillatory solution in \(|z| \leq 1\) leads to the inequality

\[
\int_{|z|=1} \|A\|_p |dz| \geq c_p.
\]

Condition (1.2) is thus sufficient to guarantee the nonoscillation of (1.1) in \( D + \partial D \). This establishes Theorem 1.1.

We end this section with a short proof of the result of Schwarz, quoted above, according to which (1.4) guarantees the nonoscillation of equation (1.1) in \( D \) if \( \|A\| \) is the matrix norm induced by an arbitrary absolute vector norm \( \|u\| \), i.e., a norm for which \( \|u\| \leq \|v\| \) if \( u = (u_1, \ldots, u_n) \), \( v = (v_1, \ldots, v_n) \) and \( |u_k| \leq |v_k| \), \( k = 1, \ldots, n \). As before, it is sufficient to consider the unit disk and to show that, if (1.4) holds, (1.1) cannot have a nontrivial solution \( w = (w_1, \ldots, w_n) \) such that \( w_k(\alpha_k) = 0 \), where
6.

\[ |a_1| = \cdots = |a_n| = r < 1. \]

Suppose, then, there exists such a solution, and denote by \( G(z, \ell) \) the Green's matrix with the following properties: \( G(z, \ell) \) is diagonal; if \( z = re^{it}, \ell = re^{is} \) and \( p, \ell \) is the solution of \( \frac{p^s}{e^s} = \frac{\ell^s}{e^s} \) such that \( p^s < \ell^s < p^{s + 2\pi} \), then the diagonal elements \( G_{kk} \) of \( G \) are \( G_{kk} = 1 \) if \( p^s < t < s \) and \( G_{kk} = -1 \) if \( s < t < \ell^s + 2\pi \). It follows from this definition (and the fact that \( ^n\nu(<^s>) = 0 \)) that, on the circle \( |z| = r \), \( w \) is a solution of the integral equation

\[
2w(\xi) = \int_{|z|=r} G(z, C) A(z)w(z) \, dz.
\]

By the triangle inequality,

\[
2\|w(C)\|_{L} = \|G Aw\|_{L} \leq 1 \|G\|_{L} \|A\|_{L} \|w\|_{L}.
\]

Since \( \| \| \) is an absolute vector norm, we have \( \|G\|_{L} \leq 1 \|G\|_{L} = 1 \) and thus \( \|G Aw\|_{L} \leq \|A\|_{L} \|w(z)\|_{L} \). If \( C \) is so chosen that

\[
M = \|w(C)\|_{L} = \max_{|z|=r} \|w(z)\|_{L},
\]

it follows that \( 2M \|G Aw\|_{L} \leq 1 \|A\|_{L} \|w\|_{L} |dz|, \)

i.e.,

\[
2 \int_{|z|=r} \|A w\|_{L} \, dz.
\]

Using, in the same manner as before, the fact that a norm of an analytic vector is subharmonic, we obtain the required contradiction to (1.4).
2. In the present section we derive two sufficient conditions for suborthogonality, the first referring to a general simply-connected domain and the second to the unit disk.

**Theorem 2.1.** Let $D$ and $A$ have the same meaning as in Theorem 1.1, and let $\|A\|$ denote the Euclidean norm of $A$. If

$$\int_{\partial D} \|A\| \, dz < \pi,$$

then equation (1.1) is suborthogonal in $D + \partial D$. The constant $\pi$ in (2.1) is the best possible.

**Theorem 2.2.** Let $A$ be an analytic $n \times n$ matrix in $|z| < 1$, and let $\|A\|$ denote the Euclidean norm of $A$. Let $p(r)$ be a function with the following properties: $p(r) \geq 0$ and $(1-r^2)p(r)$ is non-increasing on $[0,1]$;

$$\int_0^1 p(r) \, dr < \frac{\pi}{4}.$$

If

$$\|A(z)\| \leq p(|z|), \quad |z| \leq 1,$$

then equation (1.1) is suborthogonal in $|z| \leq 1$. The constant $\frac{\pi}{4}$ in (2.3) is the best possible.

