The category of ordered spaces

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THE CATEGORY OF ORDERED SPACES
by
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Based on a series of talks given in the
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In Section 3 the ordered absolutes of ordered spaces are studied, and it is shown that they are the projectives for an appropriate class of maps in the category of ordered spaces and order preserving maps.

See Herrlich \cite{H} for the definitions and properties in categorical topology.

Incorporated in this paper is most of the theorems from a paper by V. Fedorchuk. His theorems are identified by his name in parenthesis following the word "theorem". His proofs have been modified when I believed it would simplify matters or would better serve the purposes of this paper.
SECTION 1. The Category Lots.

1.1. Define LOTS to be the category of ordered spaces and order preserving (continuous) maps. The monomorphisms are the one-to-one maps and the isomorphisms are the one-to-one onto maps.

1.2. PROPOSITION. The epimorphisms are the maps with dense range.

Proof. Since every ordered space is Hausdorff, obviously a map with dense range is an epimorphism.

Conversely let \( f:X \rightarrow Y \) be an epimorphism in LOTS. Suppose \( f \) does not have dense range. Then there exists \( y_0, y_1 \in Y \) such that \( (y_0, y_1) \notin O \) and \( (y^\uparrow Y_j) \cap O \setminus f(x) = \emptyset \). If \( (y_0, y_1) \) is clopen define \( g_0:Y \rightarrow R \), and \( g_\downarrow Y \rightarrow E \) (real line):

\[
g_0(y) = \begin{cases} 0 & \text{for } y \leq y_0 \\ 1 & \text{for } y > y_0 \end{cases} \quad \text{and} \quad g_\downarrow(y) = \begin{cases} 0 & \text{for } y < y_0 \\ 1 & \text{for } y > y_0 \end{cases}
\]

Thus \( g_0, g_\downarrow \in \text{LOTS} \) and \( fg_0 = fg_\downarrow \), but \( g_0 \neq g_\downarrow \). This is a contradiction. If \( (y_0, y_1) \) is not closed, \( (y_0, y_1) \notin O \). So pick \( y^r, y^s Y'' \in (y_0, y_1) \) such that \( y^r < y'' < y^s \). Since every ordered space is normal, the proof of Tjursohn's lemma permits us to construct \( h^\uparrow: [y^r, y^s] \rightarrow [0,1] \) and \( h^\downarrow: [y^r, y^s] \rightarrow [1,0] \) such that \( h^\uparrow(y') = 0 = h_0^1(y'), h_0^1(y'') = 1 = h_0^1(y'') \), and \( h^\downarrow, h^\downarrow \in \text{LOTS} \). Then extend \( h_0, h^\downarrow \) to \( h:Y \rightarrow I \) and \( h^\downarrow \) to \( h_n:Y \rightarrow I \).
exists a unique \( g : D \to C \) such that \( \Pi_A g = g_A \) and \( \Pi_B g = g_B \),
and there exists a unique \( f : E \to C \) such that \( \Pi_A f = f_A \) and
\( n_B f = f_B \). If \( f(e_0) \wedge g(d_0) \), then
\[ a_0 = \Pi_A f(e_0) \geq \Pi_A g(d_0) = a_1. \]
If \( f(e_0) \wedge g(d_0) \), then \( b_0 = \Pi_A f(e_0) \leq \Pi_A g(d_0) = b_1 \). In either
case there is a contradiction. Thus \( A \) and \( B \) have no product.

1.4. PROPOSITION. Let \( A \) and \( B \) be non empty ordered
spaces. Then \( A \) and \( B \) have no co-product in LOTS.

Proof. Choose \( a \in A \) and \( b \in B \), and suppose there is a co-
product \( C \) of \( A \) and \( B \) in LOTS. Let \( A' \) be formed by adding
to \( A \) the end points if necessary. Form \( B' \) similarly. Let
\( A' + B' \) be the topological sum of \( A' \) and \( B' \) with the orders
induced by \( A' \) and \( B' \) and such that for all \( a \in A' \) and \( b \in B' \),
a < b. Define \( B' + A' \) similarly. Let \( i_A : A \to A' + B' \),
i \_A \to B^1 + A^2, \quad i_B : B \to A^f + B^f, \quad \text{and} \quad i_{B_2} : B \to B^1 + A^2 \quad \text{be the natural embeddings. Let} \ \text{II}_A : A \to C \quad \text{and} \quad \text{II}_B : B \to C \quad \text{be the co-product maps;}

Then there exists unique \( h : C \to *A^1 + B^1 \) \quad \text{and} \quad k : C \to B^1 + A^2 \quad \text{such that} \quad i_{A_1} = h \text{II}_A, \quad i_{B_1} = h \text{II}_B, \quad i_{A_2} = k \text{II}_A, \quad \text{and} \quad i_{B_2} = k \text{II}_B. \) Now

\[ i_{A_2}(a) > i_{B_2}(b) \quad \text{and} \quad i_{A_1}(a) < i_{B_1}(b). \] If \( \text{II}_A(a) < \text{II}_B(b), \) then

\[ i_{A_2}(a) = k \text{II}_A(a) < k \text{II}_B(b) = i_{B_2}(b). \] If \( \text{JJ}_A(a) > \text{II}_B(b), \) then

\[ i_{A_1}(a) = h \text{II}_A(a) > h \text{II}_B(b) = i_{B_1}(b). \] In either case there is a contradiction. Thus \( A \) and \( B \) have no co-product.

1.5 1.3 and 1.4 can be easily generalized as follows.

Let \( \{A_i\}^\infty_{i=1} \) be a collection of non empty ordered spaces subscripted by the set \( G. \) Its product exists iff all but one of the \( A_i \)'s is a one point space. Its co-product exists iff

\[ 5 = 1. \]

Let \( Y \) be an ordered space, and let \( y \in Y. \) Call \( y \) a left limit point if \( y \in (-\text{OD}^y), \) \quad \text{and call} \quad y \quad \text{a right limit point} \quad \text{if} \quad y \in (y,\text{OD}) . \] Then \( y \) is called a one (two) sided limit point if \( y \) is either (both) a left or (and) a right limit point. A gap in \( Y \) is a pair \( \{A,B\}_Y \) of non empty clopen subspaces such that \( A \cup B = Y \quad \text{and} \quad A < B \text{ i.e., for all} \ a \in A, \ b \in B \ a < b. \) If \( A \) has a \( \sup y^1 \) \quad \text{and} \quad \text{B has an inf} y^2 \quad \{A,B\}_Y \text{is called a} \quad \text{jump. This jump can also be denoted by the ordered pair} \quad f_{y^1},y^2. \) The
points \( y^* \) and \( y^* \) are called jump points. If both \( y^* \) and \( y^* \) are one sided limit points, \( \{y^*, y^*\} \) is called a two sided jump. 

\( (A,B) \) is called a cut if \( A \) has no sup and \( B \) has no inf. We also refer to the "hole" \( u \) between \( A \) and \( B \) as this cut.

Clearly, generalized ordered spaces need not be orderable. For example \( (0,1) \cup \{2\} \) is not an orderable subspace of \( \mathbb{R} \).

Let \( X \) be a generalized ordered space. If \( \{A^*, B^*\} \) is a pair of non empty clopen, i.e., open and closed, subspaces of \( X \) with \( A^* < B^* \) and \( A^* \cup B^* = X \), we also call this a gap. Similarly, we define jumps and cuts as we did in the ordered case.

However, if \( A^* \) has no sup but \( B \) has an inf \( x^* \) we call \( \{A^*, B^*\} \) a left cut, which is also denoted by \( \{^*x\} \). If \( A^* \) has a sup \( x \) but \( B^* \) has no inf we call \( \{A^*, B^*\} \) a right cut, which is also denoted by \( \{x,-\} \). Right cuts and left cuts are called half cuts, as are the "holes" they determine.

1.6 THEOREM. Let \( e:X \to Y \) be an epimorphism in LOTS. Then \( e \) is an extremal epi iff for all \( y \in Y \setminus e(X) \) there exists a unique \( y' \in e(X) \) such that \( y \) and \( y' \) form a two sided jump in \( Y \). Hence if \( e \) is an extremal epi and \( e(X) \) is ordered, then \( e \) is an onto map.

Proof. Let \( e:X \to Y \) be an extremal epi. If \( e(X) \) is ordered, then define \( e^1:X \to e(X) \) such that for all \( x \in X \) \( e^1(x) = e(x) \).
and $i : e(X) \rightarrow Y$ is the inclusion map. Thus the diagram commutes, and $i$ is a mono. So $i$ is an iso. Then since $e'$ is onto, $e$ is onto.

If $e(X)$ is not ordered, let $y \in Y \setminus e(X)$. Suppose $y$ is a two sided limit point of $Y$. Then $Y' = Y \setminus \{y\}$ is ordered and there exist $e^\perp : X \rightarrow Y^*$ such that for all $x \in X$ $e(x) = e'(x)$, and $i : Y^* \rightarrow Y$ is the inclusion map. Then $i$ is a mono and $ie^\perp = e$ but $i$ is not an iso. Contradiction.

Therefore, since $e(X)$ is dense in $Y$, $y$ cannot be isolated, so it is a one sided limit point. Hence there exists a unique $y^\perp$ such that $y$ and $y^\perp$ form a two sided jump, and we may assume $y < y^\perp$.

Suppose $y' \in Y \setminus e(X)$. Then $Y' = Y \setminus \{y, y'\}$ is ordered and as in the argument above, we have $e^\perp : X \rightarrow Y^*$, the inclusion $i : Y^* \rightarrow Y$, $ie^\perp = e$, and $i$ is a mono but not iso. Contradiction. So $y^\perp \in e(X)$.
Conversely, let \( e \) be an epi, \( m \) a mono, and \( f \in \text{LOTS} \) such that the diagram commutes.

If \( e(X) \) is onto, then \( m \) is onto. So \( m \) is an iso.

If \( e(X) \) is not onto, let \( y \in Y \backslash e(X) \). Let \( y^r \) be as in the hypothesis. We may assume \( y < y^r \). Since \( m \) is mono and \( y \in e(X) \), there exists a unique \( z^l \in Z \) such that \( m(z^l) = y^r \).

Since \( e(X) \) is dense in \( Y \), \( m(Z) \) is dense in \( Y \). Thus \( z^l \) is not a left limit point since \( m \) is mono and \( y^l \) is not a left limit point. Moreover, \( z^l \prec o \), and so \( z^r \) has a predecessor \( z \). Necessarily, \( m(z) = y \). So \( m \) is onto and, therefore, an iso. Thus \( e \) is an extremal epi.

1.7. EXAMPLE. There exists an extremal epi that is not onto.

Let \( X = [0,2) + (3,4), Y = [0,1] + [3,4] \) define \( e : X \rightarrow Y \) as follows:

\[
    e(x) = x \quad \text{for } x \in [0,1] \cup (3,4) \quad \text{and} \quad e(x) = 1 \quad \text{for } x \in (1,2).
\]

Then \( e(X) = [0,1] + (3,4) \) is an unordered subspace of \( Y \).
1.8 LEMMA. \textbf{If} \( m: X \rightarrow Y \) is a \underline{one-to-one order preserving} function and \( X \) and \( Y \) are \underline{ordered spaces}, then \( m \) \underline{is continuous} iff \( m(X) \) \underline{is ordered}.

