The Pareto dominant strategy-proof and equitable rule for problems with indivisible goods*

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September 27, 2009

Abstract

We study the problem of allocating indivisible goods to agents when monetary compensations are not allowed. Our central requirements are strategy-proofness and equity. For each of two different cases of this problem we identify a strategy-proof and equitable [in the sense of equal-treatment-of-equals, or envy-freeness] rule that Pareto dominates all other equitable and strategy-proof rules.

KEYWORDS: Indivisible goods, strategy-proofness, no-envy, Pareto domination.

JEL Classification: *C71; C78; D71; D78.*

1 Introduction

Consider the following problem. A central agency is to allocate a set of objects among a set of agents (for example, houses to applicants, tasks to workers, military supplies to military units) when monetary compensations are not possible. Agents do not have initial property rights over objects, but instead, all objects are collectively owned by the central agency.[1]

Our main requirement is *strategy-proofness* (i.e., no agent ever gains by misrepresenting his preferences). This property has been widely studied in numerous contexts due to its apparent desirability. Our second goal is to obtain *equity* among agents. Equitable distribution of resources, however, comes in conflict with *efficiency* in almost all economic applications. The allocation of indivisible goods is one where this conflict is felt most strongly. Even a minimal form of equity

*We thank William Thomson for his helpful comments. We would like to thank two anonymous referees for helpful comments. In particular, insightful suggestions of a referee that helped improve the exposition and motivation of the paper are gratefully acknowledged.

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[1] The early literature on indivisible good allocation focuses on the opposite case. This case is known as the *housing markets* (Shapley and Scarf [23]). Also see Abdulkadiroğlu and Sönmez [1] for more on housing markets.

is not compatible with efficiency.[2] To recover compatibility between the two requirements, a large body of literature has studied models where monetary compensations are also possible. One can easily think of applications where monetary compensation is not an option (e.g., allocation of organs, school seats, internal resource allocation among different branches of a company). For such applications the literature has thus far focused on ex ante fairness by allowing for randomization (see for example, Zhou [30] and Bogomolnaia and Moulin [5]).[3] Nonetheless, situations in which (ex post) equity is viewed as a primary concern are not uncommon. For example, bureaucratic over-centralized agencies, such as some former and current communist countries (e.g., old Soviet Union and North Korea) or central logistic headquarters of a military organization, often care about ex post distribution of resources.

In this paper we restrict attention to two equity criteria. The first one is *equal treatment of equals* which is quite weak. It simply requires that any two agents with identical preferences should be assigned the same object. An alternative criterion that has often been studied in the literature is *no-envy* (Foley [12]). It requires that no agent prefers some other agent's assignment to her own. The existence of rules meeting these two criteria necessitates the existence of an outside option available to all agents, which we call the *null object.*

We consider two interesting cases of this problem: (1) the supply of each object is exactly one; and (2) the supply of an object may be larger than one. For the former case, we identify a rule, denoted as φ^*, satisfying strategy-proofness and no-envy, that Pareto dominates any other rule satisfying the two requirements (Proposition 1). For the latter case, we identify an envy-free and strategy-proof rule, denoted as ψ^*, that Pareto dominates any other strategy-proof and envy-free rule (Theorem 1). These rules work as follows: Each agent is assigned her favorite object that passes a certain "attainability" test which is based on the preferences of all remaining agents. Loosely speaking, this test can be interpreted as a forward looking procedure that determines whether situations of 'excess demand' for an object might arise. Interestingly, such a test turns out to be a recursive procedure.

The choice of equity as a primary objective entails an obvious efficiency cost in our context. Consequently, despite their constrained efficiency properties within their specific classes, the proposed two rules are subject to welfare losses. Therefore we point out three possible interpretations of the results of our analysis from a welfare perspective. First, the Pareto dominance property of the two rules enables them to set *the* lower bound on the size of the 'sacrifice' a central agency needs to make if strategy-proofness and equity are part of the desideratum. For example, for the case with multiple supplies, if an object fails the attainability test of rule ψ^*, then no strategy-proof and envy-free rule can ever allocate this object to an agent. Second, it is plausible to think that the central agency could decide on allocation after collecting agents' preferences but before purchasing the set of objects to distribute (i.e., it would then only purchase the objects which it would actually assign to agents). This way, the agency can maintain a 'restricted efficiency' by making sure that it only acquires objects so tailored to different agents that each object is only

[2] As a simple example, suppose only a single object is available, which is desired by two agents. Then equality would imply that neither agent receives the object.