We begin the proofs of these results with the remark that an equation (1.1) which fails to be suborthogonal in the closure $D + \partial D$ of a (not necessarily simply-connected) domain $D$ must possess a nontrivial solution vector $w$ such that $\text{Re}(w(z)w(\xi)) = 0$, where $z_0$ and $\xi_0$ are points of $\partial D$. Indeed, our assumption implies the existence of points $z$ and $\xi$ in $D + \partial D$ such that $\text{Re}(w(z)w(\xi)) = 0$ for a nontrivial solution $w$. For constant $\xi$, $\text{Re}(w(z)w(\xi))$ is a harmonic function of $z$ in $D + \partial D$, and it
follows from the maximum principle that there exists a point $Z_0$ on $SD$ such that $\text{Re}\{w(Z_0)w(£)\} = 0$. Since $\text{Re}\{w(Z_0)w(£)\}$ is a harmonic function of £, a repetition of this argument shows that there exists a point $£_0$ of $SD$ such that $\text{Re}\{w(Z_0)w(£_0)\} = 0$, as asserted.

We next remark that it is sufficient to consider the case in which the solution $w = (w_1, \ldots, w_n)$ just mentioned is such that $w_k(Z_0) = 0$, $k = 1, \ldots, n-1$, $\text{Im}\{w_n(Z_0)\} = \text{Re}\{w_n(£_0)\} = 0$. To establish the truth of this assertion, we define the (constant) unitary matrix $Q$ by $Qw(Z_0) = (0, 0, \ldots, 0, ||w(Z_0)||)$ and consider the vector function $v = Qw$. By (1.1), $v$ is a solution of the equation $v^T = Bv$ where $B = QAQ^{-1}$. Since $||B|| = ||A||$, the conditions imposed on $A$ in Theorems 2.1 and 2.2 apply equally to $B$. By the definition of $v$, $v(z) = 0$, $k = 1, \ldots, n-1$ and $\text{Im}\{v_n(z_0)\} = 0$. Since $v(z_0)v(£_0) = Qw(Z_0) \cdot Qw(£_0) = w(Z_0^-)w(C_0)$, it follows from $\text{Re}\{w(Z_0)w(£_0)\} = 0$ that $\text{Re}\{v_n(£_0)\} = 0$. The solution $v$ of $v' = Bv$ thus has the indicated behavior at the points $z_0$, $£_0$ of $SD$ if (1.1) is not suborthogonal in $D + dD$.

We now prove Theorem 2.1. In view of the preceding remarks it is sufficient to show that, under the assumptions made, the equation cannot have a nontrivial solution $w = (w_1, \ldots, w_n)$ such that $w_k(Z_0) = 0$, $k = 1, \ldots, n-1$, $\text{Im}\{w_n(Z_0)\} = \text{Re}\{w_n(£_0)\} = 0$, where $|Z_0| = |£_0| = 1$. (Again, we only need consider the case in which $D$ is the unit disk.) Suppose, then, there exists a solution with these properties. The points $Z_0$ and £$_0$ divide the circle $|Z| = 1$ into two arcs $C_1$ and $C_2$, one of which -- say $C_1$ -- is such that
By (1.1), \( ||w'||^2 \leq ||A||^2 ||W||^2 \), or, if we introduce on \( |z| = 1 \) a real parameter \( t \) by \( dt = ||A|| |dz| \), \( ||w'||^2 \leq ||w||^2 \), where the dot denotes differentiation with respect to \( t \). Setting \( t = 0 \) at one of the ends of \( C_1 \) -- say at \( z_0 \) --, we then have

\[
\gamma \to \gamma (2.5) \int_{\gamma} ||w||^2 dt \leq \int_{\gamma} ||w||^2 dt,
\]

or, with \( w_k = u_k + iv_k \) (\( v^k, v_k \) real),

\[
\sum_{k=1}^{n} \left( \int_{\gamma} u_k dt - \int_{\gamma} v_k dt \right) \leq 0.
\]

