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Abstract

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. A model of this problem initially has a nonlinear, nonconcave objective. The objective is convexifled and linearized by appropriate transformations, giving a Mixed Integer Linear Program (MILP). The model uses a continuous time representation and discrete distributions for the stochastic parameters. Different representations of the disjunctive scheduling constraints are discussed. A small numerical example is presented, followed by some conclusions.

Keywords

Stochastic Scheduling; MILP Optimization; Research and Development; New Product Development

Introduction

The agricultural chemical and pharmaceutical industries typically use a screening process to test new products. A large number of candidate products are subjected to a group of tasks lesting safety, efficacy, and environmental impact. In the agricultural chemical industry, these tasks might include toxicology studies, soil dissipation studies, plant metabolism studies, and field studies, to name just a few.

Only a small percentage of the potential products pass all the tasks. Because many of these tasks arc regulatory requirements, as soon as a poieniial product fails a lask, all work on ihai produci is terminated. The investment in previous tasks has then been wasted. The tasks should be scheduled with the goal of maximizing the Net Present Value (NPV) of the new products, including the cost of research.

R&D project selection and planning has been extensively studied. Baker (1974), Souder and Mandakovic (1986), and Schmidt and Frceland (1992) all provide reviews. The literature of project scheduling is primarily concerned with minimizing the completion lime of a schedule, sometimes subject to resource constraints. Techniques such as the Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) are often used. Scheduling to maximize NPV has been considered in other contexts by Grinold (1972), Doersch and Patterson (1977), and Elmaghraby and Herroelen (1990). For an overview of stochastic scheduling, see Pinedo (1995). It appears, however, that the problem of finding an optimal, non-sequential schedule for testing of new products has not been previously addressed in the literature.

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Problem Statement

Several potential products must each undergo a scries of tasks, with given durations, costs, and probabilities of successfully passing. Whenever the tasks are completed for a product, a certain income is received, depending on the length of the testing period. The problem is to find a schedule of tasks that maximizes the expected value of the NPV, including both the cost of the testing and the income earned.

The schedule maximizing the total NPV of all products is the same as the schedule maximizing the NPV of each individual new product scheduled separately (unless there are non-renewable resource constraints). The model, therefore, will only consider the project of scheduling a single potential new product. The model should be solved for each potential new product in the R&D portfolio, to obtain an overall schedule.

Some tasks in a project may have technological precedence constraints, meaning that they rely on the results of another task which much be scheduled first. As long as these precedence constraints are respected, there is a great deal of flexibility in the schedule.

Intuitively, it would be best to schedule a task / with a low probability of success p_l first, so that if it fails, it does so with a minimum amount of investment. Similarly, tasks with a high cost c_l should be scheduled later, so they are only performed after more risky tasks have succeeded. The schedule where tasks are performed sequentially has the lowest expected value of cost. In fact, in the absence of any precedence constraints, it can be shown (Mitten, 1960, and Boothroyd, 1960) that the lowest cost sequence is to schedule tasks in order of increasing $c_i / (1 - p^\wedge)$. The analytical results do not, however, seem to generalize beyond this simple case.

A schedule where some tasks are done in parallel can be completed sooner, which will lead to greater sales of the new product. The fundamental trade-off in this scheduling problem is between the greater income from a shorter, parallel schedule and the lower expected value of cost from a longer sequential schedule.

This model is different from most scheduling problems, for it does not provide an explicit schedule with specific start times. Instead, the final product of the model is a set of precedence constraints to add to the technological precedence constraints. This concept of a schedule as a set of precedence constraints was recently used in Fernandez *et al.* (1995). The additional precedence constraints will lower the expected cost of tasks more than they will decrease the expected income by delaying the product introduction.

Some sort of policy is necessary to actually execute a project with these precedence constraints, since the task durations will be stochastic. This paper assumes an early start policy. That is, a task is started as soon as all its precedence constraints have been completed.