[3] Clearly, this does not necessarily guarantee an equitable distribution of resources *ex post.* This, in particular, is the case for one of the most common real-life allocation mechanisms, *random priority,* which works as follows. Consider a collection of distinct objects. Choose a random ordering of agents, and let the first agent choose his favorite object, next let the second agent choose his favorite object from whatever remains, and so on. Note that the outcome of such a procedure could be quite disappointing for those with a late turn in the ordering.

liked by the agents it is intended for. Third, if either rule recommends against the allocation of an object to any agent, then this does not necessarily imply immediate disposal of the object. For example, the central agency may have the option of storing any unallocated objects to use in the future when the attainability test is met as more objects become available. In other words, the kind of allocation problem studied in our model can also be thought as a snapshot of a continuous dynamic allocation process.

The paper is organized as follows: We next introduce the model. To fix ideas Section 3 analyzes the simple case when there is exactly one copy of each object. Section 4 is dedicated to the more general case. Section 5 concludes. Section 6 is the Appendix that contains all the proofs.

2 The model

Let $N \equiv \{1, 2, \ldots, n\}$, $n \geq 2$, denote the finite set of agents. Let X denote the (finite) **set of objects.** For our base model we assume that there is exactly one copy of each object. Also available to each agent is an outside option, called the **null object**. Let 0 denote this object. Let the supply of the null object be n, i.e., the null object can be assigned to any number of agents. Let $\widetilde{X} = X \cup \{0\}$. Each agent $i \in N$ is equipped with a *complete*, *transitive,* and *anti-symmetric*[4] relation R_i on $\widetilde{X}$. Let $\mathcal{R}$ denote the class of all such preferences. Let P_i be the strict relation associated with R_i. Let $R = (R_i)_{i \in N}$ be a preference profile. Then a **problem** is a preference profile $R \in \mathcal{R}^N$.

For a given problem R, a feasible **allocation** is a list $\alpha \equiv (\alpha_i)_{i \in N}$ such that for each $i \in N$, $\alpha_i \in \widetilde{X}$, no agent is assigned more than one object and no object in X is assigned to more agents than its supply (which is one in this case). Let $\mathcal{A}(X)$ be the set of all feasible allocations for X.

Given $R \in \mathcal{R}^N$, an allocation $\alpha \in \mathcal{A}(X)$ is **Pareto dominated** by another allocation $\beta \in \mathcal{A}(X)$ if $\beta_i \; R_i \; \alpha_i$ for each $i \in N$, and $\beta_i \; P_i \; \alpha_i$ for some $i \in N$.

A **rule** is a function that associates to each problem R an allocation α feasible for X. Let φ denote a generic rule.

Given a problem $R \in \mathcal{R}^N$, and an agent $i \in N$, let $\varphi_i(R)$ denote agent i's assignment at R. Let R_{-i} denote the profile $R_{N \setminus \{i\}}$. We next introduce some of the basic properties of rules.

A rule φ is **Pareto dominated** by another rule ϕ if for each $R \in \mathcal{R}^N$, each $i \in N$, $\phi_i(R) \; R_i \; \varphi_i(R)$ where the relation is strict for some $i \in N$ and $R \in \mathcal{R}^N$.

Next are the central properties studied in this paper. The first one can be seen as a minimal equity requirement: The rule should not discriminate between any two agents who have the same preferences.

Equal Treatment of Equals: For each $R \in \mathcal{R}^N$, and each $i, j \in N$, if $R_i = R_j$, then $\varphi_i(R) = \varphi_j(R)$.

The next property is a stronger equity requirement: Each agent finds her assignment at least as good as any other agent's.

No-Envy (Foley [12])**:** For each $R \in \mathcal{R}^N$ and each $i, j \in N$, we have $\varphi_i(R) \; R_i \; \varphi_j(R)$.

[4] A preference relation R_i on $\widetilde{X}$ is *anti-symmetric* if for each $x, y \in \widetilde{X}$, $x \; R_i \; y$ and $y \; R_i \; x$ implies $x = y$.

The no-envy requirement has been studied in various models such as classical exchange economies, the extension of our model to the case where monetary compensations are possible, the division of a heterogenous good, the allocation of an infinitely divisible commodity to a set of agents with single-peaked preferences etc. (see Thomson [28] for a number of other applications of this concept).

Our next property is a standard strategic requirement that has been extensively studied in a number of contexts. It requires that no agent ever gains by misreporting her preferences.

Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R_i' \in \mathcal{R}$, $\varphi_i(R)\ R_i\ \varphi_i(R_i', R_{-i})$.