Hence, there must exist an index \( ^*k \) for which either

\[
(2.6) \int_{\gamma} rY \cdot dt \leq \int_{\gamma} rY \cdot dt
\]

or

\[
(2.6 M) \int_{\gamma} rY \cdot v_k dt \leq \int_{\gamma} rY \cdot v_k dt.
\]

By assumption, all the functions \( u_1', \ldots, u_n', v_1', \ldots, v_n \) vanish either for \( t = 0 \) or for \( t = \gamma \). By classical results [cf., e.g.,1] this implies

\[
\int_{\gamma} u_k dt \leq \left( 2\gamma \pi \right)^{-2} \int_{\gamma} u_k dt,
\]

and a similar inequality for \( v_k \). Combining this with whichever of the two inequalities (2.6), (2.6') is satisfied, we find that \( 2\gamma \geq \pi \). Since \( \gamma \) is subject to the inequality (2.4), this conflicts with (2.1). Hence, if (2.1) holds, equation (1.1) is sub-orthogonal in \( |z| \, 1 \).
The fact that the constant \( \pi \) in (2.1) is the best possible follows from the example used by Schwarz [9] to show that his nonoscillation criterion is sharp. In this example the left-hand side of (2.1) has the value \( \pi + \epsilon \), where \( \epsilon \) is an arbitrarily small positive number, and it is easily seen that the equation in question is not only oscillatory but also fails to be sub-orthogonal.

Turning now to the proof of Theorem 2.2., we note that we may assume that \( A \) is analytic in \( |z| < 1 \) (replacing \( z \) by \( \rho z \), where \( \rho \) is in \((0,1)\) and sufficiently close to 1). As before, we have to show that under the assumptions made, (1.1) cannot have a nontrivial solution \( w = (w_1, \ldots, w_n) \) such that every one of the functions \( \text{Re}(w_k), \text{Im}(w_k), k = 1, \ldots, n \), vanishes at either \( z_0 \) or \( \zeta_0 \), where \( |z_0| = |\zeta_0| = 1 \). If there is such a solution, we draw through the points \( z_0 \) and \( \zeta_0 \) the (unique) circle which is orthogonal to \( |z| = 1 \), and we denote by \( C \) the arc of this circle which is in \( |z| < 1 \). We may assume (replacing, if necessary, \( z \) by \( \beta z \) where \( |\beta| = 1, \beta \) constant) that \( C \) is symmetric with respect to the imaginary axis. The transformation \( z \to (t+ir)(1-irt)^{-1} \) (with a suitable real \( r, -1 < r < 1 \)) carries \( C \) into a section of the real axis in the \( t \)-plane and maps \( |z| < 1 \) onto \( |t| < 1 \). It is easily confirmed that \( |z| > |t| \) if \( z \in C \) and that, for all \( |z| < 1 \),

\[
(2.7) \quad \frac{|dz|}{1-|z|^2} = \frac{|dt|}{1-|t|^2}.
\]

In the \( t \)-plane, the system (1.1) becomes

\[
(2.8) \quad \frac{d\eta(t)}{dt} = \frac{dz}{dt} A(z(t))\eta(t) = B(t)\eta(t),
\]
where $\eta(t) = w[z(t)]$. Since the points $z = z_o, z_o$ are carried into the points $t = \pm 1$, (2.8) has a solution $\eta = (\eta_1, \ldots, \eta_n)$ such that every one of the functions $\text{Re}[\eta_k], \text{Im}[\eta_k], k = 1, \ldots, n,$ vanishes for either $t = 1$ or $t = -1$. By (2.7), (2.8) and (2.3), we have, for $z \in C,$

$$
(1-|t|^2)\|B(t)\| = (1-|z|^2)\|A(z)\| \leq (1-|z|^2)p(|z|) \\
\leq (1-t^2)p(|t|),
$$

where the last inequality follows from $|z| > |t|$ and the assumption that $(1-r^2)p(r)$ is nonincreasing for $r \in [0,1)$. Hence,

