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## Optimal Control Model of Technology Transition

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# Optimal Control Model of Technology Transition\*

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## Abstract

This paper discusses the use of optimization software to solve an optimal control problem arising in the modeling of technology transition. We set up a series of increasingly complex models with such features as learning-by-doing, adjustment cost, and capital investment. The models are written in continuous time and then discretized by using different methods to transform them into large-scale nonlinear programs. We use a modeling language and numerical optimization methods to solve the optimization problem. Our results are consistent with findings in the literature and highlight the impact the discretization choice has on the solution and accuracy.

## 1 Introduction

Our goal is to compute an optimal transition from conventional (old) to low-emission (new) technology for energy production. The new technology has higher costs but a lower emission rate of greenhouse gases, making it possible to reduce emissions without substantial reductions in energy consumption that would be necessary using only the old technology.

We are interested in the socially optimal output schedules of both technologies; these tell us the best possible scenario that could be achieved if the entire energy industry were controlled by a (benevolent and omniscient) single agency. In reality, the energy industry consists of many independent firms, which have to be motivated by policy measures to adopt the new technology. Versions of our model could generate several important inputs to construct such a policy. First, our model determines the optimal output schedule, including the starting time for the transition to the new technology that serves as the ultimate goal of the policy. Second, our model can be used to compute the amount of needed policy intervention (tax rate, emission quota, etc.)

We seek to develop a model that can provide a realistic transition path. The output of either technology should be a continuous, but not necessarily smooth, function of time. Realistic applications are discussed in the last section.

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A gradual transition is motivated by studies of historic data on actual technology transitions that find the penetration rate of the new technology to be an S-shaped function of time—although it is increasing throughout the entire transition period, the penetration rate is convex during the early stages of transition, then passes through an inflection point, and turns concave as it approaches full adoption; see Jensen (1982) and Geroski (2000). To be consistent with these findings, we develop models with various features such as learning-by-doing in the new technology, which reduces the unit cost as the cumulative output increases; and transition costs, which penalize fast changes in output of the technologies.

Our paper considers the following economic concepts: technology diffusion, environmental policy, and learning-by-doing. Popp (2004) provides a concise overview of these concepts. The diffusion (adoption) literature studies technology adoption by individual firms and focuses on factors that cause firms to adopt at different times, and the shapes of transition paths observed in the past. Jensen (1982) quotes a number of studies that find the initial (convex) stage to be relatively short; sometimes it is not present at all, leading to a kink in the adoption path when the adoption starts and a concave shape after that. Cabral (1990) concentrates on the inflection point of an S-shaped function, arguing that it is an approximation to a discontinuous jump.

Jensen (1982, 1983) assumes some firms need more time than others to convince themselves that adoption is worthwhile; their model generates both S-shaped and concave transition paths. Tonks (1986) has a consumer cautiously discovering a new good, leading to a concave adoption path that overshoots the long-term asymptote. Cabral (1990) assumes some firms will not join until they see adoption by a certain share of the market; he generates discontinuities (vertical segments) in the adoption path. Balcer and Lippman (1984) assume multiple generations of a technology, whereas a firm will not adopt if a new technology does not offer a substantial improvement over the already installed one or if a better technology is expected to be released soon. Abstracting from economics, Geroski (2000) discusses diffusion models based on ideas from epidemiology, sociology, and information dissemination. Benhabib and Spiegel (2005) look at the diffusion of general technological knowledge between countries rather than firms; they find different rates of convergence to the “world leader” and explain them by different levels of education.

A related area of environmental economic research studies government policy measures aimed at inducing firms to behave in a socially optimal way. Milliman and Price (1989) demonstrate (in theory) how frequently suggested policies, such as emission taxes, subsidies, and tradable permits, induce innovation and diffusion of cleaner technology. Newell et al. (2006) provide empirical evidence supporting both the effectiveness of such policy measures and profit maximizing behavior predicted by the diffusion literature.

Woerlen (2004) provides an overview of learning-by-doing, discussing both the theoretical issues and evidence of the experience-driven reduction in capital cost of alternative energy.

In their classic work on correcting environmental externalities, Baumol and Oates (1988) show that an emission tax (i.e., a price on emissions) is Pareto optimal using the standard general equilibrium economic model. The magnitude of the tax should be set equal to the marginal damage

due to the emissions. This environmental price arises in our model from the constraint we impose on discounted greenhouse gas emissions integrated over the time horizon. We let the environmental urgency rate of time preference be  $a$ . From the necessary conditions arising from this constraint, it is easy to show that the optimal price on emissions will grow at a rate  $(r - a)$  where  $r$  is the rate of money time preference, i.e., return on investment in the economy (see Hanson (2007)). However, when we introduce the need in our model for a transition to new low-carbon technology, the greenhouse gas emissions tax is a necessary condition, but not sufficient. Early investment in the new technology will lower present value costs of ultimately switching to that technology. However, early investors will not fully appropriate these profits because knowledge and experience gained from developing and deploying the new technology will have public good characteristics. The social planner optimization problem will take full future benefits into account.

Our results contribute to this literature on both the economic policy and the methodological side. The different models we develop demonstrate different forms of transition behavior, suggesting that optimal transition paths are potentially implementable by policy intervention. From an economic standpoint, we demonstrate that gradual transition paths can be generated without resorting to multiple agents and are socially optimal under reasonable assumptions. We find that learning-by-doing alone leads to a discontinuous instant transition; adding adjustment costs ensures a continuous and smooth transition that starts earlier. The model with capital investment results in a transition path that is continuous and concave after the initial kink.

From a methodological standpoint, our paper offers several important insights. First, we re-examine the use of discrete-time models to approximate the continuous-time model. A typical economic paper assumes discrete time with a period of one year. This decision is driven by the annual nature of input data and an assumption that the length of the period does not affect results. We adopt a more cautious approach: we formulate the model in continuous time, then discretize it using several discretization methods, and vary the period length until all discretization methods converge. In at least one case, we demonstrate how a large period length combined with a discretization method commonly used in economics can lead to a solution that differs from the true solution in a fundamental way. Second, we use modern computational tools to solve our dynamic problems. In the studies discussed above, the optimization is performed analytically, which is possible only for highly stylized models that do not describe actual industries. Numerical methods overcome this constraint and allow us to solve more realistic problems.