3 A New rule

Consider the following rule: Assign each agent her most preferred object among the ones which no remaining agent prefers to the null object. In other words, if two agents both prefer some object a to the null object, then neither of them gets it.

We need some extra notation before we formally define this rule. Given $R \in \mathcal{R}^N$ and $X' \subseteq \widetilde{X}$, let $f(R_i, X') \equiv \{a \in X' : a\ R_i\ x \text{ for all } x \in X'\}$. That is, $f(R_i, X')$ is the favorite object of agent i in X'. Given $i \in N$, $a \in \widetilde{X}$, let $U_a(R_i) \equiv \{x \in \widetilde{X} : x\ P_i\ a\}$ and $L_a(R_i) \equiv \{x \in \widetilde{X} : a\ P_i\ x\}$. That is, $U_a(R_i)$ is the set of objects agent i prefers to a, whereas $L_a(R_i)$ is the set of objects to which i prefers a.

Rule φ^*: For each $R \in \mathcal{R}^N$ and each $i \in N$, $\varphi_i^*(R) = f(R_i, \widetilde{X} \setminus \cup_{j \neq i} U_0(R_j))$.

It is easy to see that φ^* is well-defined. Our first result suggests that rule φ^* is the most appealing one among *strategy-proof* and *envy-free* rules. Let Z be the rule that assigns the null object to each agent (i.e., for each $R \in \mathcal{R}^N$ and each $i \in N$, $Z_i(R) = 0$).

Proposition 1 Rule φ^* is strategy-proof and envy-free. Furthermore, φ^* Pareto dominates any other strategy-proof and envy-free rule.

Proposition 2 Rule Z is strategy-proof and envy-free. Furthermore, Z is Pareto dominated by any rule $\varphi \neq Z$ that is strategy-proof and envy-free.

Propositions 1 and 2 show that from a welfare standpoint rule φ^* is the best and the null rule is the worst rule within the class of rules satisfying *strategy-proofness* and *no-envy*.

Corollary 1 Rule φ^* is strategy-proof and satisfies equal treatment of equals. If any rule other than φ^* and Z also satisfies strategy-proofness and equal treatment of equals, then it is Pareto dominated by rule φ^*, and it Pareto dominates Z.

It is also worth noting that Propositions 1 and 2 still hold if one replaces *no-envy* by *anonymity*[5] (i.e., the rule does not depend on the naming of agents) which is a stronger *equity* requirement. Furthermore, rules φ^* and Z are not the only rules satisfying *strategy-proofness* and *anonymity*.

[5]Formally, φ satisfies *anonymity* if given any permutation $\pi : N \to N$ of agents, for each $R \in \mathcal{R}^N$ and each $i \in N$, we have $\varphi_i(R^\pi) = \varphi_{\pi(i)}(R)$ where $R^\pi \equiv (R_{\pi(i)})_{i \in N}$.

The following is another example: For each $R \in \mathcal{R}^N$ and each $i \in N$, if for each $\{j,k\} \in N\backslash\{i\}$, $R_j = R_k$, then $\varphi_i(R) = f(R_i, \widetilde{X} \backslash \cup_{j \neq i} U_0(R_j))$, otherwise $\varphi_i(R) = 0$.[6]

A special case of this model is when there is a single object to be assigned, i.e., $|X| = 1$. For this context Pápai [19] shows that the only *strategy-proof* and *nonbossy* rules are *hierarchical rules:* There is a group of agents and a priority order within this group such that the rule assigns the object to the agent with the highest priority among those who prefer getting the object to not getting it).[7] Rule φ^* when adopted to this context simplifies to the following rule: If there is only one agent who prefers getting the object to not getting it, then the agent is assigned the object, otherwise no agent is assigned the object.

Corollary 2 *Suppose $|X| = 1$. The rule that assigns the object only if there is exactly one agent who prefers getting it to not getting it Pareto dominates any other rule satisfying strategy-proofness and equal treatment of equals.*[8]

4 The general case: Multiple Supplies

An important extension of the problem we have considered in the previous section is when there may be multiple copies of a particular object. This extension too has attracted much attention in the recent literature. A popular application is the *school choice problem* (Abdulkadiroglu and Sönmez [2]): There is a set of schools (objects) each of which has a certain number of seats, and a set of students, each of whom seeks one seat at one of the schools.[9] Most of the papers concerning this extension assume that an exogenous *priority order* over agents is given for each object. If there is a shortage of seats at a particular school which is overdemanded, then the priority order for that school can be used to determine the allocation (more on this after Theorem 1). In contrast with these papers, we do not make an a priori distinction among agents.