$$
\|B(t)\| \leq p(|t|),
$$

where $B(t)$ is the coefficient matrix of equation (2.8) and $t \in (-1,1)$. Accordingly,

$$
\|\eta'\| \leq \|B\|\|\eta\| \leq p(|t|)\|\eta\|, \quad t \in (-1,1),
$$

and, with $ds = p(|t|)dt, s(-1) = 0, \gamma = \frac{d\eta}{ds},$

$$
\int_0^\gamma \|\eta\|^2\,ds \leq \int_0^\gamma \|\eta\|^2\,ds,
$$

where

$$
\gamma = \int_{-1}^{1} p(|t|)\,dt = 2\int_0^1 p(t)\,dt.
$$

Treating (2.9) by means of the procedure applied to (2.5), we find that $2\gamma \geq \lambda$, where $\gamma$ is the constant (2.10). Since this inequality, obtained under the assumption that (1.1) is not suborthogonal in $|z| < 1$, is incompatible with (2.2), this proves the main assertion of Theorem 2.2.

Examples showing that (2.1) is the best possible constant of its kind can be constructed in all cases in which $p(z)$ is an even analytic function in $|z| < 1,$ which is positive for real $z$ and
and such that \(|p(z)| \leq L |p(|z|)|\) (e.g., \(p\) will have these properties if \(p(z) = \sum_{m=0}^{\infty} a_m z^m, a_m > 0\)). For even \(n\), we set \(A = p(z)C\), where \(C\) is a constant matrix whose only nonzero elements \(c_{r,r+1}\) are \(c_{r,r+1} = 1, r = 1, \ldots, n-1\), and \(c_{n1} = (-1)^{n/2}\). The equation \(w = Aw\) then has a solution all of whose components are of the form 

\[ z \pm \sin CJ(z) \text{ or } z \pm \cos a(z), \text{ where } a(z) = \int_1^{n-1} p(t)dt. \]

If there is equality in (2.2), i.e., if \(\int_1^{n-1} p(t)dt = -z\), then all components vanish at either \(z = 1\) or \(z = -1\). Hence \(w(1)w(-1) = 0\), and the equation is not suborthogonal in \(|z| < 1\). A trivial modification of this example will produce an equation which fails to be suborthogonal in \(|z| < 1\). To obtain an example for odd \(n\), we define a matrix \(A_4\) by adding a row and a column of zeros to the matrix \(A\) just considered. If \(w = (w_1, \ldots, w_n)\) was the solution in the previous case, the equation \(v' = A_4v\) has the solution \(v = (w^1, \ldots, w^n, 0)\), and we evidently have \(|v_4| = |\lambda|\) and \(v(1)v(-1) = 0\).
3. It is natural to conjecture that the assumptions of Theorem
2.2 are also sufficient to guarantee the nonoscillation of (1.1) in $|z| < 1$. However, all that we are able to prove is that the
equation will be nonoscillatory if the constant $^\lambda$ in (2.2) is
replaced by a suitable smaller constant.

\textbf{Theorem 3.1.} Let $A$ be an analytic $n \times n$-matrix in $|z| < 1,$
and let $p(r) (r < 1)$ be a positive function such that $(1-r^2)p(r)
is nonincreasing and$
\begin{equation}
\sin \int p(r) \, dr < \left[ \frac{2(n-1)}{2} \right]^2.
\end{equation}

\textbf{II}

\begin{equation}
||A(z)|| \leq 1 \, p(|z|)
\end{equation}

$I(z) < 1$, then equation (1.1) is nonoscillatory in the unit disk.

We note that, for $n = 2$, condition (3.1) is equivalent to
(2.2). For $n > 2$, (3.1) is the more stringent condition.

The proof of Theorem 3.1 utilizes the following geometric
lemma.