In the past, the limited power of early computers and the large scale of dynamic problems forced computational economists to use indirect approaches, most notably replacing an optimization problem with a system of optimality conditions, and applying Gauss-Jacobi or Gauss-Seidel iterations by updating the vector of variables one component at a time. However, these methods are notoriously slow and unreliable. Moreover, using (first-order) optimality conditions requires careful assumptions about curvature. We solve the original optimization problem instead, avoiding the need both to derive optimality conditions and to implement the solution algorithm. The combination of modern computers and optimization algorithms allows us to solve problems with

thousands of variables in seconds.

Moreover, we demonstrate the ease with which a high-level modeling languages such as AMPL (Fourer et al., 2003), can be used to experiment with the models. Unlike programming languages (C, Fortran), the modeling language allows the users to specify the model in its original algebraic form.

The rest of the paper is structured as follows. Section 2 develops the various models. Section 3 explains the discretization method. Section 4 presents the results, and Section 5 discusses possible extensions.

## 2 Model Description and Background

We solve the social planner’s problem of maximizing the social welfare, with the condition that total accumulated emissions at a certain point in time will not exceed the specified level. Our variables are energy output schedules of old and new technology, where the new technology has lower emissions but is more expensive, although its cost is expected to decrease with wider adoption. We formulate a number of models with increasing detail and features.

### 2.1 Common Economic Components and Parameters

We first define the components common to all three models. Specific functional forms are chosen for demonstration; however, our approach is not limited to these, and other choices are possible.

**Time.** All our models are dynamic, with continuous time and finite horizon:  $t \in [0, T]$ . We denote functions of time as  $x(t)$  and their derivatives as

$$\dot{x}(t) = \frac{dx(t)}{dt}.$$

We use continuous discounting with the rate  $r > 0$ .

**Energy Output.** There are two technologies, old and new, and their energy output at time  $t$  is denoted  $q^o(t)$  and  $q^n(t)$ , respectively. We also define the total output to be  $Q(t) = q^o(t) + q^n(t)$ .

**Demand.** The benefit of energy to society is represented by the consumer’s willingness to pay  $\tilde{S}(Q, t)$ , computed as the integral of demand and scaled by the demand growth rate (hence the dependence on time). In our models, we use the following functional form for the consumer’s willingness to pay:

$$\tilde{S}(Q, t) = e^{bt} S(Qe^{-bt}), \tag{2.1}$$

where  $b > 0$  is the growth rate of demand. The formula for  $S(Q)$  is derived in Appendix A:

$$S(Q) = \int p(q) dq \Big|_{q=Q} = \int \frac{S_0}{q^\sigma} dq \Big|_{q=Q} = \begin{cases} S(Q) = S_0 \ln Q, & \text{if } \sigma = 1 \\ \frac{S_0}{1-\sigma} Q^{1-\sigma}, & \text{otherwise,} \end{cases}$$

where  $\sigma > 0$  is the demand parameter. The functional form of (2.1) is due to the fact that the growth factor  $e^{bt}$  is applied to the direct demand function  $q(p)$  rather than the inverse demand  $p(q)$ .

**Production Costs.** We assume constant marginal costs. Each unit of energy is produced with the old technology costs  $c_o$ . The cost of the new technology is subject to learning-by-doing; that is, the unit cost  $c_n(x(t))$  is a decreasing function of cumulative output, which we define as

$$x(t) = \int_0^t q^n(\tau) d\tau. \quad (2.2)$$

Following the economic literature (Woerlen, 2004) and discussion in Appendix A, we let

$$c_n(x) = c_n^0 \left[ \frac{x}{\bar{X}} + 1 \right]^{\log_2 \gamma}, \quad (2.3)$$

where the parameters  $\bar{X}$  and  $\gamma$  are described in Table 1.

**Greenhouse Gases Emissions.** Producing energy generates greenhouse gases at the unit rate of  $b_o > 0$  with the old technology and  $b_n \in (0, b_o)$  with the new technology. We are interested in limiting cumulative emissions at the end of the modeling period. Since earlier emissions do more damage (for example, by the irreversible melting of glaciers), we discount the emissions at an environmental time preference rate  $a$ . We use  $a \in [0, r]$ , but there also may exist justifications for  $a > r$ . The constraint on cumulative emissions is

$$\int_0^T e^{-at} (b_o q^o(t) + b_n q^n(t)) dt \leq z_T. \quad (2.4)$$

**Units and Parameters.** We make our definitions more precise by setting specific units and parameter values. The quantities ( $q(t)$ 's,  $x(t)$ ) are in “quads,” or billion of millions ( $10^{15}$ ) of BTUs<sup>1</sup>; monetary amounts (objective,  $S(Q)$ , etc.) are in billions of dollars; and emissions are measured in billion tons of carbon (tC). See Table 1.

The emission cap is computed as  $z_T = (1 - \zeta) Z_{max}$ , where  $Z_{max}$  equals the cumulative emissions when they are not constrained (which naturally leads to zero utilization of the new technology). The cost parameters  $\gamma$  and  $\bar{X}$  were selected to achieve  $c_n(x(T)) \approx 30$ , a 60% reduction in unit cost by the end of the modeling period (but still more expensive than the old technology with its unit cost of 20). The demand scale is calibrated to yield a current output level:  $q^o(0) = 60$ .

## 2.2 Model I: Basic Model

The first model takes into account only the effect of learning-by-doing and selects the energy output schedules to maximize the discounted welfare without exceeding the emission cap. The model is

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<sup>1</sup>British thermal unit (BTU) is the unit of energy used in power industry. It is equal to 1,055 joules.