First we provide the formal extension. With an abuse of notation let (X, s) denote the (finite) **set of objects** such that X is the set of **object types** available in the set of objects and for each $x \in X$, let $s_x \in \mathbb{Z}_{++}$ be the supply of x. Let $s \equiv (s_x)_{x \in X}$ be the supply vector. Let $\widetilde{X} \equiv X \cup \{0\}$. Let supply of the null object be n. Each agent $i \in N$ is equipped with a *complete*, *transitive* and *anti-symmetric* preference relation R_i over $\widetilde{X}$. Let $\mathcal{R}$ denote the class of all such preferences. Let P_i denote the strict relation associated with R_i. Let $R = (R_i)_{i \in N}$ be a preference profile. A **problem** is a pair (R, s).

For a given problem (R, s), a feasible **allocation** for (X, s) is a list $(\alpha_i)_{i \in N}$ such that for each $i \in N$, $\alpha_i \in \widetilde{X}$, no agent is assigned more than one object, and no object $x \in X$ is assigned to more agents than its supply. Let $\mathcal{A}(X, s)$ be the set of all feasible allocations for $(X$,$s)$.

A **rule** φ is a function that chooses an allocation from $\mathcal{A}(X, s)$ for each problem (R, s). Given (R, s), a **subproblem** is a pair $(R_{-N'}, s - |N'| * \mathbf{1})$ for some $N' \subset N$, $N' \neq \varnothing$, where $\mathbf{1}$ is the $|X| \times 1$ vector of 1's, obtained from (R, s) by removing all agents in N' and reducing the supply of each object by $|N'|$. *Note that we allow for an object to have a negative supply at a subproblem.*

[6] Clearly, given a problem R if for each $j, k \in N$, $R_j = R_k$, then for each $i \in N$, $\varphi_i(R) = 0$. We leave it to the reader to check that this rule satisfies the two properties.

[7] By Proposition 3 given in the Appendix, these rules do not meet *equal treatment of equals*.

[8] If *equal treatment of equals* is replaced by *anonymity*, we obtain uniqueness.

[9] See for example Ergin [10], Abdulkadiroğlu and Sönmez [2], and Kesten [16]. This application is also closely related to the college admissions problem (Gale and Shapley [13]).

4.1 Two new rules

4.1.1 The Pareto dominant envy-free rule

In this richer setting, we continue to study our strategic and equity properties. Our central ex-post equity requirement is *no-envy*. We first introduce a new rule, which we call the *unrestricted fair (UF) rule.* The outcome of this rule is calculated via repeated applications of the following UF *procedure*: Each agent initially *demands* her favorite object among the available ones. If there is excess demand for a particular object (i.e., the number of demands for the object exceeds its supply), then such an object is discarded from the set of available objects. At the next application, the UF procedure is applied to the set of remaining objects, and the procedure is repeated until no object is discarded any more. Clearly, the procedure terminates in a finite number of applications.

UF procedure: *Let a problem or a subproblem be given. Each agent demands her favorite object among the available ones. If the number of agents who demand an object is greater than its supply, then remove this object from the set of available objects. (If an object's supply is negative, it is also removed from the set of available objects since the number of agents who demand it is always non-negative.)*

Unrestricted fair rule (UF): *Let a problem* (R, s) *be given. Iteratively apply the* UF *procedure until there is no object for which demand exceeds supply. The unrestricted fair rule assigns each agent what it demands at the last application of the procedure at which there is no object for which demand exceeds supply.*

We say that *an object is eliminated by* UF in a problem if it is eliminated from the set of available objects at some application of the UF procedure to that problem. The unrestricted fair rule is envy-free. Furthermore, it is the Pareto dominant rule among all envy-free rules.

Proposition 3 *The unrestricted fair rule is envy-free. Furthermore, it Pareto dominates any other envy-free rule.*

Unfortunately, the unrestricted fair rule is not strategy-proof. To see this through a simple example, suppose there are two agents who rank two objects (each with unit supply) in the same way and above the null object. If both agents are truthful, then neither agent receives a (real) object. However, if only one agent reverses the ranking of the two objects, then each agent receives a real object. In the remainder of the paper it is our objective to recover strategy-proofness.

4.1.2 The Pareto dominant envy-free and strategy-proof rule

Before presenting our main proposal, we aim to give some insight on the implications of no-envy and strategy-proofness. Much of the intuition of our analysis can be seen through the following simple example.