\textbf{Proof.} Let \( m \) be continuous. Suppose \( m(X) \) is not ordered. Then there exists \( y \in m(X) \) and a half, say right, cut \( \{y, \ast\} \) in \( m(X) \). Since \( m \) is one-to-one, there exists a unique \( x \in X \) such that \( m(x) = y \). Since \( y/m(X) \in (y, \infty) \), \( x/(x, \infty) \). Thus \( x \) has a successor \( x' \), and \( m(x') = \min(m(X) \cap (y, \infty)) \) which is impossible since it has no min. Hence \( m(X) \) is ordered.

Conversely, let \( m \) be discontinuous. Then we may assume there exists \( x \in X \) such that \( X \notin (X, \infty) \) but \( m(x)/m(x, \infty) \). Since \( m \) is mono and \( (x, \infty) \) has
no minimum, \( m( (x, CJD) ) \) has no minimum. Thus \( \{ m(x), \cdot \} \) is a right cut in \( m(X) \), and hence \( m(X) \) is not ordered.

i.9. THEOREM. Let \( m:X\rightarrow \rightarrow Y \) be a monomorphism in LOTS. Then \( m \) is an extremal mono iff \( X \) is embedded as the largest ordered subspace of \( \overline{m(X)} \).

Proof. Let \( m:X\rightarrow \rightarrow Y \) be an extremal mono. Then by the lemma \( m(X) \) is an ordered space. Thus the diagram commutes where

\[
\begin{array}{ccc}
\text{m(X)} & \xrightarrow{\text{m'}} & \text{X} \\
\downarrow & \searrow & \downarrow \\
\text{i} & \searrow & \text{m} \\
\text{Y} & \rightarrow & \text{Y}
\end{array}
\]

\( i \) is the inclusion map and for all \( x \in X \) \( m(x) = m' (x) \). Hence \( m' \) is an epi and, therefore, an iso. So \( X \) is embedded in \( Y \).

Now let \( Y' \) be an ordered space such that \( m(X) \subseteq Y' \subseteq \overline{m(X)} \). Then \( \text{im}' = m \) where \( m':X\rightarrow Y' \) is defined such that for all \( x \in X \) \( m(x) = m' (x) \), and \( i:Y'\rightarrow Y \) is the inclusion map. Hence \( m' \) is an epi, so it is an iso. Thus \( m(X) = Y' \). So \( m(X) \) is the largest ordered space in \( \overline{m(X)} \).
Conversely, let \( m : X \to Y \) be a mono such that \( m(X) \) is the largest ordered subspace of \( m(X) \), and let the diagram commute where \( i \) is the inclusion, \( i m' = m \), and \( e \) is an epi.

We want to show the existence of \( h^T : Z \to m(X) \) such that \( ih^T = h \). To do this it is sufficient to show that \( h \) is a mono and \( h(Z) \leq m(X) \).

However, \( m(X) \leq h(Z) = h(e(X)) \leq \text{che}(X) = m(X) \). Since \( m \) is mono, \( e \) is mono. So \( e(X) \) is an ordered dense subspace of \( Z \).

Suppose there exists \( z, z' \in Z \) such that \( z < z'^T \) and \( h(z) = h(z') = y \) for some \( y \in Y \). Then since \( e(X) \) is an ordered dense subspace of \( Z \), \( \{z, z'^T\} \) is a two sided jump of \( Z \) in \( zXefX \), and \( y \) is a two sided limit point in \( \text{m}(X) \setminus \text{m}(X) \). But then \( m(X) \cup \{y\} \) is ordered, and \( m(X) \leq m(X) \cup \{fy\} \leq m(X) \). Contradiction. Thus \( h \) is a mono.

Then by the lemma \( h(Z) \) is ordered. Hence \( h(Z) = m(X) \) since \( m(X) \leq h(Z) \leq m(X) \). So there exists \( h^T : Z \to m(X) \) defined by \( h^T(z) = h(z) \) for all \( z \in Z \), i.e., \( ih^T = h \). Thus \( h^T \) is one-to-one and onto, i.e., it is an iso. Then \( m^T = h^Te \) since \( i \) is a mono and \( im^T = he = ih^Te \). Moreover, since \( m^T \) is also an iso, \( e \) is an iso. Thus \( m \) is an extremal mono.
1.10. Denote by $G\text{LOTS}$ the category of generalized ordered spaces and order preserving maps. Then the last two theorems seem to indicate that both $G\text{LOTS}$ and the subcategory in $\text{LOTS}$ of maps with ordered range would lend themselves more naturally to a categorical treatment than would $\text{LOTS}$.

1.11. EXAMPLE. There is an extremal mono in $\text{LOTS}$ with unordered range. Let $X = [0,1) + (4,5]$ and $Y = [0,1] + [2,3) + (4,5]$.

Let $m:X\to Y$ be the inclusion map. Then $m(X)$ is the greatest ordered subspace of $m(X) = [0,1] + (4,5]$.

SECTION 2. Ordered and Generalized Ordered Extensions.

2-1. Let $X, B \in (G\text{LOTS})\text{LOTS}$. Then $B$ is an (generalized) ordered extension of $X$ iff $X$ can be embedded into $B$ by a map in $(G\text{LOTS})\text{LOTS}$. Let $p \in B \setminus X$. Then a neighborhood of $p$ in $X$ is the intersection of a neighborhood of $p$ in $B$ with $X$. $B$ is called an ordered compactification of $X$ if $B$ is a compact ordered extension of $X$ in which $X$ is dense. Note that no unordered generalized ordered space is compact. $B \in G\text{LOTS}$ is
a generalized ordered realcompactification of X if B is a
realcompact generalized ordered extension of X in which X
is dense.

The set of all ordered compactifications of an ordered
space X can be partially ordered as follows. Let \( b_1X \) and
\( b_2X \) be two ordered compactifications of X. Then \( b_2X \succ \succ X \)
iff there exists a unique \( f:b_2X \rightarrow b_1X \in \text{LOTS} \) such that \( f \) is
the identity on \( X \).

Note that \( X \) is compact iff it has both end points and
no cuts.

2.2. THEOREM. (Fedorchuk). The partially ordered set of
ordered compactifications of an ordered space \( X \) is order
isomorphic to the set of all subsets of the set \( U \) of all cuts
of \( X \).

Proof. If \( U \) is empty, i.e. \( X \) has no cuts, there exists
only one ordered compactification of \( X \) obtained by adding to \( X \)
any end points it doesn't possess. Assume that \( U \neq 0 \), and let
\( bX \) be an ordered compactification of \( X \). Choose a nonterminal
point \( y \in bX \setminus X \). Then \( y \) defines a cut of \( X \) by the subsets
\( X_y^- = \{ x \in X \mid x < y \} \) and \( X_y^+ = \{ x \in X \mid x > y \} \). Thus every nonterminal
point \( y \in bX \setminus X \) defines a cut \( u \) of \( X \) such that \( x'' = x_y^-, x_y^+ = x_y^+ \),
\( y \in u, y' \in u' \),
i.e. \( u \) can be considered as containing \( y \). It is easy to see
that for a given cut \( u \) of \( X \), there is either one or two points
of \( bX \setminus X \) in \( u \). Thus the ordered compactification \( bX \) defines
a division of \( tt \) into two disjoint subsets \( lu(bX) \) and \( lu_2(bx) \), where \( U_i(bX) \) consists of those cuts \( u \in U \) containing \( i \) points of \( bx \), \( i = 1, 2 \).

We now set up a correspondence between each ordered compactification \( bX \) of \( X \) and the set \( lu_2(bX) \subset \), and show the mapping \( lu_2 \) is an order isomorphism between the set of all ordered compactifications of \( X \) and the set of all subsets of \( U \), ordered by inclusion. Since each nonterminal point of the growth of an ordered compactification lies in a cut of \( X \), we have \( U_2(b\neg X) = U_2(k\neg x) \) implies \( b\neg X = b_2\neg x \), i.e. the mapping \( lu_2 \) is one-to-one. Let \( U^T \subset U \). Consider the ordered set \( B \) obtained from \( X \) as follows:

1. by the addition, if necessary, of the end points;
2. by the addition of one point to each cut \( u \in U \setminus U^T \);
3. by the addition of an ordered pair of points to each cut \( u \in U^T \).

It is easy to see that \( B \) is an ordered compactification of \( X \) and that \( lu_2(B) = U^T \).

Thus the mapping \( lu_2 \) is onto. We show \( lu_2 \) is an order preserving mapping. Let \( b\neg X \) and \( b_2\neg X \) be two ordered compactifications of \( X \), with \( b_2\neg X \geq i\neg X \), i.e. there exists \( f : b_2\neg X \rightarrow b\neg G \) such that \( f \) is the identity on \( X \). \( f \) maps the "cut points" in \( b_2\neg x \) to the corresponding cut points in \( b\neg x \). Hence

Now we show \( lu_1^T \) is order preserving. Let \( U_2(t \gg X) \) be \( f \gg X \) and \( U_2(t_2 b \neg x) \). Then \( l\neg X \) is obtained from \( b_2\neg x \) by identifying those ordered pairs of cut points of \( b_2\neg x \) which fill the growth from the set \( U_2(b_2\neg X) \setminus U_2(b\neg X) \). Hence there exists an onto
map $f: b^X \rightarrow b^X \in \text{LOTS}$ which is the identification on $X \uparrow$ and thus $b \uparrow X \rightarrow b \uparrow X$. Hence the theorem is proved.

2.3 COROLLARY. (Fedorchuk). For every ordered space $X$ there exists a greatest ordered compactification $(B^X, B^X)$ obtained by the addition of an ordered pair of points to each cut in $X$ and by the addition, if necessary of the end points.

2.4. Let $X$ be an ordered space. Then $B^X \leq 2$ since the points in $B^X \setminus x$ are determined by cuts of $X$ which in turn are determined by pairs of subspaces in $X$.

2.5. PROPOSITION. The category of compact LOTS is an epireflective subcategory of LOTS.

Proof. Consider the following diagram where $m: X \rightarrow K \in \text{LOTS}$, $K$ compact and $i$ is the inclusion map. Since $X$ is dense in $B^X$, $i$ is an epi. So

$$
\begin{array}{ccc}
A^X & \overset{q}{\rightarrow} & K \\
\downarrow \alpha & & \downarrow \nu \\
X & \overset{m}{\rightarrow} & K \\
\end{array}
$$

if there exists $q: B^X \rightarrow K$ such that $qi = m$, then $q$ is unique.* Define $q$ as follows. For $x \in X$ let $q(x) = m(x)$. For points of
Look at the ordered pair of points \( x_1, x_2 \) formed in the cut \( \{ A^\alpha, B^\alpha \} \). This cut induces a cut \( f^m \left( A^\alpha \cup \left( B^\alpha \right)^m \right) \). Let \( q(x) = \sup_{\alpha}(m(A^\alpha)) \) and \( q(x) = \inf_{\alpha}(m(B^\alpha)) \). Let \( q(0) = 0 \) and \( q(1) = 1 \). It is easy to show \( q \) is order preserving and continuous. Hence the category of compact LOTS is an epireflective subcategory of LOTS.