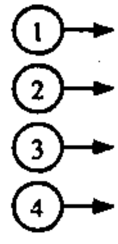
Motivating Example

Consider the problem of scheduling tasks wiih a probability p_x of task *i* succeeding. If any one task fails, then the entire project is terminated, and subsequently scheduled tasks are not performed. The expected value of task / is the cost of the task c;, limes the product of the *pj* of all tasks *j* which have finished before task i begins, or <:J"[/?, •
i

Figure 1 gives a table of the c,- and p; associated with Tour tasks, and two different schedules. The schedule on the left has the highest expected value of cost. All four tasks start immediately, so the expected value is just the total cost of the tasks, or $688,700. The sequence on the right has the lowest possible expected value of the cost. Task 1 is always performed, and if it does succeed then task 2 will be performed. The expected value of cost of task 2 is (O.8O7)($1O5,5OO)=$85.138.50, since task 2 will only be performed 80.7% of the time. Similar calculations apply to tasks 3 and 4. The expected value of this schedule is $458,381. The next section will present a model which more realistically considers both the expected income and the expected cost.

FIGURE 1. Comparison of the expected values of two schedules

	p_i	c_i
1	0.807	S75,500
2	0.775	S105,500
3	0.889	5222,700
4	0.900	S285,000



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S75.500 +
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(0.807)(0.775)(S222,700) +
(0.807)(0.775)(0.889)(S285,000) =
$458,381

ProblemFormulation

The model presented here uses a continuous time representation and a piecewise linear function for income, which decreases with project completion time. Each precedence constraint in the final solution has two effects. It will lower the expected value of cost for some tasks, but will also potentially lengthen the project, and lower the income for some scenarios. The solution of the model provides the optimal trade-off between these two effects.

The costs of the tasks, the durations of the tasks, and the income resulting from introducing a new product are not known at the time the schedule is made. Arbitrary discrete distributions are used to model these uncertainties. Distributions for the parameters can be estimated from historical data, if available, or else subjective probability distributions may be used. A scenario is a set containing one value for each uncertain parameter, pulled from the discrete distributions. The index *k* represents one of the N_k possible scenarios, where each scenario *k* occurs with probability P_k (so $\sum_k P_k = 1$).

Let the indices i_9j_9 and / represent tasks in scenario *k*. Let c_{ik} be the cost of task / in scenario *k*, p_{ik} be the probability of success of task / in scenario *k*, and d_{lk} be the duration of task / in scenario *k*. Let *r* be the discount factor, using continuously compounded interest. Let y,y be a binary variable which is 1 if there is a precedence constraint that task / must finish before task./ can begin, and 0 otherwise. Leu* be a nonnegative, continuous variable representing the overall project completion time in scenario *k*. Let s_{ik} be a nonnegative, continuous variable representing the starting time of task / in scenario *k*. The upper bound on s_{ik} is given by (EQ1) and(EQ2).

$$()\le *_{ik} <£/_{ik} \qquad \forall i,* \tag{EQi}$$

$$U_{ik} = \sum_{j * t} d_{jk} \qquad \forall i, k \tag{EQ2}$$

The income derived from introducing a new product is based on the overall completion time of the project, t_k. Clearly t_k must be greater than the completion time of each task *i* in state *k:*

$$s_{ik} + \wedge_{ik} \lesssim *k \qquad \wedge^{l\#f} * \tag{EQ 3}$$

In this model, income is assumed to be a decreasing piecewise linear function of t_h with the piecewise linear segments defined on the index *m* at times b_m.

$$b_m" *k + u_{km} \gtrsim ® \qquad V * , m \tag{EQ 4}$$

The *Ufa* are nonnegative, continuous variables denoting the excess time of t_k over b_m. This formulation is a 2-stage stochastic program with recourse. The first stage has the *y^* variables which determine the precedence constraints of the schedule. Then, in the second stage the *u^* account lor the decrease in income. The parameter A// is the maximum income from introducing the new product. The parameter/^ gives the decrease in income due to t_k exceeding time b_m in scenario *k*.