Example 1 *Let* $N = \{1, 2, 3\}$, $X = \{a, b, c\}$, $s = (2, 1, 1)$ *and the preference profile be as follows:*

$R_1: \quad c \quad a \quad b \quad 0$
$R_2: \quad b \quad a \quad 0$
$R_3: \quad c \quad a \quad 0$

Obviously, object c cannot be assigned to any agent at an envy-free allocation. We claim that no envy-free and strategy-proof rule assigns object a to agent 1 (and consequently to any other agent) for this problem. Suppose by contradiction that $\varphi_1(R) = a$ for some envy-free and strategy-proof rule φ. Let R'_i denote the preferences such that $a \ P'_i \ b \ P'_i \ c \ P'_i 0$. By strategy-proofness, we have $\varphi_1(R'_1, R_{-1}) = a$. By no-envy, $\varphi_2(R'_1, R_{-1}) \in \{a, b\}$. Again by strategy-proofness, $\varphi_2(R'_{\{1,2\}}, R_3) = \{a, b\}$. At profile $(R'_{\{1,2\}}, R_3)$, b cannot be assigned to any agent at an envy-free allocation therefore $\varphi_2(R'_{\{1,2\}}, R_3) = a$. No-envy implies that $\varphi_1(R'_{\{1,2\}}, R_3) = \varphi_2(R'_{\{1,2\}}, R_3) = a$. Again by no-envy, $\varphi_3(R'_{\{1,2\}}, R_3) \in \{a, c\}$. Then by strategy-proofness, $\varphi_3(R'_{\{1,2,3\}}) \in \{a, b, c\}$. But no-envy requires that each agent should receive the same object at this last profile. This is clearly impossible. $\diamond$

Our analysis relies on a key observation: If a rule is envy-free and strategy-proof, then it cannot assign certain objects (such as object a of the above example) even if there exist envy-free allocations at which they can be assigned. Intuitively, this observation is based on the following reasoning: an envy-free and strategy-proof rule cannot assign an object x to an agent i if either (i) its supply is not sufficiently large to avoid envy at the given profile when assigned to agent i [for example, object c for agent 1 in the above example]; or (ii) its supply is not sufficiently large to avoid envy if agent i were to change her preferences [for example, if agent 1 were to switch the positions of objects a and b in her preferences, then object a cannot be assigned to an agent at this new profile without violating no-envy]. In light of the above example, it is not much difficult to see that statement (ii) is a simple consequence of strategy-proofness. Then an equivalent restatement of the above reasoning is the following: An envy-free and strategy-proof rule cannot assign an object x to an agent i if either (i) object x is eliminated by UF at this problem; or (ii) object x is eliminated by UF at the subproblem obtained by removing agent i and reducing the supply of each object by one. But notice that since the rule is strategy-proof, one may need to iterate this reasoning to check if statement (ii) holds for even smaller subproblems obtained in a similar fashion. This suggests that the test of determining whether an object can be assigned to an agent by an envy-free and strategy-proof rule needs to be an iterative procedure. The next definition formalizes this discussion by introducing an iterative notion of "scarcity".

Definition 1: *Let a problem (R, s) be given. Object a is scarce for agent i in (R, s) iff for each $x \in U_a(R_i) \cup \{a\}$*
either (i) x is eliminated by UF in the subproblem $(R_{-i}, s - \mathbf{1})$.
or (ii) x is scarce for some agent j in the subproblem $(R_{-i}, s - \mathbf{1})$.

The above definition could be interpreted as an 'attainability' test. Given a problem (R, s), take an agent $i \in N$ and an object $a \in X$. In order to determine if a is scarce for i in (R, s), one needs to consider each object in the weak upper contour set of R_i at a. Part (i) of Definition 1 says that for each such object we first check if the object is eliminated by the UF procedure in the subproblem obtained from (R, s) by removing agent i and reducing the supply of each object by one. If this is not the case, part (ii) of Definition 1 says that we then check if there is an agent for whom the object is scarce in the same subproblem. If either one of the two parts of the scarcity definition holds for every object in the weak upper contour set of R_i at a, then we conclude that a is scarce for i in (R, s). Note that the null object is never scarce for any agent in any (sub)problem. Let $S(i, I)$ denote the set of objects that are scarce for i in (sub)problem I.

We now illustrate the scarcity definition via a simple example. (See the Appendix for a more detailed illustration of the definition.)

Example 2 *Let $N = \{1, 2, 3\}$, $X = \{a, b, c\}$, $s = (1, 2, 2)$ and the preference profile be as follows:*

$$\begin{array}{lllll} R_1: & a\ (1) & b\ (2) & 0 & c\ (2) \\ R_2: & b\ (2) & a\ (1) & 0 & c\ (2) \\ R_3: & c\ (2) & b\ (2) & 0 & a\ (1) \end{array}$$

The number in the parenthesis next to each object shows the supply of that object. Suppose we wish to determine whether $a \in S(1, (R, s))$. Since $U_a(R_1) \cup \{a\} = \{a\}$, we only need to consider a, and determine if it satisfies either (i) or (ii) of the scarcity definition.