2.6. Consider the set \( O^*(X) \) of order preserving bounded maps from an ordered space \( X \) to \( \mathbb{I} \). We form the topological product \( P^\wedge = \mathbb{R}^{O^*(X)} \) and embed \( X \) into \( P^\wedge \) by Tikhnov's method: for \( x \in X \), \( f(x) = y = (y_\alpha)_{\alpha \in \Gamma} \) where \( \Gamma \) is an indexing set for \( O^*(X) \), and each \( y_\alpha = f_\alpha(x) \) for \( f_\alpha \in O^*(X) \). \( P^\wedge \) is a partially ordered set with the following order relation:

\[
y = (t_\alpha)_{\alpha \in \Gamma} \preceq (t'_\alpha)_{\alpha \in \Gamma} = y'^\top \text{ iff } t_\alpha \preceq t'_\alpha \text{ for each } \alpha \in \Gamma.
\]

2.7. THEOREM. (Fedorchuk). Let \( f: X \rightarrow P^\wedge \) be the embedding described above. Then \( \text{cl} \ f(X) \) considered with the order relation induced by \( P^\wedge \) is isomorphic to \( (BX, P^\wedge) \).

Proof. Since each \( f_\alpha \in O^*(X) \), the order induced on \( X \) from \( P^\wedge \) coincides with the initial order of \( X \), i.e. the embedding \( f: X \rightarrow f(X) \) is an isomorphism. Clearly \( \text{cl} \ X \) is compact and we shall prove that it is an ordered compactification of \( X \).
We first show that any two points \( y' \) and \( y'' \) in \( \text{cl}_{\mathcal{P}^*} X \) are comparable in the order relation induced by \( \mathcal{P}^* \). Let \( y' \in f(X) \). If \( y' = (t'_\alpha) \) cannot be compared with \( y'' = (t''_\alpha) \), then there exist subscripts \( \alpha_1 \) and \( \alpha_2 \) such that \( t'_\alpha < t''_\alpha \), \( t'_\alpha > t''_\alpha \).

We choose a neighborhood \( \mathcal{V}y'' \) of \( y'' \) consisting of all points \( y = (t^\alpha) \) for which \( t^\alpha_1 > t'_1 \), \( t^\alpha_2 < t'_2 \). We show that \( \mathcal{V}y'' \) doesn't intersect \( X \). Since \( X \) is linearly ordered and \( y' \in X \), we have \( X \subset (-\infty, y') \cup [y', \infty) \). \( \mathcal{V}y'' \) is the intersection of the two neighborhoods \( \mathcal{V}_1y'' \) and \( \mathcal{V}_2y'' \), where \( \mathcal{V}_1y'' = \{y=(t^\alpha)|t^\alpha_1 > t'_1\} \) and \( \mathcal{V}_2y'' = \{y=(t^\alpha)|t^\alpha_2 < t'_2\} \). Clearly, \( (-\infty, y') \subset \mathcal{P}^* \setminus \mathcal{V}_1y'' \) and \( [y', \infty) \subset \mathcal{P}^* \setminus \mathcal{V}_2y'' \). We have \( X \subset (-\infty, y') \cup [y', \infty) \subset (\mathcal{P}^* \setminus \mathcal{V}_1y'') \cup (\mathcal{P}^* \setminus \mathcal{V}_2y'') = \mathcal{P}^* \setminus (\mathcal{V}_1y'' \cup \mathcal{V}_2y'') = \mathcal{P} \setminus \mathcal{V}y'' \). Thus \( X \cap \mathcal{V}y'' = \emptyset \). But \( y'' \in \text{cl}_{\mathcal{P}^*} X \). Contradiction. Hence every point \( y'' \in \text{cl}_{\mathcal{P}^*} X \) can be compared with every point \( y' \in X \). Now let \( y', y'' \in \text{cl}_{\mathcal{P}^*} X \setminus X \).

From what has just been proven \( X \subset (-\infty, y') \cup [y', \infty) \), and, repeating the above argument, we find that \( y'' \) can be compared with \( y' \). Thus \( \text{cl}_{\mathcal{P}^*} X \) is linearly ordered.

Now we shall show that the interval topology on \( \text{cl}_{\mathcal{P}^*} X \) coincides with the subspace topology. Since \( \text{cl}_{\mathcal{P}^*} X \) is compact, it is sufficient to show that the identity map from \( \text{cl}_{\mathcal{P}^*} X \) with
the subspace topology to \( \text{cl} X \) with the interval topology is continuous. Let \( y \in \text{cl} X \) and let \( V_y \) be an interval neighborhood of \( y \), i.e. \( V_y = \{ y \in \text{cl} X \mid y < y_1 < y_2 \} \) where \( y_1, y_2 \in \text{cl} X \).

Let \( I_{\alpha} \) be the \( \alpha \)-coordinate projection map. Then there exists subscripts \( a_n \) and \( a_0 \) such that \( \Pi (y_n) < \Pi (y) \) and \( n_-(y) < IL (y_\sim) \). Then \( V_y = \{ y \in \text{cl} X \mid \Pi (y) > \Pi (y) \} \) with the subspace topology such that \( \forall y \in V_y \).

Hence the identity map on \( \text{cl} X \) is continuous. As seen above \( \text{cl} X \) induces the original order on \( X \), and, therefore, it is an ordered compactification of \( X \).

Now let \( i_X : X \to \text{fo} X \) and \( i_{\text{cl} X} : X \to \text{cl} \text{P}^* X \) be the embedding maps. By Proposition 2.5, there exists a unique \( q : \text{BX} \to X \text{GLOTS} \) such that \( q i_X = i_L \). Since \( i_X \) is dense, \( q \) is dense, and since \( \text{BX} \) is compact, \( q(\text{BX}) \) is closed. Hence \( q(\text{BX})' = \text{cl} X \), i.e. \( q \) is onto. Moreover, since if \( \{A, B\}_\sim \) is a cut in \( X \), then the existence of the map \( f \in O^*(X) \) which is 0 on \( A \) and 1 on \( B \) implies that \( q \) must be one-to-one. Hence \( q \) is an isomorphism, i.e. \( \text{BX} \) is isomorphic to \( \text{cl} X \).
2.8. Fedorchuk noted \([F_2]\) that \(BX\) has characteristic properties similar to those of the Stone-Cech compactification:

(1) In order for \(X\) to be 0*-embedded in an ordered compactification \(bX\), i.e. every map in \(0^* (X)\) "has an extension to a map in \(0^* (bX)\), it is necessary and sufficient that \(bX = BX\).

(2) In order that any two convex nonintersecting closed subsets in \(X\) should have nonintersecting closures in an ordered compactification \(bX\) it is necessary and sufficient that \(bX = BX\).

(3) If \(A\) is an ordered subspace of \(X\), then \(BA = \text{cl}_{bX} A\).

Let \(S = (H_\alpha)_{\alpha \in \mathcal{A}}\) be a collection of nonempty subsets of a topological space \(X\). The collection is said to be regularly decreasing if for each \(H \in \mathcal{S}\) there exists \(L \in \mathcal{S}\) such that \(a \in \mathcal{A}\).

2.9. THEOREM. (Fedorchuk). Let \(\mathcal{X}\) be the set of all maximal regularly decreasing filters, the elements of which are convex open subsets of \(X\). Then there is a linear order on the set \(\mathcal{A}\), with respect to which \(\mathcal{A}\) is isomorphic to \(HX\).

We order the set \(\mathcal{A}\) as follows. Let \(f^f, f^r \in \mathcal{A}\). We put \(S^f < f^r\) iff there exists intervals \(H^i \in J^j\) and \(H^m \in S^m\) such that
THEOREM. (Fedorchuk). If the ordered space $X$ has weight $Y$, then $\beta X$ is the inverse limit of the directed family of the ordered compactifications of $X$ of weight $Y$. (The partial order and boundary maps for this family are defined in the second paragraph of 2.1).

Proof omitted.

In 2.11-2.16 we will consider the usefulness of the operator $6$ in showing its role in solving the metrizability problem for compact ordered spaces and in determining when certain kinds of ordered spaces are isomorphic.

The following example gives insight for Theorems 2.10, 2.15 and 2.16 as well as how $\beta X$ may be visualized for $X \subseteq LOTS$.

EXAMPLE. There is an ordered space whose cardinality and weight are $K$, but its greatest ordered compactification has cardinality and weight $\mathfrak{c}$ and is not metrizable. Let $\mathbb{Q}$ be the rationals in the unit interval $I$. Both the weight and cardinality of $\mathbb{Q}$ equal $K$. $\beta \mathbb{Q}$ is constructed from $I$ by replacing each irrational point in $I$ by an ordered pair of irrational points. Another way to construct $\beta \mathbb{Q}$ is to identify corresponding points. The rest of the proof is omitted.

This is a linear order on $\mathbb{Q}$. The rest of the proof is omitted.
rational points in $I \times \{0,1\}$ (lexicographic product). $B\mathbb{O}^1$ is compact and separable but both the cardinality and weight of $B\mathbb{O}^1$ equal $c$. Hence $B\mathbb{O}^1$ is not metrizable.

2.12. An ordered space $X$ is said to be **minimal** if it has no end points and no two sided jumps.

2.13. **THEOREM.** (Fedorchuk). If $X$ and $Y$ are minimal ordered spaces, and $BX$ and $BY$ are isomorphic to each other, then $X$ is isomorphic to $Y$.

**Proof.** Let $f: BX \to BY$ be an isomorphism. Then $f$ maps end points to end points and two sided jumps to two sided jumps. Now $BX\setminus x$ and $BY\setminus y$ consists of two sided jumps with the possible exception of the two end points. Since $X$ and $Y$ are minimal, $f$ maps $BX\setminus X$ onto $BY\setminus Y$. Hence $f|_x$ is a one-to-one onto map from $X$ to $Y$, i.e. an isomorphism. So $X$ is isomorphic to $Y$.

2.14. **EXAMPLES.** Minimal ordered spaces $X$ and $Y$ may be **homeomorphic** while $BX$ and $BY$ are not isomorphic. For let $X$ be the discrete space $\omega^\ast + a >$ and let $Y$ be the discrete space $\omega^\ast + \omega + \omega^\ast + \omega^0 + \omega^0 \setminus \setminus$, where $\setminus \setminus$ is the ordinal $\setminus \setminus$ with the reverse order. Then $BX = (\omega + 1)^\ast + (\omega + 1)$, which has two limit points, and $BY = (\omega_0 + 1)^\ast + (\omega_0 + 1) + (\omega_0 + 1)^\ast + (\omega_0 + 1)$, which has four limit points.
Conversely, minimal ordered spaces $X$ and $Y$ may not be homeomorphic, while $B_X$ and $B_Y$ are homeomorphic. For let $X$ be the discrete space $\omega^* + \sum_{n=1}^{\infty} (\omega^+ \cup \omega^\om)$, and let $Y$ be the nondiscrete space $\omega^* + \sum_{n=1}^{\infty} (\omega^+ \cup \omega^\om)$. Then $B_X = (\omega^* + \sum_{n=1}^{\infty} (\omega^+ \cup \omega^\om)) + 1$ and $B_Y = Y + 1$, which can easily be shown to be homeomorphic.

2.15. THEOREM. (Fedorchuk). A compact ordered space $B$ is metrizable iff there exists a separable space $X$ with no two sided jumps such that $B = \text{fit}_X$ and $B_X \subseteq \langle K \rangle$.