Table 1: Parameter values common to all models

Parameter	Unit	Notation	Value
Discount rate	-	$r$	0.05
Demand exponent	-	$\sigma$	2.0
Demand scale	\$B	$S_0$	98,000
Demand growth rate	-	$b$	0.015
Environmental rate	-	$a$	0.02
Emissions, old tech.	tC/mBTU	$b_o$	0.02
Emissions, new tech.	tC/mBTU	$b_n$	0.001
Unconstrained emissions	BtC	$Z_{\max}$	61.9358
Emission reduction %	-	$\zeta$	0.5
Production cost, old tech.	\$/mBTU	$c_o$	20
Starting cost, new tech.	\$/mBTU	$c_n^0$	50
Learning rate	-	$\gamma$	0.85
Initial experience	quad	$x_0$	0
Experience unit size	quad	$\bar{X}$	300

an optimal control problem. Energy output amounts  $q^o(t), q^n(t)$  are the controls, and the state variables are the experience level  $x(t)$  and the accumulated emissions  $z(t)$ :

$$\begin{aligned} & \underset{\{q^o, q^n, x, z\}(t)}{\text{maximize}} && \int_0^T e^{-rt} \left[ \tilde{S}(q^o(t) + q^n(t), t) - c_o q^o(t) - c_n(x(t)) q^n(t) \right] dt && (2.5a) \end{aligned}$$

$$\text{subject to} \quad \dot{x}(t) = q^n(t), \quad x(0) = x_0 = 0 \quad (2.5b)$$

$$\dot{z}(t) = e^{-at} (b_o q^o(t) + b_n q^n(t)), \quad z(0) = z_0 = 0 \quad (2.5c)$$

$$z(T) \leq z_T \quad (2.5d)$$

$$q^o(t) \geq 0, \quad q^n(t) \geq 0. \quad (2.5e)$$

The objective (2.5a) is discounted welfare, computed as the difference between consumer's willingness to pay and production cost. Constraint (2.5b) defines the cumulative output  $x(t)$  and is a transformation of (2.2) into a differential equation; this transformation has the advantage that the discretized equations are sparse, allowing us to use the large-scale nonlinear programming (NLP) solvers. Constraints (2.5c) and (2.5d) represent the cumulative emission cap (2.4); (2.5c) is again a differential constraint that defines  $z(t)$  to be cumulative emissions at time  $t$ , and (2.5d) imposes the cap. Constraint (2.5e) requires that output amounts be nonnegative. We do not need to impose a nonnegativity constraint on  $x(t)$  because (2.5b) and (2.5e) guarantee it.

### 2.3 Model II: Adjustment Cost

Model I is rather simplistic and does not take adjustment costs into account. As a consequence, the resulting transition shows an instantaneous switch, or bang-bang control. Hence, we refine the model by imposing an adjustment cost on the increase in new energy output,  $q^n(t)$ . Keeping the

rest of model components the same, we define

$$y(t) = \max\{0, \dot{q}^n(t)\}, \quad (2.6)$$

the positive change in  $q^n(t)$ . We use it to penalize the rate of change in the output of the new technology by subtracting the adjustment cost  $c_y y^2(t)$  from the objective. The power on  $y(t)$  makes the adjustment cost convex to reflect the difficulty of rapid change. The only new parameter is the scale of the adjustment cost, which we set as  $c_y = 10$ . The model is given by

$$\begin{aligned} & \underset{\{q^o, q^n, x, z, y\}(t)}{\text{maximize}} && \int_0^T e^{-rt} \left[ \tilde{S}(q^o(t) + q^n(t), t) - c_o q^o(t) - c_n(x(t)) q^n(t) - c_y y^2(t) \right] dt \end{aligned} \quad (2.7a)$$

$$\text{subject to} \quad \dot{x}(t) = q^n(t), \quad x(0) = x_0 = 0 \quad (2.7b)$$

$$\dot{q}^n(t) \leq y(t), \quad q^n(0) = 0 \quad (2.7c)$$

$$0 \leq y(t) \quad (2.7d)$$

$$\dot{z}(t) = e^{-at} [b_o q^o(t) + b_n q^n(t)], \quad z(0) = z_0 = 0 \quad (2.7e)$$

$$z(T) \leq z_T \quad (2.7f)$$

$$q^o(t) \geq 0, \quad q^n(t) \geq 0. \quad (2.7g)$$

We retain all the model components from the previous subsection; objective (2.7a) now includes the adjustment cost term  $c_y y^2(t)$ .

The new constraints are (2.7c) and (2.7d), which are an equivalent way of stating (2.6) because the objective term ensures that at least one of the constraints (2.7c)–(2.7d) does bind. The initial condition on  $q^n(0)$  is necessary for a well-defined problem but does not affect the solution because we have  $q^n(t) = 0$  for sufficiently small  $t$ 's.

## 2.4 Model III: Capital Investment

The penalty on the rate of increases in output is a somewhat artificial construct. The resulting adjustment cost was motivated by the need to build up capacity in the new technology. Here, we model this capital investment directly. To simplify the notation, we index the technologies by  $j \in \{o, n\}$ .

**Capital and Investment.**  $K^j(t)$  is the amount of capital in technology  $j$  at time  $t$ . It is increased by an investment  $I^j(t)$  and depreciates at a constant annual rate of  $\delta \in (0, 1)$ . We denote the initial capital level as  $\bar{K}_0^j$ .

**Costs.** Instead of the constant unit costs ( $cq$ ) used in the previous models,  $C^j(q, K)$  now denotes the total variable cost of producing energy quantity  $q$  with technology  $j$  and capital  $K$ . We expect  $C^j(q, K)$  to be increasing and possibly convex in  $q$  (increasing marginal costs) and decreasing and concave in  $K$  (decreasing marginal productivity of capital). We also expect  $C^o(q, K) < C^n(q, K)$  to

reflect the higher costs of the new technology. As explained in Appendix A, the specific functional form that satisfies these conditions can be derived from the Cobb-Douglas production function:

$$C^j(q, K) = A_j \left[ \frac{q}{K^{\alpha_j}} \right]^{1/\beta_j}, \quad (2.8)$$

where  $\alpha_j, \beta_j \in (0, 1)$  and  $\alpha_j + \beta_j = 1$  represent relative intensities of capital and other inputs and  $A_j > 0$  is a generic scaling factor.