(a.i) We check if a is eliminated by UF in $(R_{-1}, s - \mathbf{1})$.

$$\begin{array}{lllll} R_2: & b\ (1) & a\ (0) & 0 & c\ (1) \\ R_3: & c\ (1) & b\ (1) & 0 & a\ (0) \end{array}$$

***UF* Procedure**:

1st application: Agents 2 and 3 demand b and c respectively. The demand for neither of these objects exceeds the corresponding supply. Thus, no object is eliminated by UF in $(R_{-1}, s - \mathbf{1})$.

(a.ii) We check if a is scarce for some agent in $(R_{-1}, s - \mathbf{1})$. The only candidate is agent 2. To show that $a \in S(2, (R_{-1}, s - \mathbf{1}))$, we consider each $x \in U_a(R_2) \cup \{a\} = \{a, b\}$ one at a time, and determine if either (i) or (ii) of the scarcity definition is satisfied.

(a.ii.b.i) We check if b is eliminated by UF in $(R_{-12}, s - 2 * \mathbf{1})$.

$$\begin{array}{lllll} R_3: & c\ (0) & b\ (0) & 0 & a\ (\text{-}1) \end{array}$$

***UF* Procedure**:

1st application: Agent 3 demands c. The demand for c exceeds its supply.

2nd application: Agent 3 demands b. The demand for b exceeds its supply.

3rd application: Agent 3 demands 0. The demand for 0 never exceeds its supply.

Object b is eliminated by UF in $(R_{-12}, s - 2 * \mathbf{1})$.

(a.ii.a.i) We check if a is eliminated by UF in $(R_{-12}, s - 2 * \mathbf{1})$. Consider the case $(a.ii.b.i)$. Note that even though 3 does not demand a, it is eliminated by UF in $(R_{-12}, s - 2 * \mathbf{1})$ since its supply is negative.

Conclusion 1: Steps $(a.ii.b.i)$ and $(a.ii.a.i)$ show that $a \in S(2, (R_{-1}, s - \mathbf{1}))$.

Conclusion 2: Since (ii) of the definition of the scarcity of a for 1 in (R, s) is satisfied and $U_a(R_1) \cup \{a\} = \{a\}$, we have shown that $a \in S(1, (R, s))$. $\diamond$

We are now ready to present our main solution to the problem considered in this study. The scarcity notion defined above is an essential part of this rule. In each problem, this rule assigns each agent her favorite object among the ones that are not scarce for her in that problem.

Rule ψ^*: *For each problem* (R, s) *and each agent* $i \in N$, ψ^* *assigns agent* i *her most favorite object among the ones that are not scarce for* i *in* (R, s).

We defer to the appendix the issue of ψ^* being a well-defined rule. Lemma 3 in the appendix is that ψ^* recommends a feasible allocation for each problem. Our main result is that ψ^* is the Pareto dominant rule within the class of strategy-proof and envy-free rules.

Theorem 1 *Rule* ψ^* *is strategy-proof and envy-free. Furthermore, it Pareto dominates any other envy free and strategy-proof rule.*

Rule ψ^* reduces to φ^* in the single supply case. The following remark presents the equivalence between the two rules when there is only one copy of each object. Thus, Proposition 1 is a corollary to Theorem 1 and Remark 1. Nonetheless, Proposition 2 does not hold in the multiple supply case. The null rule is still strategy-proof and envy-free but there are strategy-proof and envy-free rules which do not Pareto dominate the null rule[10].

Remark 1 $\psi^*(R, s) = \varphi^*(R)$ *for each* (R, s) *such that* $s = \mathbf{1}$.