Proof. Sufficiency. Let $X$ be a separable space with no two sided jump points such that $B_X \subseteq \langle K \rangle$. It is sufficient to show that $B_X$ is second countable. Let $X = (x^\omega, x^{\cdot \cdot \cdot}, x^\omega)$, a countable dense subset of $X$. Let $B_X = \{y_1^\omega y_2^\omega \cdots s_{Y_n}^\omega \cdots \}$. We renumber the points of $X \cup B_X = \{z^\omega, z_2^\omega, \cdots, z_n^\omega, \cdots \}$. Since $X$ has no two sided jump points, it is easy to verify that all
sets of the type \([0^\ominus), (z^\ominus z^\cdot), \text{or\ } (z^\cdot,1),\]\ where \(z^i < z^j\), form a countable base for \(RX\).

Necessity. Let \(B\) be a metrizable compact ordered space. Consider the ordered space \(X\) obtained by deleting all two sided jumps of \(B\). Since \(B\) has a countable base, there are at most a countable number of two sided jump points in \(B\).

Hence \(B \setminus X < H_\omega\). We shall show that \(X\) is dense in \(B\). Suppose it is false. Then there exist a nonempty open interval \(C\) of \(B\) contained in \(B \setminus X\) such that \(C\) consists of two sided jumps and hence is perfect and totally disconnected. Thus since \(B\) is compact metric, \(C\) contains the Cantor set whose cardinality is \(c\). Contradiction.

Now \(BX = B\) since \&X is formed by placing an ordered pair of points to each cut in \(X\). But these are precisely the points removed from \(B\) to form \(X\) since \(X\) is dense in \(B\). Moreover, the construction of \(X\) and its density in \(B\) implies that \(X\) has no two sided jumps. Since \(B\) is compact metric, it has a countable base, and hence \(X\) is separable.

2.16. THEOREM. (Fedorchuk). A compact ordered space \(B\) is metrizable iff there exists an ordered space \(X\) with a point countable base such that \(B = RX\) and \(BX \setminus X < C_{fc}\).
Proof. The necessity is obvious, since we can take $X$ as $B$. Now let $X$ be an ordered space with point countable base such that $B = BX$ and $BX\setminus X < \aleph_0$. First we show $BX$ satisfies the first axiom of countability. Suppose this is false. Then there exists $y \in BX\setminus X$ which doesn't have a countable neighborhood base. Hence there is a monotonic (for example, increasing) net $\{x \mid x \in X, cx < \omega \}$ converging to $y$, where $\omega$ is a regular uncountable ordinal and the set $\{x \mid x \in X, cx \in \omega \}$ is a closed subset of $X$. Consider the part $\{x \mid a < a+1 \}$ of this net. Since $X$ has a point countable base, it is first countable. Hence the net $\{x \mid a < a+1 \}$ converges to some point $y' \in BX\setminus X$. Since $BX\setminus X < \aleph_0$, then at most a countable number of intervals $[x, x+1]$ contain points of $BX\setminus X$. Hence there exists an ordinal $a^0 < a^1$ such that the interval $[x^{a^0}, y^{a^1})$ is contained in $X$ as a closed subset. Now $[x^{a^0}, y^{a^1})$ contains as a closed subspace the nonparacompact space $\{x^{a^0}, a < a+1 \}$ of order type $\omega_1$.

Thus $X$ is not paracompact. Then $X$ doesn't have a point countable base $[F_1$ or $B]$. Contradiction. Hence $BX$ is first countable.

A. Mishchenko [M] proved that a compact Hausdorff space with a point countable base is metrizable. Hence to prove that $BX$ is metrizable it is sufficient to show that it has a point countable base. Let $C = (v)$ be a point countable for $X$ whose
members are convex. For each \( V \in \mathcal{C} \) we denote by \( V^T \) the maximal interval of \( B_X \) such that \( V^t \cap X = V \). Then \( C_T = \{ V^t \} \) remains point countable at all points of \( X \). Now let \( y \in \bigcap X \) and \( (x_n) \) be a sequence in \( X \) converging to \( y \). Consider the set \( C_y \) consisting of all intervals \( v^* \in C^* \) containing \( y \). Each such interval \( V^t \) contains some point \( x_n \). Hence \( C = \bigcup_{y \in \bigcap X} C_y \) and each \( C_{x_n} \) is countable. Therefore, \( C \) is also countable, and thus \( C_T \) is point countable throughout all of \( B_X \). Since \( B_X \) is first countable and \( \bigcap X \) is a countable set consisting of the elements of a neighborhood base for each point in \( B_x \). Hence \( B_X \) is metrizable.

2.17. For any Hausdorff space, sequential compactness implies countable compactness which in turn implies pseudo-compactness. Conversely, for ordered spaces pseudo-compactness implies sequential compactness. To prove this last statement note that if an ordered space \( X \) has a sequence with no convergent subsequence, then one can find a monotonic subsequence which is a copy of \( \mathbb{N} \) (by mapping the sequence in an order preserving not necessarily continuous fashion into \( R \)). Since \( X \) is normal, Tietze's extension theorem shows that any closed subspace of \( X \) is \( C^* \) embedded in \( X \).

Recall that \( u \) is the Hewitt realcompact operator \([G-J] \).
2.18. THEOREM. Let $X$ be an ordered space. Then $\ell X$ is orderable iff $X$ is sequentially compact. If $X$ is sequentially compact, then $\bar{\ell}X = vX = BX$.

Proof. Assume $X$ is not sequentially compact. Then it is easy to show there is a monotone (for example, increasing) sequence $\{x_n\}$ which does not converge in $X$. Hence this sequence is a closed set isomorphic to the natural numbers (GT). By Tietze's extension theorem, $fx$ is C-embedded in $X$. Thus, $\text{cl}_{\ell}X = \bigcup_{n=1}^{\infty} x_n$ is isomorphic to $\mathbb{N}$. Hence $\{x_n\}$ does not converge in its closure and hence not in $\bar{\ell}X$. So $\bar{\ell}X$ is not sequentially compact. By 2.17 $\bar{\ell}X$ is not orderable.

Conversely, assume $X$ is sequentially compact. It is sufficient to show that $X$ is C-embedded in $BX$. First show that if $A$ and $B$ are disjoint closed subsets of $X$ then $(BX \setminus X) \cap \text{cl} A \cap \text{cl} B = 0$. So choose $p \in BX \setminus X$, and let $A$ and $B$ be disjoint closed subsets of $X$. We may assume that $p$ is a left limit point of $BX$. Since $X$ is sequentially compact no sequence in $X$ can converge to $p$. Suppose $p \in \text{cl}A \cap \text{cl}B$. Then there exists an increasing sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in A$ for $n$ odd and $x_n \in B$ for $n$ even. Then $\{x_n\}$ converges to $x < p$, where $x \in EX$, since $X$ is sequentially compact. Since $A$ and $B$ are closed, $x \in A \cap B$. Contradiction. Hence $p \notin \text{cl}A \cap \text{cl}B$. Thus $(BX \setminus X) \cap \text{cl}A \cap \text{cl}B = 0$. 


Now show for every $f \in C(X, l)$ there exists $x \in (-\infty, p) \cap X$ such that $f(y(x, p))$ is constant. Since $X$ is sequentially compact and $[x, p]$ is closed in $X$ for every $x < p$ in $X$, $f([x, p])$ is sequentially compact in $\{y\}$ and hence compact. So the nested intersection $\bigcap_{x < p} f([x, p])$ is non-empty. Choose $r_p$ in this intersection. Then $f^{-1}(r_p)$ is closed in $X$ and $p \in \text{cl}_{X, f^{-1}(r_p)} l$.

For every $n \in \mathbb{N}$, the closed set $(x \in (-\infty, p) | |f(x) - r_{p^n}| > \frac{1}{n})$ is disjoint from $f^{-1}(r_p)$. Hence by the above paragraph, this set has an upper bound $x \in (-\infty, p)$. Thus $\sup_{n \in \mathbb{N}} x = x$ exists in $X$ and is less than $p$. Thus $f(x, p) = r_p$.

Thus $f$ extends to $f_p : X \cup \{p\} \to \mathbb{R}$ such that $f_p(p) = r_p$. Similarly, we can extend $f$ to $f' : X \cup \{p'\} \to \mathbb{R}$ for each $p' \in X \setminus \{x\}$. Let $f'_{p'}$ be the induced extension of the $f_p$'s.

Obviously, $f'$ is order preserving. Moreover $f'$ is continuous since any net in $(X \setminus \{x\}$ converging to $p' \inSX \setminus \{x\}$ is eventually in $[x, p')$ and thus the image of the net is eventually equal to $r_{p'} = f'(p')$. Hence $X$ is $C$-embedded in $\mathcal{J}X$ and thus $\mathcal{J}X = -\mathcal{J}X = BX$. So $\mathcal{J}X$ is orderable.

2.19. After writing this paper I was told that M. Venkatajaman, M. Rajogopolan, and T. Soundararajan has also shown in a paper not yet published that if $\mathcal{J}X$ is orderable, then $X$ is
countably compact. However, the first half of the proof above is more concise than is their proof.

Let $X$ be a generalized ordered space. In a similar manner to 2.6 we let $0^*(X)$ be all order preserving maps from $X$ to $\mathbb{R}$ and define an embedding $f:X \rightarrow P^*$, where $P^* = K_0^*(X)$. Then similar to the proof of Theorem 2.7, $\text{cl}_P f(X)$ is the greatest ordered compactification $BX$ in the sense that $X$ is $0^*$-embedded in $BX$. Note that a generalized ordered space is compact iff it has its end points and has neither cuts nor half cuts. Hence a compact generalized ordered space is ordered. Similar to the proof of Theorem 2.2, $BX$ is constructed by the addition of an ordered pair of points to each cut, by the addition of a single point to each half cut, and by the addition, if necessary, of the end points. Then similar to the proof of Proposition 2.5 we have, the category of compact LOTS is an epireflective subcategory of GLOTS.

D. J. Lutzer pointed out to me that for a topological space $X$, $\beta X$ is orderable iff $X$ is a sequentially compact generalized ordered space. Noting that 2.17 also holds for generalized ordered spaces, there is a proof of this almost identical to that of Theorem 2.18. Similarly, we also have that $\text{iff } X$ is a sequentially compact generalized ordered space then $\beta X = BX = \omega X$. 
Let $X$ be a topological space. A point $p \in X$ is called a **P-point** iff every map $f: X \to \mathbb{I}$ is constant in a neighborhood of $p$. If $X$ is an ordered space then $p \in X$ is a P-point iff no monotone sequence in $X \setminus \{p\}$ converges to $p$ [G-J, problem 5.0]. This characterization can be extended to generalized ordered spaces— Hence for a topological space $X$, $\mathcal{X}$ is orderable iff $X$ is a generalized ordered space and every point of $(\mathcal{X} \setminus X$ is a P-point of $\mathcal{X}$.

**2.20. THEOREM.** Let $X$ be a generalized ordered space and let $\mathcal{X}$ be nonmeasurable. Then $uX \subseteq \mathcal{X}$ and $uX = X \cup T$, where $T = \{x \in X \mid x \notin \mathcal{X} \setminus (X \cup T)\}$ for every pair of disjoint closed subspaces $A, B$ of $X$.