The objective of maximizing the expected value of the NPV is equivalent to minimizing the expected cost plus the negative of the expected income. This could be modeled in a nonlinear, nonconcave form as:

$$\min \; -M_t + \sum_k p_k\left(\sum_i c_{ik} e^{-rs_{ik}} \prod_j q_{ijk} + \sum_m f_{km} u_{km}\right) \qquad \text{(EQ 5)}$$

where $q_{ijk} = \begin{cases} 1 & \text{if } y_{ji} = 0 \\ p_{jk} & \text{if } y_{ji} = 1 \end{cases}$. Let $q_{ijk} = e^{v_{ijk}}$ and substitute into (EQ 5), neglecting the constant $-M_t$:

$$\min \sum_k p_k\left(\sum_i c_{ik} e^{\left(-rs_{ik} + \sum_j v_{ijk}\right)} + \sum_m f_{km} u_{km}\right) \qquad \text{(EQ6)}$$

The objective is now equivalent to minimizing the expected cost and decrease in income. Introduce new variables w_{ik} and use the fact that $v_{ijk} = \ln(p_{jk})\, y_{ji}$ to obtain a nonlinear, concave form:

$$\min \sum_k p_k\left(\sum_i c_{ik} e^{w_{ik}} + \sum_m f_{km} u_{km}\right) \qquad \text{(EQ7)}$$

$$w_{ik} = -r\, s_{ik} + \sum_{j \neq i} \ln(p_{jk})\, y_{ji} \qquad \forall i, k \qquad \text{(EQ8)}$$

The nonlinearity of (EQ 7) can be eliminated, since the nonlinear term $e^{w_{ik}}$ involves a single variable. Using the technique of separable programming, the exponential is approximated by $n-1$ piecewise linear segments between the grid points a_{ikn}. The standard lambda formulation is used in (EQ 9) through (EQ 12), which is given in many textbooks such as Nemhauser and Wolsey (1988). The resulting approximation is very close when 9 or 10 grid points a_{ikn} are used.

$$\min \sum_k p_k\left(\sum_i c_{ik} \sum_n \left(e^{a_{ikn}} \lambda_{ikn}\right) + \sum_m f_{km} u_{km}\right) \qquad \text{(EQ9)}$$

$$\sum_n a_{ikn} \lambda_{ikn} = -r\, s_{ik} + \sum_j \ln(p_{jk})\, y_{ji} \qquad \forall i, k \qquad \text{(EQ 10)}$$

$$\sum_n \lambda_{ikn} = 1 \qquad \forall i, k \qquad \text{(EQ11)}$$

$$\lambda_{ikn} \geq 0 \qquad \forall i, k \qquad \text{(EQ 12)}$$

For a pair of tasks (i,j) with $i < j$, there are three possibilities. Either task i starts after task j is complete in all scenarios, task j starts after task i is complete in all scenarios, or the relationship between i and j is undetermined and varies between scenarios. These three possibilities are given in the disjunction of (EQ 13).

$$\begin{bmatrix} y_{ij} \wedge \neg y_{ji} \\ s_{ik} + d_{ik} \leq s_{jk}, \forall k \end{bmatrix} \vee \begin{bmatrix} y_{ji} \wedge \neg y_{ij} \\ s_{jk} + d_{jk} \leq s_{ik}, \forall k \end{bmatrix} \vee \begin{bmatrix} \neg y_{ij} \wedge \neg y_{ji} \end{bmatrix} \quad \forall (i,j) \mid (i < j) \qquad \text{(EQ 13)}$$

The disjunction can be used directly in a Disjunctive Programming (DP) approach to solving the problem, or it can be converted to integer programming constraints. The "Big M" constraint of (EQ 14) is the simplest way to model the disjunction, in terms of the fewest number of variables and constraints. It does suffer, however, from a weak Linear Programming (LP) relaxation.