The equivalence between the two rules in the single supply case follows from the following observation: $U_{\varphi_i^*(R)}(R_i) \subseteq S(i, (R, s)) \subseteq \cup_{j \neq i} U_0(R_j)$ for each $i \in N$ and each (R, s) such that $s = \mathbf{1}$. To see this, let $i \in N$ and (R, s) be such that $s = \mathbf{1}$. For the first relation, note that by definition of φ^*, $U_{\varphi_i^*(R)}(R_i) \subseteq \cup_{j \neq i} U_0(R_j)$. Note also that each $x \in \cup_{j \neq i} U_0(R_j)$ is eliminated by UF in $(R_{-i}, s - \mathbf{1})$ because each has a zero supply. The previous two statements imply that each $x \in U_{\varphi_i^*(R)}(R_i)$ is eliminated by UF in $(R_{-i}, s - \mathbf{1})$, which in turn implies that $U_{\varphi_i^*(R)}(R_i) \subseteq S(i, (R, s))$. For the second relation, let $a \in S(i, (R, s))$. Then either (i) a is eliminated by UF in $(R_{-i}, s - \mathbf{1})$ in which case, demand for a exceeds its supply (zero) at some application of the UF procedure to $(R_{-i}, s - \mathbf{1})$ which implies that there is $j \neq i$ and $a \ P_j \ 0$ or (ii) a is scarce for some $j \neq i$ in $(R_{-i}, s - \mathbf{1})$, in which case $a \ P_j \ 0$ because $S(j, (R_{-i}, s - \mathbf{1})) \subseteq U_0(R_j)$.

It is also illustrative to compare ψ^* with other strategy-proof and envy-free rules. Consider the following rule, which is a straightforward generalization of φ^* to the case with multiple supplies. For each (R, s) and each $i \in N$, $\psi_i(R, s) = f(R_i, Y(R, s))$ where $Y(R, s) \equiv \left\{ a \in \widetilde{X} : |\{j \in N : a \ R_j \ 0\}| \leq s_a \right\}$. It is easy to check that ψ is strategy-proof and envy-free. By Theorem 1, ψ is indeed Pareto inferior to ψ^*. We contrast the two rules' outcomes in the next example.

Example 3 *Let* $N = \{1, 2, 3\}$, $X = \{a, b, c, d, e, g, x, y, z\}$, $s = (2, 2, 2, 2, 2, 1, 3, 3, 3)$ *and the preference profile be as follows.*

$$\begin{array}{lcccccccc}
R_1 : & \boxed{a} & b & d & e & [x] & c & 0 & \\
R_2 : & \boxed{b} & c & e & d & g & [y] & a & 0 \\
R_3 : & \boxed{c} & a & d & e & g & [z] & b & 0
\end{array}$$

[10] We are thankful to a referee for pointing out this.

The allocations recommended by ψ^* and ψ are shown in boxes and brackets respectively. Clearly, the allocation recommended by ψ^* Pareto dominates that by ψ. For each of a, b, c, d, e, and g, the number of agents that prefer this object to the null object exceeds its supply, hence none of these objects is allocated by ψ. Rule ψ^* however allocates each agent her most favorite. This is because, none of a, b and c are scarce for agents 1, 2, and 3 in (R, s) respectively. Observe that a is preferred to null by three agents, and its supply is only two. Based on this information, ψ behaves cautiously by not allocating a at all. On the other hand, for ψ^* this is not the case. This is because, only one of the agents, namely agent 2, prefers y to a, and y is abundant [in the sense that its supply is at least as many as the number of agents] in the problem (R, s). One can show that it is this abundance of y that prevents a being scarce for 1 in (R, s). $\diamond$

5 Conclusion

We have looked for equitable rules that are immune to strategic maneuvers. The proposed rules φ^* and ψ^* stand out as the most appealing rules in their corresponding settings. For other indivisible good allocation problems such as *house allocation with existing tenants* (e.g., Abdulkadiroğlu and Sönmez [4]) and *school choice problems* (e.g., Abdulkadiroğlu and Sönmez [2]), it is commonly assumed that for each object, there is an exogenously given *priority order* over agents that needs to be respected while assigning the objects. Given a collection of priority orders, a rule is said to be *fair* if no agent ever envies some other agent for an assignment for which it has higher priority. There is a well-known fair rule for this context: The *student-optimal stable rule (SOSM).*[11] Rule ψ^* and SOSM bear striking resemblances. First, both try to serve the best interests of the agents. That is, both rules try to assign agents their favorite choices as much as possible. When there is competition for a particular object among a certain group of agents, SOSM resolves this conflict using the *exogenous* priority structure, whereas rule ψ^* does this in an *endogenous* way using the preferences of all agents but the competing ones. The second and indirect resemblance is in terms of the properties they satisfy. SOSM is strategy-proof and fair and Pareto dominates any other strategy-proof and fair rule whereas rule ψ^* is strategy-proof and envy-free and Pareto dominates any other strategy-proof and envy-free rule.

6 The Appendix

We first give a more detailed example to illustrate the scarcity definition. It will also be used in the sketch of the proof of Lemma 5.