**Proof.** Let $X$ and $T$ be as in the hypothesis. To show $uX = X \cup T$ we prove that there is a bijective correspondence between the real free $\mathbb{Z}$-ultrafilters on $X$ and their limits—the points of $T$, and that every $f \in C(X, \mathbb{I})$ can be continuously extended over $X \cup T$. First we show that no real free $\mathbb{Z}$-ultrafilter on $X$ can converge to a point in $(\mathcal{X} \setminus (X \cup T))$. Let $p \in (\mathcal{X} \setminus (X \cup T))$. We may assume that $p$ is a left limit point. Suppose there is a real free $\mathbb{Z}$-ultrafilter $A_p$ on $X$ converging to $p$. First
suppose that $p$ is not a $P$-point. Then there exists an increasing sequence $\{x_n\}_{n=0}^\infty$ in $X$ converging to $p$. Then $p^T$ since $\cap_{2n+1}^{\infty} n^0$ are disjoint closed sets in $X$ both of whose closures in $BX$ contain $p$. Moreover, $A_p$ must contain the collection $\{[x_n,p] | n \in \omega\}$. Hence $A_p$ doesn't have the countable intersection property and therefore, it is hyper-real. Contradiction. Thus if a real free $\mathbb{Z}$-ultrafilter converges to $p \in X \setminus T$, then $p$ must be a $P$-point. Also all the points of $T$ are $P$-points.

Now suppose $p$ is a $P$-point in $BX \setminus (X \cup T)$. Since $p \in T$, there exist nonempty disjoint closed sets $A$ and $B$ in $X$ such that $p \in \text{Cl}_X^A \cap \text{Cl}_X^B$. We can construct an increasing net $\{x_\alpha \mid \alpha < \omega_1\}$ in $X$ converging to $p$, where $\omega_\alpha$ is an initial regular ordinal and for all $\xi < \omega_\alpha$ $x_\xi \in A$ and $x_{\xi+1} \in B$. (Note for a limit ordinal $\alpha$, $2\alpha = \alpha$). Since $p$ is a $P$-point, $\alpha > \omega_\alpha$ [The remainder of this paragraph is from the proof in G-H of Theorem 10.3(2)]. For each limit ordinal $\alpha < \omega_\omega$ the limit of the subnet $\{x_\xi \mid \xi < \alpha\}$ is a cut or left cut $u_\xi$. Then the increasing net $\{u_\xi \mid \xi < \alpha\}$ of cuts and left cuts, obtained as above, "converge" to $p$. The intervals $J^*_\xi = (u_\xi, \xi, l)$ are clopen and their union $J = \bigcup_{\xi < \omega_\alpha} J^*_\xi$ is a clopen interval with $\sup p$. Hence $J \in A_p$. Consequently if $C$
and D are any two complementary subsets of \( O \), then
\[ J = \bigcup_{p \in C} J_p, \quad J = \bigcap_{p \in D} J_p \]
are clopen and exactly one of them is in \( A \). Now denote by \( S \) the set of intervals \( \{J_j^{\pm} \} \), every subset of \( J \) is of the form \( 3L_r^\pm = \langle \pm \rangle \). Define a finitely additive two valued measure \( m \) on the family of all subsets of \( 3 \), by putting \( m(J_j) = 1 \) iff \( J \in A \). Since for each \( \varepsilon < \omega \), points have zero measure i.e. \( J > 4^\omega \). Moreover \( m(\#) = 1 \). Since \( fc \) is nonmeasurable, the measure \( m \) cannot be countably additive. Hence there exists a countable family \( \{U \}_{n=1}^\infty \) of subsets of \( 5J \) of measure 1, whose intersection \( \bigcap_{n=1}^\infty U \) is of measure zero. Then \( J \in A \). Therefore \( j \in A \). Hence \( \{J \in A \} \) is a countable family of zero sets of \( A \) having empty intersection. Thus \( A \) is hyper-real. Contradiction.

Hence no real free \( Z \)-ultrafilter on \( X \) converges to any point in \( RX \setminus (X \cup T) \).

Before we show that for each point in \( T \) there is a unique real free \( Z \)-ultrafilter converging to it, we show that any map \( fc(X \setminus R) \) can be continuously extended to \( X \). Let \( p \in T \). First we show \( f \) is constant on a neighborhood of \( p \) in \( X \), and to do this we first show that \( f \) is bounded on a neighborhood of \( p \).
in X. If not $Z^+ = \{x \in X | f(x) > n+1\}$ and $Z^- = \{x \in X | |f(x)| < n\}$ are closed disjoint subsets in X for each $n \in \mathbb{N}$. Since $p \in \bigcap_{n \in \mathbb{N}} Z^+_n$, each $x_n = \sup_{\mathbb{R}^+} (Z^- \cap (-\infty, p))$ is less than $p$.

Hence, since $p$ is a P-point, $\sup_{n \in \mathbb{N}} x_n < p$. Then, since $p$ is a left limit point, $(x^*, p)$ is a non-empty neighborhood of $p$ in X on which $f$ has the value of OD or $-CD$ which is not in $R$. Contradiction. Thus, $f$ is bounded on a neighborhood of $p$. Now we show that $f$ is constant on a neighborhood of $p$ in X. Since $f$ is bounded on a neighborhood of $p$, $f([x, p))$ is compact for $x \in (x^*, p)$. Hence the nested family $\{f([x, p))| x \in (x^*, p)\}$ has a non-empty intersection. Thus there exists $r \in \bigcap_{x \in (x^*, p)} f([x, p))$. Moreover, $r \in (x^*, p)$, since $p \not\in f([x, p))$, otherwise we could construct a map unbounded in a neighborhood of $p$ in $X$. Thus $A = f^{-1}(r)$ is a non-empty closed subspace of $X$ and $\text{peel}_A$. In addition, for each $n \in \mathbb{N}$, $Z = (x \in (x^*, p) | |f(x)| < p \land (n+1))$ is closed in $X$ and disjoint from $A$. So $p \not\in \bigcap_{x \in (x^*, p)} Z_n$. For $Z$ empty define $\sup_{x \in (x^*, p)} x = x^*$. Then for each $n \in \mathbb{N}$, $\sup_{x \in (x^*, p)} Z_n = x_n < p$. Hence, $\sup_{x \in (x^*, p)} x_n = x^- < p$, and $(x^*, p)$ is non-empty. Thus $f^{-1}((x, p)) = r$. Therefore, $f$ is constant on a neighborhood in $X$ of each point of $T$, and hence $f$ extends to $X \cup T$. 

To finish the proof we show $p$ is the limit in $BX$ of a unique real free $Z$-ultrafilter in $X$. Note there is a free $Z$-ultrafilter converging to $p$. By the preceding paragraph $p$ is not the limit of a hyper-real $Z$-ultrafilter in $X$, since every map is bounded in a neighborhood of $p$ [see G-J]. Suppose $A$ and $A'$ are distinct $Z$-ultrafilters in $X$ converging to $p$. Then there exists disjoint zero sets $Z_eA$, $Z'G A'$ and peel $Z \cap \overline{Z'}$. Contradiction to $p \in T$. Therefore, each point in $T$ is the limit of a unique $Z$-ultrafilter in $X$, in fact, a free real $Z$-ultrafilter. Hence there is a one-to-one correspondence between the free real $Z$-ultrafilters on $X$ and their limits, the points of $T$. Hence $uX = XUT$.

2.21. COROLLARY. Let $X$ be a generalized ordered space, and let $T$ be the set of $P$-points in $BX \setminus X$. Then $XUT = uX$ iff whenever $A$ and $B$ are disjoint closed subspaces of $X$, $TOCI_{RXA} \cap C_{EXB} = 0$.

2.22. Recall that a cardinal $K$ is regular iff it is not the supremum of less than $K$ cardinals, each less than $N$. An ordinal is regular if it is a regular cardinal. Let $\alpha$ be a regular initial ordinal, whose cardinal is $K$, where $a$ is an ordinal number. Then a monotone net $(x_\xi)_{\xi<\alpha}$ in a linearly ordered space $X$ is called a $Q$-net [G-H], if for every nonzero
limit ordinal $\alpha < \omega_\alpha$, the limit in $\mathfrak{B}_X$ of the segment $t^\xi \in \mathfrak{B}_X$. In particular every $\omega_\alpha$-sequence and every $\omega_\beta$-sequence are $\mathfrak{Q}$-nets. If $\omega_\alpha$ is (non) measurable, then every $\omega_\alpha$-sequence is a (non) measurable $\mathfrak{Q}$-net. A point in $\mathfrak{B}_X \setminus \chi$ is called a (non) measurable $\mathfrak{Q}$-net. A point in $\mathfrak{B}_X \setminus \chi$ is a $\mathfrak{Q}$-point if it is the limit of a $\mathfrak{Q}$-net in $X$.

Let $p \in (\mathfrak{B}_X \setminus \chi$. Then $p$ is a non $\mathfrak{Q}$-point iff for every pair $A$ and $B$ of disjoint closed subspaces of $X$, $p \not\in \text{cl}_{\omega_\alpha} A \cap \text{cl}_{\omega_\alpha} B$. The proof of Theorem 2.20 shows that if there exists a distinct pair $A$ and $B$ of disjoint closed subspaces in $X$ such that $p \not\in \text{cl}_{\omega_\alpha} A \cap \text{cl}_{\omega_\alpha} B$, then $p$ is a $\mathfrak{Q}$-point. Conversely, if $p$ is a $\mathfrak{Q}$-point then there is an ordinal $\alpha$ and a $\mathfrak{Q}$-net $(x^\xi \in X)$ converging to $p$. Let $A = \text{cl}_{\omega_\alpha} A \cap \text{cl}_{\omega_\alpha} B$ and $B = \text{cl}_{\omega_\alpha} B$. Then clearly $A$ and $B$ is a pair of distinct disjoint closed subspaces in $X$ and $p \in \text{cl}_{\omega_\alpha} A \cap \text{cl}_{\omega_\alpha} B$.

Gillman and Henriksen [G-H, pp. 359-360] proved that if $X$ is a linearly ordered space with no measurable $\mathfrak{Q}$-net, then $uX = X \cup T$, where $T$ is the set of non $\mathfrak{Q}$-points in $\mathfrak{B}_X \setminus \chi$. Hence if $\mathfrak{B}_X$ is nonmeasurable this statement is Theorem 2.20.

Let $X$ be a generalized ordered space. Denote by $\text{O}(X)$ all order preserving maps from $X$ to $E$. If $Y$ is a generalized ordered space containing $X$, then $X$ is said to be $\text{O}$-embedded in $Y$ if every map in $\text{O}(X)$ can be extended to a map
in \(0(Y)\). Let us call \(v_X\) the greatest generalized ordered real-
compact extension of \(X\) in the sense that \(v_X\) is the greatest
subspace of \(B_X\) in which \(X\) \(\mathcal{J}\)-\(0\)-embedded. Then clearly,
\(v_X = X \cup T\), where \(T\) is the set of all \(P\)-points in \(fX\). Hence, if \(X\) is nonmeasurable, then \(u_X < v_X < (B_X)\).
Moreover, the last condition in Corollary 2.21 holds, then \(u_X = v_X\).