Further, we assume that the costs of new technology are affected by the learning:

$$\tilde{C}^n(q, K, x) = \left[ \frac{x}{\bar{X}} + 1 \right]^{\log_2 \gamma} C^n(q, K).$$

Table 2: Additional parameter values: Model III (capital)

Parameter	Unit	Notation	Value
Old tech. scale	\$B	$A_o$	36.52
New tech. scale	\$B	$A_n$	91.30
Old tech. capital exponent	-	$\alpha_o$	0.2
New tech. capital exponent	-	$\alpha_n$	0.2
Old tech. other exponent	-	$\beta_o$	0.8
New tech. other exponent	-	$\beta_n$	0.8
Old tech. initial capital	-	$\bar{K}_0^o$	1900
New tech. initial capital	-	$\bar{K}_0^n$	1
Depreciation rate	-	$\delta$	0.1
Unconstrained emissions	tC	$Z_{\max}$	61.4698

**Parametrization.** The additional parameter values are listed in Table 2. The cost functions  $C^j(q, K)$  and capital accumulation process  $(\delta, f(I), K_0^j)$  are calibrated to match statistics of the basic model:  $c^o(0) = 20$ ,  $c^n(T) \approx 30$ ,  $q^o(0) \approx 60$ .

$K_0^n = 1$  is a token level to avoid division by zero in the cost function; setting  $K_0^o$  much higher implies that we start out with fully developed old technology.  $Z_{\max}$  is changed because, despite our best efforts at calibrating the models to be similar, the different structure of the capital model implies a change to the optimal output level when emission are unconstrained (which is how  $z_T$  is determined).

**Model.** To keep the notation concise, we define several aggregate variables.

$$\begin{aligned} Q(t) &= q^o(t) + q^n(t) & C(t) &= C^o(q^o(t), K^o(t)) + \tilde{C}^n(q^n(t), K^n(t), x(t)) \\ I(t) &= I^o(t) + I^n(t) & K(t) &= K^o(t) + K^n(t) \end{aligned}$$

The new optimal control problem is given by

$$\begin{aligned} & \underset{\{q^j, K^j, I^j, x, z\}(t)}{\text{maximize}} && \left\{ \int_0^T e^{-rt} \left[ \tilde{S}(Q(t), t) - C(t) - I(t) \right] dt + e^{-rT} K(T) \right\} \end{aligned} \quad (2.9a)$$

$$\text{subject to} \quad \dot{x}(t) = q^n(t), \quad x(0) = x_0 = 0 \quad (2.9b)$$

$$\dot{K}^j(t) = -\delta K^j(t) + I^j(t), \quad K^j(0) = \bar{K}_0^j \quad j \in \{o, n\} \quad (2.9c)$$

$$\dot{z}(t) = e^{-at} [b_o q^o(t) + b_n q^n(t)], \quad z(0) = z_0 = 0 \quad (2.9d)$$

$$z(T) \leq z_T \quad (2.9e)$$

$$q^j(t) \geq 0, \quad j \in \{o, n\} \quad (2.9f)$$

$$I^j(t) \geq 0, \quad j \in \{o, n\}. \quad (2.9g)$$

Our objective function (2.9a) now computes production costs including investment. The new term ( $e^{-rT}K(T)$ ) is used to avoid the terminal effects that arise because we use a finite-horizon approximation to an infinite-horizon problem. Without this correction, we would see the investment fall to zero as we approach  $t = T$ , causing a decline in capital and subsequent slowdown or decline in  $Q$ . The correction method we use can be interpreted as getting back the investment at the end of modeling period. This is not the only possible correction; see Barr and Manne (1967) and Lau et al. (2002) for a discussion of correction terms.

Constraint (2.9c) defines the law of motion for the capital—it is increased by investment and decreased by depreciation. Constraints (2.9d)–(2.9f) are identical to previous models; (2.9g) adds nonnegativity constraint on investment. While (2.8) requires  $K^j(t) > 0$ , we do not impose that constraint because (2.9c) and (2.9g) ensure that it holds.

### 3 Discretized Optimal Control Problem

In practice, we cannot expect to solve the optimal control problems like (2.5), (2.7), and (2.9) explicitly. We therefore approximate them by a finite-dimensional problem obtained by discretizing the time-dependent functions and evaluating the discretization at a finite number of points. This approach is referred to as collocation and results in NLPs that we can solve using large-scale optimization methods; see e.g. by Betts (2001).

We discretize time  $t \in [0, T]$  by  $N + 1$  equally spaced points  $t_i = ih$ , where  $h := T/N$  is the step size and  $i = 0, 1, 2, \dots, N$ . Our convention is that  $q_i^n$  approximates  $q^n(t)$  at  $t_i$ , namely,  $q_i^n = q^n(t_i) = q^n(ih)$ , with a similar convention for other variables.

Next, we discretize the differential equations such as (2.5b) by applying one of three Runge-Kutta methods and collocating at the discretization points,

$$x_{i+1} = x_i + hq_i^n, \quad x_{i+1} = x_i + hq_{i+1}^n, \quad \text{and} \quad x_{i+1} = x_i + h \frac{q_i^n + q_{i+1}^n}{2},$$

for the explicit Euler scheme, the implicit Euler scheme, and the trapezoidal method, respectively. We note that the implicit Euler method and the trapezoidal method are implicit schemes. Since

we are solving optimization problems, however, the complexity is not significantly increased. The Euler methods are both first-order accurate, while the trapezoidal rule is second-order accurate. For illustration, we use explicit Euler for Model I, implicit Euler for Model II, and trapezoidal for Model III. We note, however, that our AMPL models are set up so that each model can use any discretization.