$$s_{ik} + d_{ik} \le s_{jk} + (U_{ik} + d_{ik})(1 - y_{ij}) \qquad \forall (i, j,$$

Balas (1985) showed that a tighter LP relaxation can be obtained when the variables are disaggregated. The superscripted variables s^1_{ijk}, s^2_{ijk}, and s^3_{ijk} correspond to the three terms in the disjunction. After disaggregation, **(EQ 14) is replaced by (EQ 15) through (EQ 22):**

$$s_{ijk} = s^1_{ijk} + s^2_{ijk} + s^3_{ijk}$$

$$s_{jik} = s^1_{jik} + s^2_{jik} + s^3_{jik} \qquad \forall (i, j, k) \mid (i < j) \qquad \text{(EQ16)}$$

$$s^1_{ijk} - s^1_{jik} \le -d_{ik} y_{ij} \qquad \forall (i, j, k) \mid (i < j) \qquad \text{(EQ17)}$$

$$s^2_{jik} - s^2_{ijk} \le -d_{jk} y_{ji} \qquad \forall (i, j, k) \mid (i < j) \qquad \text{(EQ18)}$$

$$s^1_{jik} \le U_{jk} y_{ij} \qquad \forall (i, j, k) \mid (i < j) \qquad \text{(EQ19)}$$

$$s^2_{ijk} \le U_{ik} y_{ji} \qquad \forall (i, j, k) \mid (i < j) \qquad \text{(EQ20)}$$

$$s^3_{ijk} \le U_{ik} (1 - y_{ij} - y_{ji}) \qquad \forall (i, j, k) \mid (i < j) \qquad \text{(EQ21)}$$

$$s^3_{jik} \le U_{jk} (1 - y_{ij} - y_{ji}) \qquad \forall (i, j, k) \mid (i < j) \qquad \text{(EQ22)}$$

As discussed **in Raman** and Grossmann (1994), a disjunction is **w-MIP** representable if and only if it **has** a convex hull representation without disaggregated variables, and **all** solutions satisfying the disjunction imply integer^. If a disjunction is not w-MIP representable, then it is a weaker representation when modeled as an MILP. Unfortunately, (EQ 13) is not w-MIP reprcsentable.

As previously mentioned, there may be technological precedence coastraints in the problem. These constraints actually make the model easier to solve. If task *i* must precede task), simply fix $y_{ij} = 1$ and $y_{ji} = 0$.

The final schedule of precedence constraints must be acyclic. The constraints of (EQ 23) prevent cycles of length 2 from forming, while equations (EQ 24) and (EQ 25) prevent cycles of length 3. These equations arc not necessary, but they are valid integer cuts which greatly strengthen the LP relaxation of the model.

$$y_{ij} + y_{ji} \le 1 \qquad \forall (i,j) \mid i < j \qquad \text{(EQ 23)}$$

$$y_{ij} + y_{jl} + y_{li} \le 2 \qquad \forall (i,j,l) \mid i < j < l \qquad \text{(EQ24)}$$

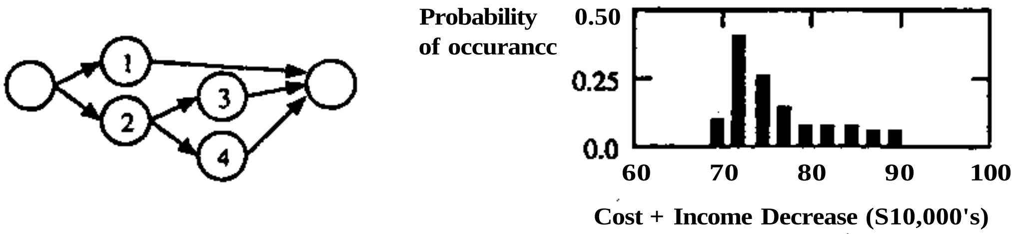
$$y_{ji} + y_{il} + y_{lj} \le 2 \qquad \forall (i,j,l) \mid i < j < l \qquad \text{(EQ25)}$$