Now let \(a_X\) be the family of maximal filters, described in
Theorem 2.9, on the generalized ordered space \(X\). Call a maximal
filter in \(a_X\) real iff it has the countable intersection prop-
erty. Then there is a one-to-one correspondence between the real
maximal filters in \(a_X\) and their limit points, the points of \(v_X\).
Moreover, similar to Theorem 2.9, there is an order on this sub-
family of \(a_X\) for which it is isomorphic to \(v_X\).

Now let \(JP = R^\mathcal{O}(X)\). Then as in 2.6 and Theorem 2.7 we can find an
embedding \(f: X \rightarrow P\) such that \(\text{cl}_P f(X)\) is isomorphic to \(v_X\).

We call a generalized ordered space ordered realcompact
iff \(X = v_X\).

2.23 PROPOSITION. The category of ordered realcompact
GLOTS is an epireflective subcategory of GLOTS.

Proof. Consider the diagram where \(i_{v_X}, i_X\), and \(i_Y\) are
inclusion maps, \( f : X \rightarrow^\gamma Y \rightarrow \text{GLOTS} \), and \( Y \) is ordered and real-compact. Then by 2.19 there exists a unique \( q : 8X \rightarrow \beta Y \) such that \( q_i \circ i = i \circ f \). We shall show there exists \( h : \nu X \rightarrow \nu Y \) such that \( h \circ i = f \). It is sufficient to show that \( q_i \nu X \) maps onto \( Y \).

Suppose this is false. Then there exists \( x \in \nu X \) and \( y \in (\beta Y \setminus Y) \) such that \( q_i \nu X(x) = y \). We may assume that \( x \) is a left sided limit point of \( \nu X \). Then there exists an increasing chain net \( \langle x_\alpha \rangle \) in \( X \) converging to \( x \), and \( q_i \nu X(\langle x_\alpha \rangle) \) is an increasing chain net in \( Y \) converging to \( y \). Since \( y \) is not a \( P \)-point in \( (\beta Y, \nu) \), we can choose an increasing cofinal subsequence \( q_i \nu X(\langle x_\alpha \rangle) \) converging to \( y \). Hence \( f \circ \langle x_\alpha \rangle \) is an increasing sequence in \( X \) converging to \( x \). But this is impossible since \( x \) is a \( P \)-point of \( \nu X \). Hence \( q_i \nu X \) maps onto \( Y \). So we can define \( h : \nu X \rightarrow \nu Y \) such that for all \( x \in \nu X \), \( h(x) = q_i \nu X(x) \). Hence \( h_i \circ i = f \), and since \( i \) is an epi, \( h \) is unique, and we are done.

2.24. EXAMPLES. There exists a topologically realcompact ordered space \( X \) such that \( \beta X \setminus X \) contains a \( P \)-point of \( (\beta X \setminus X) \). Consider the space \( \mathfrak{u}_i \) of all ordinals less than the first uncountable ordinal. For each limit ordinal \( a < \omega_1 \) replace \( a \) by \( \tau^{a} \). Call this space \( \mathfrak{u} \). This is the required space since
one can easily show \(A\) is discrete and the greatest element 1 of \(Rui\) is in \(K\); Hence, since \(\omega_1 = K\) is a nonmeasurable cardinal, \(uX = X\). Therefore, \(1^uX\).

If there exists a measurable cardinal, then there exists an ordered space \(X\) such that \(uX\) is not an ordered extension of \(X\). Let \(0^\ast\) be the space of all ordinals less than the first measurable ordinal. For each limit ordinal \(a < \alpha\), replace \(a\) by \(\alpha^\ast\). Call this space \(\alpha^m\). Then \(\alpha^m\) is discrete of cardinality \(K\), the first measurable cardinal. The greatest point \(1\) of \(B_\alpha^m\) is in \(^\wedge\{A\}^m\) and for any subset \(A \subseteq X\) whenever \(K = K\), then \(1 \in \text{cl}_{\alpha^m}A\). Then there is a free real \(\mathbb{Z}\)-ultrafilter in \(0^\ast\) converging to 1 in \(^\wedge\{\}\). However, the map \(f \in \text{O}(\alpha^m)\), where \(f = 0\) on the points of \(\alpha\) from \(\alpha^m\) and \(f = 1\) otherwise, has no extension to \(\alpha^m \cup \{1\}\). Hence \(1^uX\), and \(uX\) is not an ordered extension of \(X\).

2.24. Note that although an extension \(Y\) of an ordered space \(X\) may not be an ordered extension, \(Y\) may still be orderable. A new ordering may make \(Y\) ordered while inducing an unordered generalized order on \(X\). For example let \(X = (0,1) + (2,3)\) and \(Y = [0,1] + (2,3)\). Reorder \(Y\) as \((2,3) + [0,1]\). Then with the new ordering \(Y\) is ordered and \(X\) is the unordered subspace \((2,3) + (0,1)\).
SECTION 3. Projectives.

3.1. Let $C$ be a category and let $P$ be a class of morphisms in $C$. An object $P$ of $C$ is called $p$-projective iff for each morphism $f: P \to Y$ and for each $P$-morphism $g : X \to Y$ there exists a morphism $h : P \to X$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow{f} \\
P & \xrightarrow{h} & X
\end{array}
\]

commutes. A morphism $f$ is called $P$-essential ($P^*$) iff $f \in P$ and $f \circ g \in P$ implies $g \in P$ for each morphism $g \in C$. $f : P \to X$ is called a $P$-protective cover of $X$ iff $P$ is $P$-projective and $f \in P^*$.

Note that $P^*$ is closed under compositions. Since the essential morphisms and projective covers generated by $P$ and its closure under compositions are identical, it is convenient to choose $P$ to be closed under composition.

It will be shown in this section that in LOTS if $P$ is the class of closed onto maps $P^*$ is the class of irreducible maps. Also we shall show that for every $X \in \text{LOTS}$ there exists a unique $P$-projective cover $\Pi_X : \langle kX \rangle \to X$, but that no class intersecting the complement of $P$ has this property.
3.2. PROPOSITION. Let \( X \) and \( Y \) be ordered spaces, and let \( f : X \rightarrow Y \) be an onto order preserving function. Then \( f \) is continuous iff \( f^{-1}(y) \) is closed for every \( y \in Y \).

Proof. Necessity is obvious. Now assume \( f^{-1}(y) \) is closed for every \( y \in Y \). It is sufficient to show that \( f^{-1}((-\infty, y]) \) and \( f^{-1}([y, \infty)) \) are closed for each \( y \in Y \). Let \( Y_0 \in Y \). Now
\[ f^{-1}((-\infty, y_0]) = f^{-1}((-\infty, y_0)) \cup f^{-1}(y_0), \]
which is a ray in \( X \) from the nonempty closed convex set \( f^{-1}(y) \) to \(-\infty \). Hence \( f^{-1}((-\infty, y_0]) \) is closed. Similarly, \( f^{-1}([y, \infty)) \) is closed.

3.3. PROPOSITION. Let \( f : X \rightarrow Y \) be an onto map in LOTS. If \( f^{-1}(y) \) is compact for every \( y \in Y \), then \( f \) is a closed map.

Proof. We prove the contrapositive. Assume \( f \) is not a closed map. Then there exists \( A \) closed in \( X \) such that \( f(A) \) is not closed. Hence there exists \( y \in Y \) such that \( y \in f^{-1}(A) \setminus f(A) \). We may assume that \( y \) is a right limit point of \( f(A) \), i.e. \( \{y \in \mathbb{C} \setminus \partial f(A) \} \cap f(A) \). Hence, \( f^{-1}(y) \) has no inf, and, therefore, \( f^{-1}(y) \) has no sup. Thus \( f^{-1}(y) \) is not compact.

3.4. For topological spaces \( X \) and \( Y \) recall that a map \( f : X \rightarrow Y \) is irreducible iff \( f \) is onto and for all proper closed
subspaces \( A \) in \( X \), \( f(A) \uparrow Y \). An ordered space \( Y \) is called an ordered absolute iff whenever \( f:X \rightarrow Y \) is irreducible in LOTS, then \( f \) is an isomorphism.

3.5. PROPOSITION. Let \( f:X \rightarrow Y \) be an onto map in LOTS, then \( f \) is irreducible iff for all \( y \in Y \) (1) \( f^{-1}(y) \leq 2 \) and (2) \( \exists \, \text{ a two sided jump in } X \).

Proof. Let (1) and (2) hold. Suppose there exists a proper closed subspace \( A \) in \( X \) such that \( f(A) = Y \). Then there exists \( x \in X \setminus A \) and \( x \in x^f \) such that \( f^{-1}(x) = \{x, x^f\} \), which is a two sided jump in \( X \). Then there exists \( x < x \) such that \( (x, x^f] \) is a neighborhood of \( x \) contained in the open set \( X \setminus A \). Necessarily, \( (x^f, x] \uparrow N \). Hence, since (1) holds, there exists \( y \in Y \) such that \( f^{-1}(y) \subseteq (x^f, x] \). This is a contradiction since \( (x^f, x] \subseteq X \setminus A \), but \( f(A) = Y \). Therefore, \( f \) is irreducible.

Conversely, let \( f \) be irreducible. Suppose there exists \( y \in Y \) such that \( f^{-1}(y) > 2 \). Then there exists a proper open subinterval \( A \subseteq f^{-1}(y) \). Hence \( X \setminus A \) is closed and \( f(X \setminus A) = Y \). Contradiction. So (1) holds.

Now let \( f^{-1}(y) = 2 \) for some \( y \in Y \). Then there exists \( x, x^f \in X \) such that \( f^{-1}(y) = \{x, x^f\} \). Obviously \( \{x, x^f\} \) is a jump. Suppose
it is not a two sided jump. Then one of the points, say \( x \),
is isolated. So \( X\setminus\{x\} \) is closed, and \( f(X\setminus\{x\}) = Y \). Contra-
diction. Therefore,, (2) also holds.

3.6. Note that if \( f: X \rightarrow Y \) is irreducible in LOTS, then
for \( y \in Y \) whenever \( f'^{-1}(y) = 2 \), \( y \) is a two sided limit point.
Whenever \( f'^{-1}(y) = 1 \), both \( y \) and \( f'^{-1}(y) \) are either isolated
points, left limit points, right limit points, or two sided limit
points in \( Y \) and \( X \) respectively. Note also that Propositions
3.3 and 3.5 imply \( f \) is a closed map.

3.7. THEOREM. (Fedorchuk). Let \( X \) be an ordered space.
Then there exists \( \Pi:\langle X \rightarrow X \rangle \in \text{LOTS} \) such that \( \langle X \) is an ordered
absolute and \( \Pi \) is irreducible. If \( f: X \rightarrow Y \) is irreducible
in LOTS, then there exists an isomorphism \( h: OX \rightarrow 6tf \) of ordered
absolutes such that \( f\Pi = \Pi h \).

Proof. Let \( X^\perp \) be the set of all two sided limit points
of \( X \). Consider the ordered space \( c^\perp X \) obtained by replacing
each point \( x^\perp X^\perp \) by an ordered pair of points \( \{ x, x^\perp \} \), which
is clearly a two sided jump in \( ciX \). Define \( \Pi^\perp: AX \rightarrow X \) as follows,
\[
\Pi^\perp (x) = x \text{ for } x \in X \setminus \Delta
\text{ and } \Pi^\perp (x_o) = \Pi^\perp (x_1) = x \text{ for } x \in X_\Delta.
\]
Clearly \( \mathbb{I}_X \) is order preserving and onto. So by Proposition 3.2, \( \mathbb{I}_X \) is continuous, and so by Proposition 3.5, \( \mathbb{II}^\uparrow \) is irreducible.