### 3.1 Discretization of Model I (Basic)

The model variables are  $\{q_i^o, q_i^n, x_i, z_i\}_{i=0}^N \in \mathbb{R}^{4(N+1)}$ . To simplify the model statement, we define

$$\begin{aligned} W_i &= e^{-rhi} \left[ \tilde{S}\{q_i^o + q_i^n, t_i\} - c_o q_i^o - c_n(x_i) q_i^n \right] \\ g_i &= e^{-ahi} [b_o q_i^o + b_n q_i^n]. \end{aligned}$$

With the explicit Euler discretization, the continuous problem (2.5) becomes

$$\begin{aligned} \underset{q_i^o, q_i^n, x_i, z_i}{\text{maximize}} \quad & h \sum_{i=0}^{N-1} W_i \end{aligned} \tag{3.1a}$$

$$\text{s.t.:} \quad x_{i+1} = x_i + h q_i^n, \quad i = 0, \dots, N-1; \quad x_0 = 0 \tag{3.1b}$$

$$z_{i+1} = z_i + h g_i, \quad i = 0, \dots, N-1; \quad z_0 = 0 \tag{3.1c}$$

$$z_N \leq z_T \tag{3.1d}$$

$$q_i^o \geq 0, \quad q_i^n \geq 0, \quad i = 0, \dots, N. \tag{3.1e}$$

### 3.2 Discretization of Model II (Adjustment Cost)

The model variables are  $\{q_i^o, q_i^n, x_i, z_i, y_i\}_{i=0}^N \in \mathbb{R}^{5(N+1)}$ , and the objective term becomes

$$W_i = e^{-ihr} \left\{ \tilde{S}\{q_i^o + q_i^n, t_i\} - c_o q_i^o - c_n(x_i) q_i^n - c_y y_i^2 \right\},$$

where  $y_i$  is the discretization of  $y(t)$ . The implicit Euler discretization of model (2.7) is given by

$$\underset{q_i^o, q_i^n, x_i, z_i, y_i}{\text{maximize}} \quad h \sum_{i=0}^{N-1} W_i \tag{3.2a}$$

$$\text{s.t.:} \quad x_{i+1} = x_i + h q_{i+1}^n, \quad i = 0, \dots, N-1; \quad x_0 = 0 \tag{3.2b}$$

$$q_{i+1}^n \leq q_i^n + h y_{i+1} \quad i = 0, \dots, N-1; \quad q_i^n = 0 \tag{3.2c}$$

$$0 \leq y_i, \quad i = 0, \dots, N-1; \tag{3.2d}$$

$$z_{i+1} = z_i + h g_{i+1}, \quad i = 0, \dots, N-1; \quad z_0 = 0 \tag{3.2e}$$

$$z_N \leq z_T \tag{3.2f}$$

$$q_i^o \geq 0, \quad q_i^n \geq 0, \quad i = 0, \dots, N. \tag{3.2g}$$

The differential inclusion (3.2c) is a new type of constraint, but it poses no difficulty for our optimization method.

### 3.3 Discretization of Model III (Capital Investment)

The model variables now are  $\{q_i^j, I_i^j, K_i^j, x_i, z_i\}_{i=0}^N \in \mathbb{R}^{7(N+1)}$ , and the new objective term is

$$W_i = e^{-ihr} \left\{ \tilde{S}\{q_i^o + q_i^n, t_i\} - C^o(q_i^o, K_i^o) - C^n(q_i^n, K_i^n) - I_i^o - I_i^n \right\}.$$

We also define

$$k_i^j := -\delta K_i^j + I_i^j.$$

The trapezoidal discretization of model (2.9) is

$$\begin{array}{l} \text{maximize} \\ \{q_i^j, K_i^j, I_i^j, x_i, z_i\} \end{array} \left\{ h \sum_{i=0}^{N-1} W_i + e^{-rNh} (K_N^o + K_N^n) \right\} \quad (3.3a)$$

$$\text{subject to} \quad x_{i+1} = x_i + h \frac{q_i^n + q_{i+1}^n}{2} \quad x_0 = 0 \quad (3.3b)$$

$$K_{i+1}^j = K_i^j + h \frac{k_i^j + k_{i+1}^j}{2}, \quad K_0^j = \bar{K}_0^j \quad j \in \{o, n\} \quad (3.3c)$$

$$z_{i+1} = z_i + h \frac{g_i + g_{i+1}}{2}, \quad z_0 = 0 \quad (3.3d)$$

$$z(T) \leq z_T \quad (3.3e)$$

$$q_i^j \geq 0, \quad j \in \{o, n\} \quad (3.3f)$$

$$I_i^j \geq 0, \quad j \in \{o, n\}. \quad (3.3g)$$

## 4 Numerical Solution and Results

All models are coded in the AMPL modeling language (Fourer et al., 2003). and solved by using KNITRO (Byrd et al., 2006). Unless specified otherwise, we use trapezoidal discretization with  $h = 0.1$ . The AMPL code is available online and is briefly described in Appendix B.

Several advantages accrue from attacking the problem directly as an optimal control problem, instead of following the Hamiltonian formalism. For example, we can easily include bounds on the controls in the optimal control problem. Further, tackling the optimization directly gives the numerical algorithms a better chance at finding maximizers instead of minimizers, or other stationary points, that are not distinguishable from the first-order conditions.

Figure 1 presents the optimal output quantities ( $q^j(t)$ ) for our three models. We achieve transition in all cases, in the sense that  $q^n(0) = 0$  and  $q^o(T) = 0$ . The common parametrization of demand ensures that  $q^o(0)$  is similar for all models. Looking at  $q^n(T)$ , we see that the addition of adjustment costs in Model II predictably decreases it; Model III is not directly comparable to the other models.

Our key interest is the shape of the output schedules. Model I shows an instant transition;

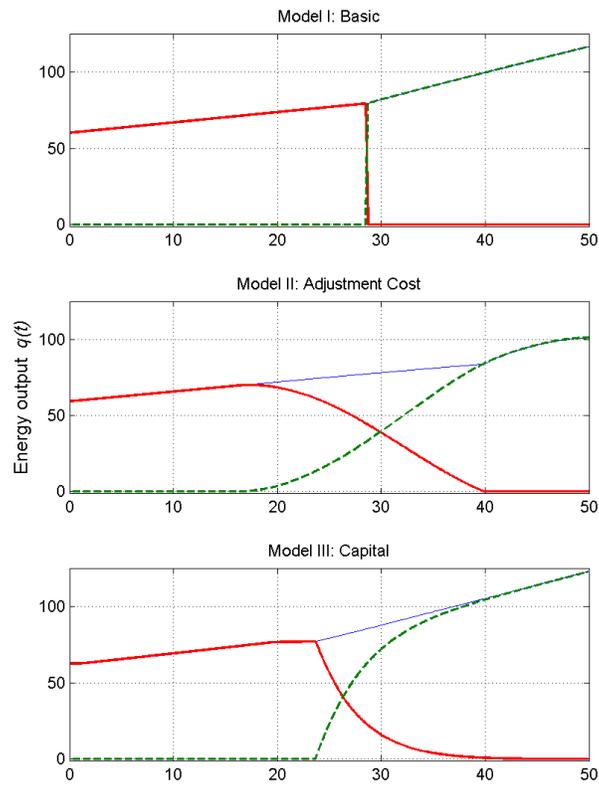


Figure 1: Optimal energy output schedules ( $q^o(t), q^n(t)$ ) for the three models. Emission reduction rate  $\zeta = 0.5$ .