Let $u^k = (u_{k1}, u_{k2}, \ldots, u_{kn})$, $k = 1, \ldots, n$, $n > 2$ be (constant)
complex unit vectors. If $u^k = 0$ for $k = 1, \ldots, n$, then there
exist integers $k, m$ such that $1 \leq k < m \leq n$ and
\begin{equation}
||u^{(k)} u^{(m)}|| \leq \varepsilon^f .
\end{equation}

For $n = 2$, (3.3) is trivial since $u^{(1)}$ and $u^{(2)}$ are, in
this case, orthogonal. Suppose that, for $n > 2$, (3.3) fails for
all permissible pairs $k, m$, i.e.,
\[ \frac{n-2}{n-1} < \sum_{v=1}^n |Z u_k v u_m v| \]
for \( k \neq m, 1 \leq k, m \leq n \). Since \( u_{kk} = u_{mm} = 0 \), this implies

\[
2 \left( \frac{n-2}{n-1} \right) < \sum_{k \neq m} |u_{kv}|^2 + \sum_{k \neq m} |u_{mv}|^2
= 2 - |u_{km}|^2 - |u_{mk}|^2,
\]

i.e.,

\[
\frac{2}{n-1} > |u_{km}|^2 + |u_{mk}|^2.
\]

Adding over all \( \frac{1}{2} n(n-1) \) pairs \( k, m \) (and observing that \( u_{kk} = 0, k = 1, \ldots, n \)), we obtain

\[ n > \sum_{k, m=1}^{n} |u_{km}|^2 = n. \]

Hence, there must exist a pair \( k, m \) for which (3.3) holds.

Suppose now that (1.1) has an oscillatory solution \( w = (w_1, \ldots, w_n) \), i.e., suppose there exist points \( z_1, \ldots, z_n \) in \( |z| < 1 \) such that \( w_k(z_k) = 0, k = 1, \ldots, n \). If \( u(z) \) is defined by \( w(z) = u(z) \|w(z)\| \), we then may apply the lemma to the unit vectors \( u(z_k) \). Accordingly, there exist subscripts \( k, m \) such that

\[ |u(z_k)u(z_m)| \leq (n-2)(n-1)^{-1}. \]

Next, we subject \( z \) to a Möbius transformation \( z \rightarrow t \) which carries the points \( z_k, z_m \) into two points \( t_1, t_2 \) on the real axis in the \( t \)-plane. As shown in the proof of Theorem 2.2, this can be done in such a way that the coefficient matrix of the transformed equation \( \eta' = B\eta \) satisfies the inequality \( \|B(t)\| \leq \rho(|t|) \) if (3.2) holds. We note that for the function \( v \) defined by

\[ \eta(t) = v(t) \|\eta(t)\| \]

we have \( |v(t_1)v(t_2)| = |u(z_k)u(z_m)| \leq (n-2)(n-1)^{-1}. \)

We set \( \eta = \alpha + i\beta, B = B_1 + iB_2 \) (\( \alpha, \beta, B_1, B_2 \) real), and denote the real 2\( n \)-dimensional vector \( (\alpha, \beta) \) by \( \gamma \). If the real \( 2n \times 2n \) matrix \( D \) is defined by
the equation $T' = B T$ is equivalent to the real equation $y' = D y$;
and we have $\|B\| = \|D\|$, $\|T\| = \|y\|$. If $\vec{6}$ is a constant unit vec-
tor in $E_2^n$ and $\vec{a}$ is defined by $y = a \|y\|$, then, as shown in
[5], the function $x = \vec{6} \vec{a}$ is subject to the inequality

$$\frac{1}{\sqrt{1-x^*}} \leq \|D\|.$$ 

Since $\vec{6} \vec{a}$ can be written in the form $\text{Re}\{cv\}$, where $c$ is a com-
plex $n$-dimensional unit vector, and since $\|p\| = \|B\|$, this is equiv-
alent to

$$j t^2 \wedge i \mathbb{R}^n,$$

where $<p = \text{Re}(cv)$. Integrating this inequality from $t_1$ to $t_2$
(where the subscripts are so chosen that $t_1 < t_2$), and setting
$c = v(t_1)$, we obtain

$$| \text{arc cos } \text{Re}v(t_1) v(t_2) | \leq \int_{t_1}^{t_2} \|B\| dt \leq \int_{t_1}^{t_2} \|B\| dt \leq 2 \int_{\mathbb{R}} p(t) dt.$$