Now let \( g:Z \to aX \) irreducible in LOTS. Since \( aX \) has no two sided limit points, then by 3.6, \( g \) must be one-to-one. Then since \( g \) is onto, it is an isomorphism. Hence \( aX \) is an ordered absolute.

The rest of the proof will be given in 3.17.

3.8. Let \( f:X \to Y \) be a non-closed map in LOTS. Then there is a clopen ray \( R \) in \( X \) with no initial point whose image in \( f(X) \) is a ray whose initial point is in \( \text{cl}_X(f(R)) \setminus f(R) \).

3.9. We now begin to show that the projectives are the ordered absolutes for the class \( P \) defined below.

**Lemma.** In LOTS let \( P \) be the closed onto maps. Then the \( P \) essential morphisms \( (P^*) \) are the irreducible maps.

**Proof.** First note that \( P \) is closed under composition. Let \( f:Y \to Z \) be irreducible in LOTS, \( g:X \to Y \) LOTS, and \( f \in P \). Then by 3.6, \( f \) is closed, and hence \( f \in P \). We must show \( g \in P \). First suppose \( g \) is not onto. Then there exists \( y \in Y \setminus g(X) \).
Since \( f \) is irreducible and \( fg \) is onto, \( f^{-1}(y) \) is a two sided jump consisting of \( y \) and some \( y' \in Y \), and hence \( f(y) \) is a two sided limit point. We may assume that \( y < y' \). Now \( g^{-1}((-\infty, y]) = g^{-1}((-\infty, y)) \) is clopen, and \( f g g^{-1}((-\infty, y)) = (-\infty, f(y)) \). However, \( f(y) \in (-\infty, f(y)) \setminus (-\infty, f(y)) \), which implies \( fg \) is not closed. Contradiction. Hence \( g \) is onto.

Now suppose \( g \) is not closed. Then there exists a ray,\(^*\) say \((-\infty, u)\), where \( u \) is a cut, and there exists \( y \in Y \) such that \( y \in g((-\infty, u)) \setminus g((-\infty, u)) \). Then since \( y \) is a left limit point and \( f \) is irreducible, \( f(y) \in g((-\infty, u)) \setminus g((-\infty, u)) \). Thus since \((-\infty, u)\) is closed, \( fg \) is not closed. Contradiction. Hence \( g \) is closed. So \( geP \), and hence \( feP^* \).

Conversely, let \( f:Y \to \mathbb{Z} \in P \) not be irreducible. Then there is a proper closed subspace \( B \) in \( Y \) such that \( f(B) = Z \). We will find an ordered space \( B^\tau \supseteq B \) such that \( B^\tau \) is also a proper subset of \( Y \). If \( B \) is ordered let \( B^e = B \). If \( B \) is not ordered, then there exists a half cut, say a left cut \((C, D)\).
Let $b = \min D$. Since $B$ is closed, $C$ has no sup in $Y$.

Hence $C' = \{ y \in Y | y < c \text{ for some } c \in C \}$ and $D' = X \setminus C'$ determine a cut

\[
\begin{array}{c}
\text{Y} \\
\text{u} \\
\text{in } Y \setminus B \\
\end{array}
\]

u in $Y$. So $u < b$, and $(\bar{u}, B)$ is in $Y$. Hence there exists $y, y', y'' \in Y \setminus B$ such that $u < y < y'' < y' < b$. Let $B' = (-\infty, y] \cup [y', \infty)$. Then since $y'' \in B'$, $B'$ is a proper closed ordered subspace of $Y$ containing $B$. Hence $f(B') = Z$.

Now let $i: B' \rightarrow B$ be the inclusion map. Thus $i \in P$, since $B'$ is closed. However, $i$ is not onto. So $i^{\sim} \in P^*$. Hence $f \notin P^*$.

3.10. THEOREM. Let $X$ be LOTS. Then $\Pi_X$ is a $P$-projective cover, where the $P$-morphisms are the closed onto maps.

Since $\Pi_X$ is irreducible, then by Lemma 3.9, $\Pi_X \in P^*$. So we need only show that $\Pi_g (XX)$ is $P$-projective. So let $f: \alpha X \rightarrow Y \in LOTS$, and let $g: Z \rightarrow Y e P$
Then we must find \( h: \text{X} \rightarrow \text{Z} \) which makes the diagram commute.

Claim: If \( x \in \text{X} \) is a left (right) limit point such that \( \min (\max) f^{-1}f(x) = x \), then \( \min (\max) g \circ f(x) \) exists and is a left (right) limit point. Let \( x \) be a left limit point in \( \text{A X} \) such that \( \min f^{-1}f(x) = x \). Then

\[
f(x) \in (-\infty, f(x)) \setminus (-\infty, f(x)).
\]
Thus \( f(x) \) is a left limit point.

Suppose \( g^{-1}f(x) \) has no minimum. Then since \( g \) is continuous, \( g^{-1}f(x) \) has no inf in \( \text{Z} \). Hence \( (-\infty, g^{-1}f(x)) = \{ z \in \text{Z} | c < z \text{ for all } z \in g^{-1}f(x) \} \) has no sup, and it is non empty since \( g \) is onto. In addition, since \( g \) is onto,

\[
f(x) \in g(-\infty, g^{-1}f(x)) \setminus g(-\infty, g^{-1}f(x)),
\]
i.e. \( g \) is not closed. Contradiction. Hence \( g^{-1}f(x) \) has a min.

Moreover, since \( f(x) \) is a left limit point and \( g \) is onto, then \( \min g^{-1}f(x) \) is a left limit point. Use the dual argument if \( x \) is a right limit point. Hence the claim is proved.
Now we begin to construct \( h: \langle xX \rangle \to Z \). Decompose \( X \) into the collection \( \{ f^{-1}(y) \mid y \in \text{ef}(xX) \} \). For \( y \in \text{ef}(aX) \) such that \( f^{-1}(y) \geq 2 \), there is a gap \( u^*_Y \) in \( f^{-1}(y) \), which is either a cut or a jump since \( \langle xX \rangle \) has no two sided limit points. Let \( u^-_Y = \{ x \in f^{-1}(y) \mid x < u^*_Y \} \), and let \( u^*_Y = \{ x \in f^{-1}(y) \mid x > u^*_Y \} \). Now for each \( x \in \text{ef}(\langle xX \rangle) \), let \( z^*_Y = \inf g^{-1}(y) \) if it exists, and let \( z^+_Y = \sup g^{-1}(y) \) if it exists. Choose a point \( z^*_Y \in g^{-1}(y) \). Define \( h: \langle xX \rangle \to Z \) as follows. For \( y \in \text{ef}(\langle xX \rangle) \) and \( f^{-1}(y) = 1 \), there exists a unique \( x \in x \) such that \( f(x) = y \). Then

1. If both \( \sup \) and \( \inf \) of \( g^{-1}(y) \) exist, then for all \( x \in u^*_Y \) \( h(x) = z^*_Y \). For all \( x \in u^*_Y \) \( h(x) = z^*_Y \).

2. If \( \inf \) of \( g^{-1}(y) \) exists but \( \sup \) doesn't exist, then for all \( x \in f^{-1}(y) \) \( h(x) = z^*_Y \).

3. If \( \sup \) of \( g^{-1}(y) \) exists but \( \inf \) doesn't exist, then for all \( x \in f^{-1}(y) \) \( h(x) = z^+_Y \).

4. If neither \( \sup \) nor \( \inf \) of \( g^{-1}(y) \) exist, then for all \( x \in f^{-1}(y) \) \( h(x) = z^*_Y \).

For \( \text{ef}(\langle xX \rangle) \) and \( f^{-1}(y) = 1 \), there exists a unique \( x \in x \) such that \( f(x) = y \). Then

1. If \( x \) is a left sided limit point, then by the claim \( z^*_Y \) exists. So \( h(x) = z^*_Y \).
(2) If \( x \) is a right sided limit point, then by the claim \( z_y \) exists. So \( h(x) = z_y \).

(3) If \( x \) is isolated, \( h(x) = z_y \).

This definition is complete and the function is well defined since \( AX \) has no two sided limit points.

Clearly \( h \) is order preserving and \( gh = f \). To prove \( h \) is continuous it is sufficient to show that whenever a monotone, say increasing, chain net \( \{x_\alpha\} \) in \( <xX \) converges to \( xcoX \), then \( h(\{x_\alpha\}) \) converges to \( h(x) \). If \( x \) is not a left limit point, then \( \{x_\alpha\} \) eventually is equal to \( x \), and hence \( h(\{x_\alpha\}) \) is eventually equal to \( h(x) \). If \( x \) is a left limit point and \( \min f^{-1}(y) = x \), where \( y = f(x) \), then by the claim \( z_\neg y \) exists, is a left limit point, and \( h(x) = z_\neg y \). Clearly \( f(\{x_\alpha\}) \) is an increasing chain net converging to \( y \). Thus since \( g \) is onto, \( g^{-1} f(\{x_\alpha\}) \) is an increasing chain net of convex sets converging to \( z_\neg y \), and hence \( h(\{x_\alpha\}) \) converges to \( z_\neg y = h(x) \). If \( x \wedge \min f^{-1}(y) \), where \( y = f(x) \), then \( f^{m}(y) \nearrow 2 \), and \( XGU_y \) or \( XGU_y \). So if \( xeu(y) \), then \( \{x_\alpha\} \) is eventually in \( u^{m}(u) \). Hence \( h(\{x_\alpha\}) \) is eventually equal to \( h(x) \). Dually, we can show that if \( f{x_\alpha} \) is a decreasing chain net converging to \( X\in<XX \), then \( h(\{x_\alpha\}) \) converges to \( h(x) \). Hence, \( h \) is
continuous, and \( h: \langle x \rangle \to \text{ZeLOTS} \). Therefore \( \langle x \rangle \) is \( P \)-projective, and \( \Pi_X : \langle f \rangle \to \langle x \rangle \) is a \( P \)-projective cover.

3.11. PROPOSITION. Let \( \Pi_X : \langle t \rangle \to \langle x \rangle \in \text{ZeLOTS} \) and let \( P \) be a class of morphisms for which \( n_v : \text{Os} \to \langle x \rangle \) is a \( P \)-projective cover. If \( f : \langle y \rangle \to \langle x \rangle \in P \), then \( f \) is a closed onto map.

Proof. Consider the diagram below. Let \( \Pi_X : \langle a \rangle \to \langle x \rangle \) be a \( P \)-projective cover, and let \( f : \langle x \rangle \to \langle y \rangle \in P \). Then there exists \( h : f \langle x \rangle \to \langle y \rangle \) such that \( fh = \Pi_X \). Then since \( \Pi_X \) is onto, \( f \) must be onto.