that is, there is no point in time where both  $q^o(t)$  and  $q^n(t)$  are positive. This is clearly not a realistic model. Model II has a continuous and smooth S-shaped transition, which is one of the common shapes of transition paths observed in the past (Jensen, 1982). Model III has a continuous transition with a kink at the start, followed by a concave segment, which is another historic shape described by Jensen (1982). Growth in total output is virtually unaffected by the transition, and appears to be linear, which is consistent with what we have seen in Model I.

Table 3: Computation time for various discretization methods and period lengths ( $h$ ).

	Model I			Model II			Model III		
Step length $h$	1.0	0.5	0.1	1.0	0.5	0.1	1.0	0.5	0.1
Number of variables	200	400	2000	249	499	2499	400	800	4000
Number of constraints	100	200	1000	149	299	1499	202	402	2002
Computation time, s									
Explicit Euler	0.040	0.168	3.487	0.141	0.443	7.191	0.052	0.105	0.642
Implicit Euler	0.092	0.460	3.483	0.143	0.422	8.191	0.053	0.108	0.647
Trapezoidal	0.058	0.196	3.841	0.218	0.445	10.400	0.060	0.119	0.684

Table 3 shows the computation time for the three models and various discretization methods and step lengths. We see that step length is the main determining factor of the computation time. This result is expected because  $h$  determines the number of periods in the discretization scheme ( $N = T/h$ ) and hence the number of variables and constraints. Given the same  $h$ , the discretization method does not matter nearly as much; moreover, no one method turns out to be the fastest across all step lengths. In the remainder of the section, we discuss the results of models I and III.

#### 4.1 Discussion of Model I (Basic)

Figure 2 presents the solution to Model I given different discretizations methods and period lengths ( $h$ ). We present the output only for the new technology, because the output for the old technology evolves in a symmetric fashion.

Clearly, the “smooth” transition that we observe for larger period lengths is an artifact of the discretization. As  $h \rightarrow 0$ , the solution approaches the instantaneous transition, which corresponds to the solution of the continuous-time model (2.5). We note that Euler methods carry a discretization error proportional to  $O(h)$ , making solutions with  $h = 1$  or coarser time-steps questionable.

Further, the explicit Euler method results in qualitative error; while the trapezoidal and implicit Euler methods achieve transition in a single time-step, the explicit Euler method takes 2–3 periods, creating the illusion of a smooth transition. The explicit Euler method stands out because the two other discretization methods are both implicit. Economically, an implicit method means that the firms begin to enjoy the benefits of learning before the period is over; that is  $q_i^n$  affects the total cost ( $c_n(x_i)q_i^n$ ) both directly and through  $x_i$ . Numerically, we prefer the use of implicit methods because they are better suited to stiff differential equations and differential algebraic

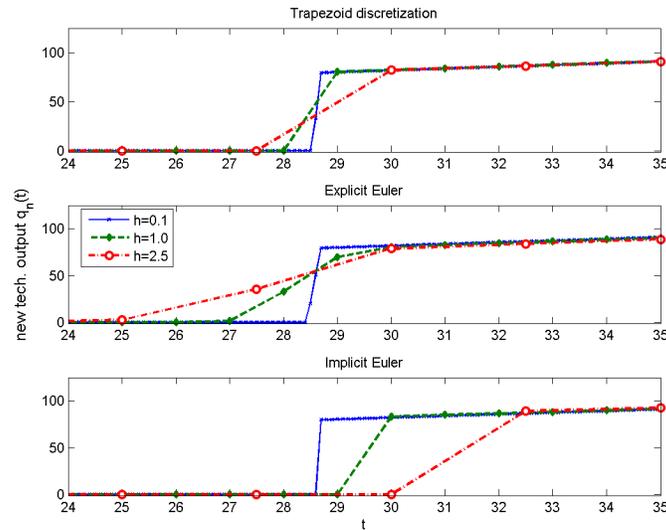


Figure 2: Model I (Basic). Optimal output for the new technology ( $q^n(t)$ ) for various discretization methods and period lengths ( $h$ ).

equations that are found in optimal control problems.

## 4.2 Discussion of Model III (Capital)

Figure 3 presents the detailed solution to Model III, which is substantially different from the other ones. The total energy output continues to increase throughout the transition from old to new technology. This can be achieved despite a period where there is investment in neither old nor new technology by increasing the utilization of existing capacity and using price to mitigate demand growth during the transition. The finite-horizon correction successfully eliminates terminal effects.

The investment schedule is discontinuous for both technologies, which seems reasonable, unlike the jump in outputs. Prior to the start of transition, there is period of zero investment in both technologies, as the model waits for the old capital to depreciate before replacing it with the new technology. Once the transition starts, investment into the new technology accelerates as the model tries to ramp up output to reap benefits of learning-by-doing.

The ease of computing the solutions allows us to study the impact of parameter values. For example, we can investigate the sensitivity to the emission reduction percentage by changing  $\zeta$  illustrated in Figure 4. The left panel shows that smaller emission reductions require lower use of new technology and permit the transition to happen later. The right panel shows the social cost of reducing emissions. The convex shape is consistent with increasing marginal abatement costs commonly observed in economics.