Example Continued

Continue with the same four tasks from the previous example. The value of the *M/* parameter does not affect the optimal solution, so it will be set to 0. The problem is then reduced to minimizing the expected value of costs and the decrease in income. The income is decreasing in two piecewise segments with 6j=0, /plCMXX), & 2=12'/2=15,000-The task durations arc stochastic: $d_1 = \{12,13,14\}$ with probability {0.2,0.6, 0.2}, $d_2 = \{4,5,6,9,10,11\}$ with probability {0.3,0.4,0.1,0.075,0.1,0.025}, $d_3=\{4,5,7\}$ with probability {0.3,0.5,0.2}, and $d_4=\{6, 8, 10\}$ with probability {0.25,0.5, 0.25}. If all possible combinations of the durations are considered, then there are N_k = (3)(6)(3)(3) = 162 scenarios. A discount rate r=0.0075 was used, which would correspond to 9% annually, if the durations arc in months.

Figure 2 shows the optimal schedule, and a histogram giving the distribution of the objective. The expected value of the objective was $731,650. For comparison, the parallel schedule of Figure 1 had an expected value of $806,400, while the sequential sequence of Figure 1 was $873,720. The "Big M" formulation took 1410 CPU seconds on an HP712/80 workstation, after examining 67 nodes. The convex hull form took 3214 CPU seconds and 28 nodes. In practice, the convex hull form has a better relaxation. However, that benefit is usually outweighed by the additional variables and constraints used in the disaggregation.

FIGURE 2. Optimal Schedule and Distribution of total schedule cost for Example



Conclusions

This paper introduces an interesting, industrially relevant scheduling problem. The model presented can optimally schedule tasks in the New Product Development process. There is still, however, much room for future research. The model exhibits weak lower bounds, which causes the CPU limes to grow rapidly as the number of tasks increases. Scheduling more than 10 tasks becomes computationally prohibitive. Preliminary work using cutting planes based on minimal cover inequalities to strengthen the lower bounds has been promising (Schmidt and Grossmann, 1995). The major challenge of this problem, then, is to handle the large number of tasks in a real R&D department. There may typically be over 100 tasks for each product. Also, the number of scenarios, Af'» will be huge for industrial problems.

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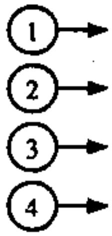
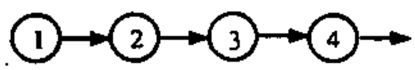


$688,700



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In this model, income is assumed to be a decreasing piecewise linear function of t_k, with the piecewise linear segments defined on the index m at times b_m.

$$b_m + u_{km} \ge t_k \qquad \forall k, m \tag{EQ 4}$$

The u_{km} are nonnegative, continuous variables denoting the excess time of t_k over b_m. This formulation is a 2-stage stochastic program with recourse. The first stage has the y_{ij} variables which determine the precedence constraints of the schedule. Then, in the second stage the u_{km} account for the decrease in income. The parameter M is the maximum income from introducing the new product. The parameter f_{km} gives the decrease in income due to t_k exceeding time b_m in scenario k.

The objective of maximizing the expected value of the NPV is equivalent to minimizing the expected cost plus the negative of the expected income. This could be modeled in a nonlinear, nonconcave form as:

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The objective is now equivalent to minimizing the expected cost and decrease in income. Introduce new variables w_{ik} and use the fact that $v_{ijk} = \ln(p_{jk})\, y_{ji}$ to obtain a nonlinear, concave form:

$$\min \; \sum_k p_k \left(\sum_i c_{ik} e^{w_{ik}} + \sum_m f_{km} u_{km} \right) \qquad \text{(EQ7)}$$

$$w_{ik} = -r\, s_{ik} + \sum_{j \neq i} \ln(p_{jk})\, y_{ji} \qquad \forall i, k \qquad \text{(EQ8)}$$

The nonlinearity of (EQ 7) can be eliminated, since the nonlinear term $e^{w_{ik}}$ involves a single variable. Using the technique of separable programming, the exponential is approximated by $n-1$ piecewise linear segments between the grid points a_{ikn}. The standard lambda formulation is used in (EQ 9) through (EQ 12), which is given in many textbooks such as Nemhauser and Wolsey (1988). The resulting approximation is very close when 9 or 10 grid points a_{ikn} are used.