Now suppose \( f \) is not closed. Then there exists a ray, say \( \langle u, aD \rangle \), where \( u \) is a cut in \( Y \) and there exists a point \( x \in \langle x \rangle \) such that \( f(\langle -OD, u \rangle) = \langle -OD, X \rangle \) and \( x \in \langle -OD \rangle \setminus \langle -OD \rangle \).

\[ \begin{array}{ccc} \alpha_X & \xrightarrow{h} & \Pi_X \\ \downarrow & & \downarrow \\ \langle x \rangle & \xrightarrow{f} & \langle y \rangle \\ \downarrow & & \downarrow \\ \langle ax \rangle & \xrightarrow{\alpha_X} & \langle x \rangle \\ \end{array} \]
Since \( f \) must be onto, \( F^{-1}(x) \) is nonempty and \( \inf \ F^{-1}(x) = u \).

Since \( \Pi^{-1}(x) \leq 2 \), \( \inf \ F^{-1}(x) = x \) exists, and it is a left limit point (since \( x \) is such a point). However, \( M^x_o > u \) and \( h((-0D,x_o)) \subset (-0D,u) \), which implies \( h \) is not continuous. Contradiction. Hence \( f \) must be closed, and the Proposition is proved.

3.12. From now on let \( P \) be the closed onto maps in LOTS.

We will now look at connectivity properties of ordered absolutes.

An ordered space \( X \) is called ordered extremely disconnected if for any open interval \( V \subset X \), \( \overline{V} \) is open. Obviously, if \( X \) is extremely disconnected, then \( X \) is ordered extremely disconnected.

3.13. EXAMPLE. There is an ordered space \( X \) which is ordered extremely disconnected but not extremely disconnected. Let \( X = \{0\} + 1 \). Then clearly the closure of any open interval in \( X \) is open. However, \( V = (2n|n \in 0) \) is an open set, but \( V \) is not open in \( X \).

3.14. THEOREM. (Fedorchuk). Let \( X \) be an ordered space. Then the following are equivalent:
(1) \(X\) is ordered absolute.

(2) \(X\) has no two sided limit points.

(3) \(X\) is order extremely disconnected.

Proof. First we show that not (3) implies not (2) implies not (1). So assume \(X\) is not order extremely disconnected. Then there exists an open interval \(V \subseteq X\) and \(x \in X\) such that \(x \in V \setminus (X \setminus V)\). Hence \(x\) is a two sided limit point. By splitting \(x\) in two, we obtain a nonisomorphic irreducible map onto \(X\). Thus \(X\) is not an ordered absolute.

Now we show that not (1) implies not (2) implies not (3). So assume \(X\) is not an ordered absolute. Then since \(ax\) is obtained by splitting the two sided limit points in \(X\), then there exists a two sided limit point \(x \in X\). Hence \((x^{ao})\) is an open interval in \(X\), but \((X^{CD})\) is not open. Therefore, \(X\) is not ordered extremely disconnected.

3.15. The next two propositions give equivalences for P-projectives and P-projective covers, most of which are true in any category. With slight alteration these propositions were stated for another topological category with perfect onto maps as P-morphisms by H. Herrlich [H-\(^{2}\) Theorem 4.3].
PROPOSITION. Let \( X \in \text{LOTS} \). Then the following are equivalent:

1. \( X \) is \( p \)-projective.
2. Any \( P \)-morphism \( f: Y \to X \) is a retraction
3. For all \( f \in \text{P}^* \), \( f: Y \to X \), \( f \) is an isomorphism, i.e. \( X \) is an ordered absolute.
4. \( X \) is ordered extremely disconnected.

Proof. For all categories (1) is equivalent to (2) which in turn implies (3). By Theorem 3.14, (3) is equivalent to (4), and by Theorems 3.7 and 3.10, (3) implies (1).

3.16. PROPOSITION. Let \( f: P \to X \in \text{P} \). Then the following are equivalent:

1. \( f: P \to X \) is a \( P \)-projective cover.
2. \( f \in \text{P}^* \) and for all \( g \) such that \( f \in \text{P} \), \( g \) is an isomorphism.
3. \( P \) is \( P \)-projective, and if \( g \in \text{P} \) such that \( gh = f \) and the domain of \( g \) is \( P \)-projective, then \( h \) is an isomorphism.
4. \( P \) is \( P \)-projective and if \( g \) is \( \text{LOTS} \) such that \( gh = f \), \( h \) is onto, and the domain of \( g \) is \( P \)-projective, then \( h \) is an isomorphism.

Proof. In every category (1), (2), and (3) are equivalent. Obviously (3*) implies (3). To finish the proof we show (1) implies (3*). So let \( f: P \to X \) be a \( P \)-projective cover, \( h : P \to Y \)
an onto map, \( Y \) \( P \)-projective, \( g: Y \rightarrow X \), and \( gh = f \). Since \( f \) is irreducible, it is at most a two-to-one map, so \( h \) is at most a two-to-one map. Now it is sufficient to show that \( h \) is a one-to-one map. Suppose this is false. Then there exists \( p, p' \in P \) such that \( f(p) = gh(p) = gh(p') = f(p) \). Hence \( p \) and \( p' \) form a two-sided jump, and since \( g \) is at most a two-to-one map, \( h(p) \) is a two-sided limit point. But this is impossible since \( P \) is an ordered absolute. Thus (1) implies (3*).

3.17. Note that in Proposition 3.16 that (3*) implies that \( g \in P \). Also Lemma 3.9 and Proposition 3.15 imply the last part of Theorem 3.7. For let \( f: X \rightarrow Y \in P^* \). Since \( n_y \in P^* \subseteq P^* \) and \( \alpha X \) is \( P \)-projective, there exists

\[
\begin{array}{ccc}
\alpha X & \xrightarrow{h} & \Pi X \\
| & | & | \\
\downarrow & h & \downarrow \\
X & \downarrow & X \\
\downarrow & \downarrow & \downarrow \\
0.Y & \xrightarrow{\Pi_Y} & Y \\
\end{array}
\]

\( h: AX \rightarrow X \) such that \( II h = \text{fl} \). Thus \( h \) is in \( P^* \), since \( P^* \) is closed under composition. Then it is easy to show \( h \in P^* \).

(In fact in any category if \( g \in P^* \) and \( g \in P^* \) then \( k \in P^* \).) Then since \( \langle XY \rangle \) is an ordered absolute and \( h \) is irreducible, \( h \) is
an isomorphism. Hence the last part of Theorem 3.7 is proved.

Note that the proof above shows that any \( f : X \rightarrow \text{YeP} \) can be lifted to \( f^! : \text{OX} \rightarrow \text{<xY}. \)

Also note that for each \( X \in \text{LOTS}, \) the \( \text{P-protective cover} \)
\( \text{H : <XX} \rightarrow \text{X} \) JLS unique up to isomorphism, since in any category the projective covers are unique.

3.18. In the last part of this section we look at more properties of ordered absolutes. In particular we look at the importance of minimal ordered spaces and the functor \( (B \) in relation to ordered absolutes.

THEOREM. (Fedorchuk). Let \( X \) be an ordered absolute and \( bx \) an ordered compactification of \( X. \) Then \( bx \) is an ordered absolute iff \( bx = ftx. \)

Proof. Necessity. Let \( bx \) be an ordered absolute. Clearly the canonical map \( f: (BX) \rightarrow bx \) is irreducible. So \( f \) is an iso, and hence \( (EX) = bx. \)

Sufficiency. Let \( bx = (EX). \) Since \( X \) is an ordered absolute, it has no two sided limit points. Since \( (EX) \) is constructed by the addition of an ordered pair of points to each cut of \( X, \) \( (EX) \) has no two sided limit points either. Hence \( (EX) \) is an ordered absolute.
3.19. COROLLARY. (Fedorchuk). \( \llcorner \mathbb{B} \mathbb{X} = \mathbb{B} \mathbb{A} \mathbb{X} \). 

Proof. By Theorem 3.16 \( \mathbb{B} \mathbb{A} \mathbb{X} \) is an order absolute. Now there exists a unique \( q : \mathbb{B} \mathbb{A} \mathbb{X} \rightarrow \mathbb{B} \mathbb{X} \) such that \( q_i \mathbb{A} = \mathbb{X} \). 

\[
\begin{array}{c}
\mathbb{B} \mathbb{A} \mathbb{X} \\
\downarrow q \\
\mathbb{B} \mathbb{X}
\end{array} 
\]

To prove the Corollary it is sufficient to show \( q \in \mathbb{P}^* \). Since \( \mathbb{B} \mathbb{A} \mathbb{X} \) is compact, \( q \) is closed. Thus since \( q(\llcorner \mathbb{B} \llcorner \mathbb{X}) \) is a subspace of \( \mathbb{B} \mathbb{X} \), then \( \mathbb{B} \mathbb{X} \) is onto. Since \( \llcorner \mathbb{X} \llcorner \) is dense in \( \mathbb{S} \mathbb{A} \mathbb{X} \), \( q_i \mathbb{A} = \mathbb{X} \) is a subspace of \( \mathbb{B} \mathbb{X} \), and \( q_i \mathbb{A} \) is at most a two-to-one map, then no point of \( \mathbb{B} \mathbb{E} \mathbb{J} \mathbb{X} \llcorner \mathbb{A} \mathbb{X} \) maps to a point of \( \mathbb{X} \) in \( \mathbb{B} \mathbb{X} \), i.e., \( q(\mathbb{B} \mathbb{E} \llcorner \mathbb{X} \llcorner \mathbb{X}) = \mathbb{B} \mathbb{X} \). Hence if \( A \) is closed in \( B \llcorner \mathbb{X} \) and \( q(A) = \mathbb{B} \mathbb{X} \), then \( A \mathbb{Z} \mathbb{D} \mathbb{X} \). So \( B \llcorner \mathbb{X} \Rightarrow A \Rightarrow \mathbb{X} = \mathbb{B} \mathbb{A} X \). Hence \( q \in \mathbb{P}^* \). Since \( \mathbb{P} \)-projective covers are unique \( \mathbb{B} \mathbb{A} \mathbb{X} = \llcorner \mathbb{X} \mathbb{B} \mathbb{X} \). 

3.20. B.V.S. Thomas (T) has formed categorical proofs of the last two theorems using definitions that I do not wish to consider in this paper.
3.21. THEOREM. (Fedorchuk). If \( X \) and \( Y \) are ordered minimal spaces and \( \times X \) is isomorphic to \( \times Y \), then \( X \) is isomorphic to \( Y \).

Proof. Let \( h : aX \to \times Y \) be an isomorphism. We will show that \( f = n h \circ Y \) is an isomorphism from \( X \) to \( Y \). First we show that \( f \) is single valued. Let \( X \subseteq X \). If \( H_1(x) = 1 \), then \( f(x) = 1 \). If \( \Pi_1(x) = 2 \), then \( \Pi_1(x) \) is a two sided jump in \( aX \). Hence \( h \Pi_1(x) \) is a two sided jump in \( \times Y \). Hence if \( \Pi_1(x) \) consists of two points, it is a two sided jump in \( Y \), which is impossible, since \( Y \) is minimal. Hence \( f(x) = 1 \), and so \( f \) is single valued. In a similar manner it is proved that \( f^{-1} \) is single valued, i.e., \( f \) is one-to-one, by using the fact that \( h^{-1} \) is an isomorphism. Since \( Uy \circ h \) and \( Y \) are onto, then \( f \) is onto. Hence \( f \) is an isomorphism, and \( X \) is isomorphic to \( Y \).
BIBLIOGRAPHY