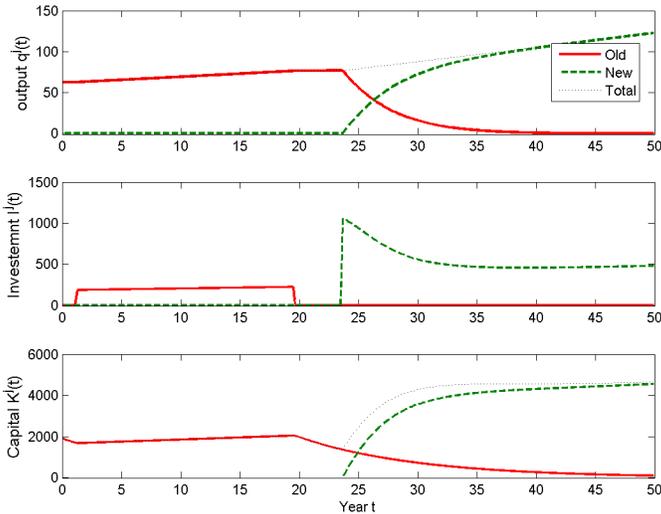


Figure 3: Model III (Capital). Optimal output, investment, and capital given 50% reduction in emissions.

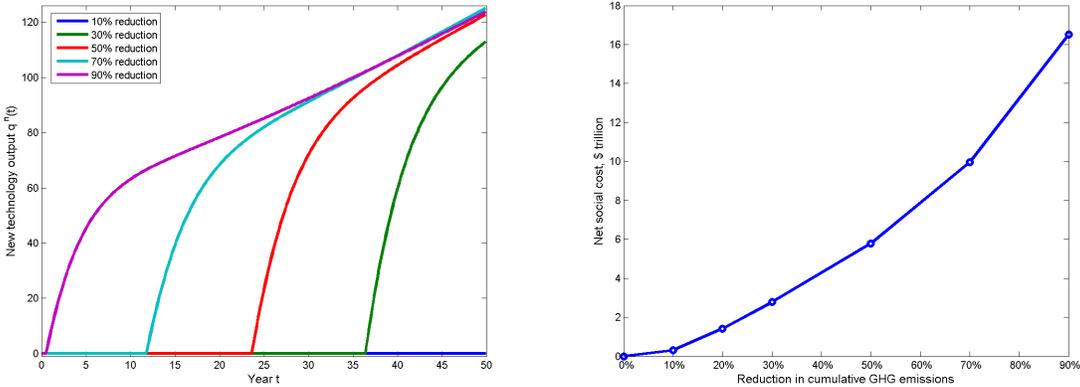


Figure 4: Model III (Capital). Left: the optimal  $q^n(t)$  for various reductions targets  $\zeta$ . Right: the corresponding abatement costs. the  $x$ -axis plots  $\zeta$ , the  $y$ -axis the reduction in maximized objective from the case of  $\zeta = 0$ .

## 5 Comments and Extensions

The primary goal of this paper is to develop solution methods for use in policy work. To do this, we will need to expand the model to include realistic features such as additional technologies and capital conversion. Our modeling approach allows us to add new technologies with minimal effort. For example, Figure 5 presents the solution to the capital model that includes a third technology, say, nuclear fusion, that is emission free but even more expensive ( $A_3 > A_n$ ). We observe that industry first transitions to the intermediate technology but eventually adopts the cleanest one.

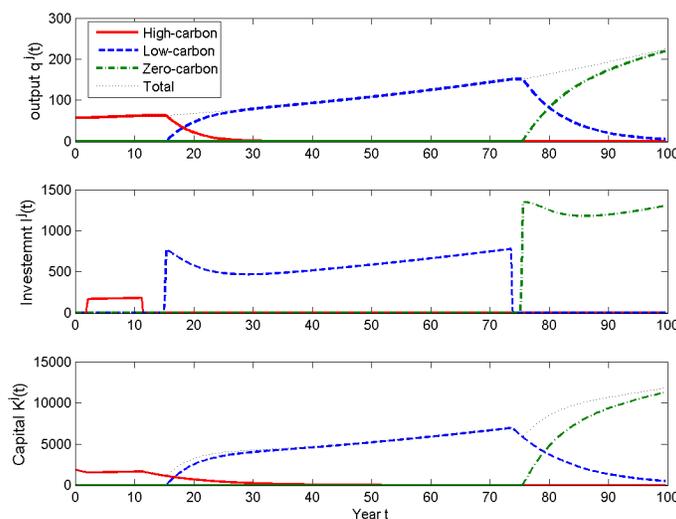


Figure 5: Capital model III with three technologies. Optimal output, investment, and capital given 80% reduction in emissions.

We note that our solutions have few areas of large curvature and are relatively smooth elsewhere. On such smooth segments, we can achieve adequate precision with relatively large periods, which means fewer variables and increased computation speed. But we still need a small step length during the transition period. Adaptive mesh refinement methods have been developed to automatically identify the areas where extra precision is needed. Betts (2001) describes how the computation time can be reduced by a factor of 2.

Another direction for further development of the model is the combination of learning-by-doing, transition costs, and capital accumulation in a single model but in a more industry-specific way. As Woerlen (2004) noted, learning-by-doing effects apply to the production of equipment, rather than to the energy generation itself. This fact suggests that experience should reduce investment rather than production cost, replacing  $I^j$  in (2.9a) with an investment cost function  $f(I^j, K^j)$  that is increasing in  $I$  but decreasing in  $K$ .

Further, it may be reasonable to allow the (costly) conversion of capital from one technology to the other (e.g., from fossil to biofuel). To avoid instant conversion, we could constrain the

conversion rate from above or make the conversion costs convex. We might, however, need to take into account the shutdown time necessary to perform the conversion.

The methodology demonstrated in this paper can be used to solve medium to large scale optimization problems, without needing to solve difficult nonlinear systems of model equations and necessary conditions for the dynamic costate variables (shadow prices). Applications using the methodology can provide insight on important energy and climate policy questions pertaining to low-carbon electricity and fuels production. Coal is the largest source of electricity and also the most carbon intensive. To continue to use coal in a carbon constrained world requires carbon capture and storage (CCS). The near-term least cost technology is to retrofit existing plants with CO<sub>2</sub> capture. However, CO<sub>2</sub> capture and pipeline compression use a large amount of electricity, resulting in low overall efficiency. Advanced technologies with CO<sub>2</sub> capture, such as integrated gasification combined cycle (IGCC) and Coal-to-Liquids (CTL) for transportation fuels, are making gradual progress in cost reduction and efficiency improvement. A limiting factor on the development of all the competing CO<sub>2</sub> capture technologies including both retrofits and advanced technologies is overcoming barriers to geological CO<sub>2</sub> sequestration (mapping and approving sites; establishing property rights to pore space in subsurface geological formations; institutionalizing long-term monitoring and verification; implementing state and federal regulations, especially regarding liability). The methodology in our paper can help to understand and quantify the social planner problem including a CO<sub>2</sub> emissions constraint, technology development, learning-by-doing, and adjustment costs. Other important social planner problems exist for promoting the market penetration for renewable and nuclear energy.