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$$\sum_n \lambda_{ikn} = 1 \qquad \forall i, k \qquad \text{(EQ11)}$$

$$\lambda_{ikn} \geq 0 \qquad \forall i, k \qquad \text{(EQ 12)}$$

For a pair of tasks (i,j) with $i < j$, there are three possibilities. Either task i starts after task j is complete in all scenarios, task j starts after task i is complete in all scenarios, or the relationship between i and j is undetermined and varies between scenarios. These three possibilities are given in the disjunction of (EQ 13).

$$\begin{bmatrix} y_{ij} \wedge \neg y_{ji} \\ s_{ik} + d_{ik} \leq s_{jk}, \forall k \end{bmatrix} \vee \begin{bmatrix} y_{ji} \wedge \neg y_{ij} \\ s_{jk} + d_{jk} \leq s_{ik}, \forall k \end{bmatrix} \vee \begin{bmatrix} \neg y_{ij} \wedge \neg y_{ji} \end{bmatrix} \quad \forall (i,j) \mid (i < j) \qquad \text{(EQ 13)}$$

The disjunction can be used directly in a Disjunctive Programming (DP) approach to solving the problem, or it can be converted to integer programming constraints. The "Big M" constraint of (EQ 14) is the simplest way to model the disjunction, in terms of the fewest number of variables and constraints. It does suffer, however, from a weak Linear Programming (LP) relaxation.

$$s_{ik} + d_{ik} \le s_{jk} + (U_{ik} + d_{ik})(1 - y_{ij}) \qquad \forall (i, j,$$

Balas (1985) showed that a tighter LP relaxation can be obtained when the variables are disaggregated. The superscripted variables s^1_{ijk}, s^2_{ijk}, and s^3_{ijk} correspond to the three terms in the disjunction. After disaggregation, (EQ 14) is replaced by (EQ 15) through (EQ 22):

$$s_{jik} = s^1_{jik} + s^2_{jik} + s^3_{jik} \qquad \forall (i,j,k) | (i<j) \qquad \text{(EQ16)}$$

$$s^1_{ijk} - s^1_{jik} \le -d_{ik} y_{ij} \qquad \forall (i,j,k) | (i<j) \qquad \text{(EQ17)}$$

$$s^2_{jik} - s^2_{ijk} \le -d_{jk} y_{ji} \qquad \forall (i,j,k) | (i<j) \qquad \text{(EQ18)}$$

$$s^1_{jik} \le U_{jk} y_{ij} \qquad \forall (i,j,k) | (i<j) \qquad \text{(EQ19)}$$

$$s^2_{ijk} \le U_{ik} y_{ji} \qquad \forall (i,j,k) | (i<j) \qquad \text{(EQ20)}$$

$$s^3_{ijk} \le U_{ik}(1 - y_{ij} - y_{ji}) \qquad \forall (i,j,k) | (i<j) \qquad \text{(EQ21)}$$

$$s^3_{jik} \le U_{jk}(1 - y_{ij} - y_{ji}) \qquad \forall (i,j,k) | (i<j) \qquad \text{(EQ22)}$$

As discussed in Raman and Grossmann (1994), a disjunction is w-MIP representable if and only if it has a convex hull representation without disaggregated variables, and all solutions satisfying the disjunction imply integer^. If a disjunction is not w-MIP representable, then it is a weaker representation when modeled as an MILP. Unfortunately, (EQ 13) is not w-MIP reprcsentable.