Since $| \text{Re}v(t_1) v(t_2) | \leq \int_{\mathbb{R}} v(t_1) v(t_2) | - (n-2) (n-1) \leq 1$, this implies

$$\text{arc cos } \sum_{\mathbb{R}} 2 \int_{\mathbb{R}} p(t) dt,$$

i.e.,

$$\sin \int_{\mathbb{R}} p(t) dt \leq \frac{1}{2} (2n-1) \frac{1}{2}.$$

The assumption that (1.1) is oscillatory in $|z| < 1$ has thus led
to an inequality which is incompatible with (3.1), and Theorem 3.1
is proved.
4. In this final section we derive two necessary conditions for the suborthogonality of analytic systems (1.1) in the unit disk. Necessary conditions for the nonoscillation of such systems can be found in a recent paper by M. Lavie [4].

**Theorem 4.1.** Let $\Lambda(A) = \max _{\|a\| = 1} aAa$, for $\|a\| = 1$, i.e., let $\Lambda(A)$ be the largest eigenvalue of the Hermitian part of the matrix $A$. If the analytic system (1.1) is suborthogonal in $|z| < 1$, then

\[
\Lambda[A(z)] \leq \frac{1}{1 - |z|^2}.
\]

The inequality is sharp for all $|z| < 1$.

**Theorem 4.2.** If (1.1) is suborthogonal in $|z| < 1$, then

\[
\|A(z)\| \leq \frac{\sqrt{2}}{1 - |z|^2}.
\]

We begin with the remark that it is sufficient to prove both (4.1) and (4.2) for $z = 0$. To see this, we map the unit disk onto itself by the transformation $z \rightarrow t$, where $z = (t + \alpha)(1 + \bar{\alpha}t)^{-1}$ and $\alpha (|\alpha| < 1)$ is a constant. In this mapping, (1.1) is transformed into the equation $\frac{d\sigma}{dt} = B(t)\sigma$, where

\[
w(z) = \sigma(t), \quad B(t) = \frac{1 - |\alpha|^2}{(1 + \bar{\alpha}t)^2} A[t + \bar{\alpha}t].
\]

Since $B(0) = (1 - |\alpha|^2)A(\alpha)$ and the transformation does not affect the suborthogonality of the equation, this establishes our assertion.

To prove Theorem 4.1, we observe that it follows from the suborthogonality of the equation that $\text{Re}[w(z)w(\bar{z})] > 0$ for any $|z| < 1, |\beta| = 1$. Using the power series expansion of the vector $w$,
we have \( w(pz) = w(o) + pzw(o) + O(z^2) \), and a similar expression for \( w(p^*z) \). Hence

\[
h(z) = w(kz)w(kz^*) = \|w(o)\|^2 + 2z\text{Re}\{pw\cdot(o)w(o)\} + O(z^2).
\]

The (scalar) function \( h(z) \) is analytic in \( |z| < 1 \) and has there a positive real part. By a classical result, this implies \( |h^\top(o)| \leq 2 \text{Re}h(o) \). Thus, if \( \arg p \) is so chosen that \( pw'(o)w(o) \geq 0, |w'(o)w(o)| \leq |w(o)||^2 \), or, by (1.1),

\[
|w(o)A(o)w(o)| \leq \|w(o)\|.
\]

Since \( w(o) \) may be taken to be an arbitrary non-zero vector, we obtain \( A[A(o)] \leq 1 \). As remarked above, this is equivalent to (4.1).