An equally interesting and challenging problem arises in the development and deployment of biomass-based fuels to supplement petroleum-based gasoline and diesel production. Replacing corn-based ethanol, second generation biofuels will need to produce a substantial amount of bio oils to meet fast growing transportation freight and jet fuel demand. These products are currently produced from petroleum middle distillates. Pyrolysis to produce bio crude oil and biomass gasification are technologies under development. The social planner problem will be to optimally accelerate the development and deployment of both bio feedstocks and conversion technologies. Further, these models can be used to sweep through parameter uncertainties.

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## A Functional Forms and Model Parameters

**Demand Function** The area  $S(q)$  under a constant price elasticity demand function reduces to either a power function (with negative exponential) or a log function. We use an indefinite integral to compute this area; neither formulation would be defined if we took a definite integral from 0 to  $Q$ .

**Demand Growth** is represented by (2.1) above; here we show its derivation. We assume that the demand growth takes the following form (so if price were constant, demand would grow at a rate  $b$ )

$$\tilde{q}(p, t) = e^{bt}q(p),$$

where  $q(p)$  represents demand at  $t = 0$ , and is a monotone decreasing function. Hence, there exist inverse demand functions  $p(q)$  and  $\tilde{p}(q, t)$  that are related as follows:

$$\tilde{p}(q, t) = p(q/e^{bt}).$$

$\tilde{S}(q, t)$  is defined as an indefinite integral of inverse demand

$$\tilde{S}(q, t) = \int \tilde{p}(q, t) dq = \int p(q/e^{bt}) dq = e^{bt} \int p(q/e^{bt}) d(q/e^{bt}) = e^{bt} S(q/e^{bt}) + C,$$

where  $S(q) = \int p(q) dq$  and  $C$  is a constant of integration. Since  $\tilde{S}$  is part of the objective, the constant does not affect the solution of the optimization problem.

**Unit Cost Function**  $c_n(x)$  (also called “learning curve”) is given by

$$c_n(x) = c_n^0 \left[ \frac{x}{\bar{X}} + 1 \right]^{\log_2 \gamma}. \quad (\text{A.1})$$

The interpretation of the exponent is that whenever experience doubles,  $c(x)$  is multiplied by  $\gamma$ . The constant  $c_n^0$  represents the cost at the top of the learning curve ( $x = 0$ ). The unit size of experience  $\bar{X}$  scales  $x$ . The scaled experience is shifted by unity to ensure that  $c_n(x)$  is decreasing for  $x \geq 0$ .

**Total Cost Function** (2.8) is used with the capital model (2.9). Let us assume that the energy production process uses two types of inputs— $K$  units of capital and  $B$  units of other inputs (e.g., materials). The Cobb-Douglass production function translates these into output amount  $q$ :

$$q = DK^\alpha B^\beta. \quad (\text{A.2})$$

Our model accounts for costs of capital via investment, so any additional cost of production is associated with other input  $B$ . Assume its price is  $p_B$ . Then the cost of producing  $q$  is

$$\begin{aligned} C(q, K) &= p_B B \\ &= A \left[ \frac{q}{K^\alpha} \right]^{1/\beta}, \end{aligned}$$

where  $A = p_B D^{-1/\beta}$ .

## B Description of the AMPL Code

The AMPL code for all three models is available at [www.mcs.anl.gov/~leyffer/OptTechPen](http://www.mcs.anl.gov/~leyffer/OptTechPen). In this appendix, we briefly describe the code for Model I and use it to highlight key features of AMPL language. The code consists of several files.

**Z\_run.ampl** is the “run file” that incorporates all commands necessary to set up the model, solve it, and produce output. The run file also allows the user to change the discretization method and period length. To run it, one types `ampl Z_run.ampl` at the command prompt.

**Z\_optpen.mod** is the main model file that sets the parameter values, declares model variables, and defines the constraints and other relationships that are independent of the discretization method.

**Z\_EE.mod**, **Z\_EI.mod**, **Z\_Tr.mod** are three additional model files describing the model components specific to the explicit Euler, implicit Euler, and trapezoidal discretization, respectively. Only one of these files should be used; the user can change the discretization method by (un)commenting lines in `Z_run.ampl`.

Unlike programming languages (C, Fortran), which concentrate on computing outputs from inputs, modeling languages such as AMPL define the algebraic relationships between variables; the model is then submitted with a single command to one of many available solvers. AMPL has several types of statements; we illustrate them using examples of code from Model I.

**Parameters** are the constants that cannot be changed by the solver. One can set parameters equal to specific numbers, for example  $T = 50$  and  $N = 500$ , or compute them from other parameters, such as  $h = T/N$ .

```
param tf := 50;
param hn := 500;
param h := tf/nh;
```

**Variables** can be declared along with constraints on them or default values, such as the outputs  $(q_i^j, i = 1, \dots, N)$ , which are declared to be nonnegative and given a default value of zero. Variables can also be declared to be a function of other variables and parameters, thus creating an implicit constraint, such as computation of total output  $Q_i$ .

```
var qo {0..nh} >=0, :=0;
var qn {0..nh} >=0, :=0;
var Q {i in 0..nh} = qo[i] + qn[i];
```

**Objective** The objective (3.1a) is represented as

```
maximize DiscWelf: sum {i in 0..nh-1} h*welf[i];
```

where `welf[i]` stands in for  $W_i$  and `DiscWelf` is the variable that receives the maximized value of objective.

**Constraints** are directly specified. For example, the constraint (3.1b) becomes

```
subject to ode_x {i in 0..(nh-1)}: x[i+1] = x[i] + h * qn[i];
```

where `ode_x` is the name assigned to that constraint (we could use it, for example, to retrieve the shadow cost of this constraint).

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