As previously mentioned, there may be technological precedence coastraints in the problem. These constraints actually make the model easier to solve. If task *i* must precede task), simply fix $y_{ij} = 1$ and $y_{ji} = 0$.

The final schedule of precedence constraints must be acyclic. The constraints of (EQ 23) prevent cycles of length 2 from forming, while equations (EQ 24) and (EQ 25) prevent cycles of length 3. These equations arc not necessary, but they are valid integer cuts which greatly strengthen the LP relaxation of the model.

$$y_{ij} + y_{ji} \le 1 \qquad \forall (i,j) | i<j \qquad \text{(EQ 23)}$$

$$y_{ij} + y_{jl} + y_{li} \le 2 \qquad \forall (i,j,l) | i<j<l \qquad \text{(EQ24)}$$

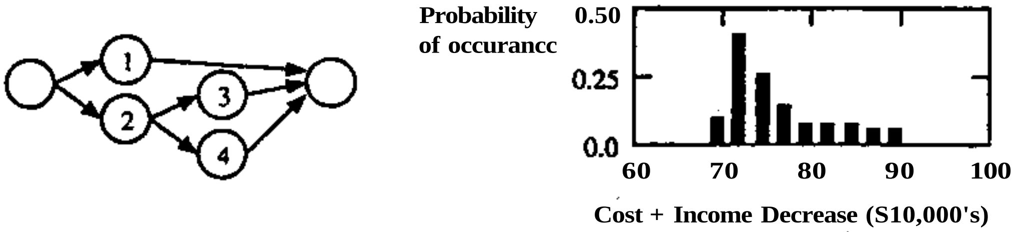
$$y_{ji} + y_{il} + y_{lj} \le 2 \qquad \forall (i,j,l) | i<j<l \qquad \text{(EQ25)}$$

Example Continued

Continue with the same four tasks from the previous example. The value of the *M/* parameter does not affect the optimal solution, so it will be set to 0. The problem is then reduced to minimizing the expected value of costs and the decrease in income. The income is decreasing in two piecewise segments with 6j=0, /plCMXX), & 2=12,/2=15,000. The task durations arc stochastic: d_1 = {12,13,14} with probability {0.2,0.6, 0.2}, d_2= {4,5,6,9,10,11} with probability {0.3,0.4,0.1,0.075,0.1,0.025}, d_3={4,5,7} with probability {0.3,0.5,0.2}, and d_4={6, 8, 10} with probability {0.25,0.5, 0.25}. If all possible combinations of the durations are considered, then there are N_k = (3)(6)(3)(3) = 162 scenarios. A discount rate r=0.0075 was used, which would correspond to 9% annually, if the durations arc in months.

Figure 2 shows the optimal schedule, and a histogram giving the distribution of the objective. The expected value of the objective was $731,650. For comparison, the parallel schedule of Figure 1 had an expected value of $806,400, while the sequential sequence of Figure 1 was $873,720. The "Big M" formulation took 1410 CPU seconds on an HP712/80 workstation, after examining 67 nodes. The convex hull form took 3214 CPU seconds and 28 nodes. In practice, the convex hull form has a better relaxation. However, that benefit is usually outweighed by the additional variables and constraints used in the disaggregation.

FIGURE 2. Optimal Schedule and Distribution of total schedule cost for Example



Conclusions

This paper introduces an interesting, industrially relevant scheduling problem. The model presented can optimally schedule tasks in the New Product Development process. There is still, however, much room for future research. The model exhibits weak lower bounds, which causes the CPU limes to grow rapidly as the number of tasks increases. Scheduling more than 10 tasks becomes computationally prohibitive. Preliminary work using cutting planes based on minimal cover inequalities to strengthen the lower bounds has been promising (Schmidt and Grossmann, 1995). The major challenge of this problem, then, is to handle the large number of tasks in a real R&D department. There may typically be over 100 tasks for each product. Also, the number of scenarios, Af'» will be huge for industrial problems.

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