To show that the inequality (4.1) is exact, we consider the equation \( w^\top = Aw \) with \( A = (1-z^2)^{-1}I \), where \( I \) is the nxn unit matrix. The general solution of this equation is \( w = f(z)a, -1^{1/2} \)

where \( f(z) = [(1+z)(1-z)]^{-1/2} \) and \( a \) is an arbitrary constant vector. The equation is suborthogonal in \( |z| < 1 \). Indeed, \( \text{Re}\{w(z)w(\ell)\} = \|a\|^2 \text{Re}\{f(z)f(\ell)\} \), and it is easily confirmed that the latter expression is positive for \( |z| < 1, |\ell| < 1 \). Since \( \|A(z)\| = |1-z|^2 I^h \), we have equality in (4.1) for all \( z \in (-1,1) \).

Theorem 4.2 is a consequence of the following result.

If \( w(z) \) is an analytic vector in \( |z| < 1 \) such that \( \text{Re}\{w(z)w(\ell)\} > 0 \) for \( |z| < 1, |\ell| < 1 \), then

\[
|w'(o)| \leq \sqrt{2} \|w(o)\|.
\]

The constant \( \sqrt{2} \) is the best possible.
Indeed, if (1.1) is suborthogonal in $|z| < 1$ then (4.3) shows that, for any nontrivial solution $w$, $\|A(o)w(o)\| \leq \sqrt{2} \|w(o)\|$. Since $w(o)$ is arbitrary, we have $\|A(o)\| \leq \sqrt{2}$, and this is equivalent to (4.2).

To prove (4.3), we note that, for any $t$ in the unit disk, $\Re\{w(tz)w(\overline{t}z)\} > 0$. If $w(z) = a_o + a_1 z + \ldots$ is the power series expansion of $w$, we thus have

$$0 < \Re\left[ \frac{1}{2\pi} \int_{|z|=1} \left[ \sum_{k=0}^{\infty} a_k (zt)^k \right] \left[ \sum_{k=0}^{\infty} a_k (\overline{zt})^k \right] |dz| \right]$$

$$= \Re\left[ \sum_{k=0}^{\infty} \|a_k\|^2 t^{2k} \right]$$

for $|t| < 1$. Since the function in braces has a positive real part, it follows from the result quoted above that $\|a_1\|^2 \leq 2\|a_o\|^2$. Because of $a_o = w(o)$, $a_1 = w'(o)$, this proves (4.3).

We now construct an example which shows that the inequality (4.3) is sharp. We denote by $P(z)$ a polynomial of degree $n - 1$, and we define the vector $w = (w_1, \ldots, w_n)$ by $w_k = P(w^k z)$, $k = 1, \ldots, n$, where $w = \exp(2\pi i n^{-1})$. If $P(z) = b_o + b_1 z + \ldots + b_{n-1} z^{n-1}$, it follows from the properties of the roots of unity that

$$w(z)w(\zeta) = n \sum_{k=0}^{n-1} |b_k|^2 (z\overline{\zeta})^k.$$

Accordingly, we shall have $\Re\{w(z)w(\zeta)\} > 0$ if the polynomial $R(z) = |b_o|^2 + |b_1|^2 z + \ldots + |b_{n-1}|^2 z^{n-1}$ has a positive real part for $|z| < 1$. A polynomial with this property is

$$R(z) = 1 + 2 \sum_{k=1}^{n-1} \frac{n-k}{n} z^k.$$
Indeed, $R(z)$ is the $n$-th Fejér mean of the partial sums of the function $f(z) = (1+z) (1-z)^{-1}$, for which $\text{Re}(f(z)) > 0$ in $|z| < 1$, and the Fejér means of a function with positive real part share this property [6]. We thus may set $b^{2-2kn^{-1}}1/2$, $k = 1, \ldots, n-1$, $b_0 = 1$. The corresponding vector $w$ will then satisfy $\|w(o)\|^2 = n$, $\|w(n)\|^2 = 2n(1-n^{-1})$, i.e.,

$$\|w(n)\| = \sqrt{2(1-\cdot)} \|w(0)\|,$$

and this shows that the constant $VT$ in (4.3) cannot be replaced by a smaller number.

This, of course, does not imply that the constant $\|f-2$ in (4.2) is also the best possible. All we can say is that the true constant is $\geq 1$, since $A(A) \leq \|A\|$ and (4.1) is a sharp inequality.
References