Example 4 *Let* $N = \{1, 2, 3, 4, 5\}$, $X = \{a, b, c, d, e\}$, $s = (1, 2, 3, 3, 5)$ *and the preference profile be as follows.*

[11] Its outcome is calculated via the well-known *deferred acceptance algorithm*. For the two-sided matching context, it yields the most preferred stable allocation for each agent. See Roth and Sotomayor [22] for a comprehensive account. This rule is central to all the papers mentioned in footnote 10.

R_1	R_2	R_3	R_4	R_5
a	b	c	d	e
e	a	b	c	d
0	0	0	0	b
				0

We determine if $a \in S(1,(R,s))$. Note that $U_a(R_1) \cup \{a\} = \{a\}$.

(a.i) We determine if a is eliminated by UF in $(R_{-1}, s-1)$.

The supply vector is $s - \mathbf{1} = (0,1,2,2,4)$ and the preference profile is as follows:

R_2	R_3	R_4	R_5
b	c	d	e
a	b	c	d
0	0	0	b
			0

***UF* procedure:**

1st application:agents $2, 3$,4 and 5 demand objects b, c, d and e respectively. Demand for none of these objects exceeds its supply, therefore none is eliminated by UF procedure.

Object a is not eliminated by UF in $(R_{-1}, s-1)$.

(a.ii) We determine if a is scarce for some agent in $(R_{-1}, s-\mathbf{1})$. We determine if $a \in S(2,(R_{-1}, s-\mathbf{1}))$. Note that $U_a(R_2) \cup \{a\} = \{a,b\}$.

(a.ii.b.i) We determine if b is eliminated by UF in $(R_{-12}, s-2*1)$.
The supply vector is $s - 2*\mathbf{1} = (-1,0,1,1,3)$ and the preference profile is as follows:

R_3	R_4	R_5
c	d	e
b	c	d
0	0	b
		0

***UF* procedure:**

1st application: agents 3, 4 and 5 demand objects c, d and e respectively. Demand for none of these objects exceeds its supply therefore none is eliminated by UF procedure. Specifically, b is not eliminated by UF procedure.
Note that even though a is not demanded, it is eliminated by UF because its supply is negative.

(a.ii.b.ii) We determine if b is scarce for some agent in $(R_{-12}, s-2*1)$. We determine if $b \in S(3,(R_{-12}, s-2*\mathbf{1}))$. Note that $U_b(R_3) \cup \{b\} = \{b,c\}$. We consider each object in $\{b,c\}$ one at a time and check if it satisfies either (i) or (ii) of the definition of scarcity.

(a.ii.b.ii.c.i) We determine if c is eliminated by UF in $(R_{-123}, s-3*1)$.

The supply vector is $s - 3*\mathbf{1} = (-2,-1,0,0,2)$ and the preference profile is as follows:

R_4	R_5
d	e
c	d
0	b
	0

***UF* procedure:**
1st application: agents 4 and 5 demand d and e respectively. Demand for d exceeds its supply. Object d is eliminated by UF procedure.
2nd application: agents 4 and 5 demand c and e respectively. Demand for c exceeds its supply. Object c is eliminated by UF procedure.
3rd application: agents 4 and 5 demand 0 and e respectively. Demand for none of these objects exceeds its supply.
Note that objects a and b are eliminated by UF because each has a negative supply.
(a.ii.b.ii.b.i) We determine if b is eliminated by UF in $(R_{-123}, s - 3 * 1)$.
Consider the case $(a.ii.b.ii.c.i)$. Object b is eliminated by UF in $(R_{-123}, s - 3 * 1)$.
Conclusion 1: Steps $(a.ii.b.ii), (a.ii.b.ii.c.i)$ and $(a.ii.b.ii.b.i)$ show that $b \in S(3, (R_{-12}, s-2*\mathbf{1}))$.

(a.ii.a.i) We determine if a is eliminated by UF in $(R_{-12}, s - 2 * 1)$.
Consider the case $(ii.b.i)$. Object a is eliminated by UF in $(R_{-12}, s - 2 * 1)$.
Conclusion 2: Conclusion 1 and $(a.ii.a.i)$ show that $a \in S(2, (R_{-1}, s - \mathbf{1}))$.
Conclusion 3: Since (ii) of the definition of scarcity of a for 1 in (R, s) is satisfied and $U_a(R_1) \cup \{a\} = \{a\}$, we have shown that $a \in S(1, (R, s))$.

It can be easily shown that $S(1, (R, s)) = \{a\}$, $S(2, (R, s)) = \varnothing$, $S(3, (R, s)) = \varnothing$, $S(4, (R, s)) = \varnothing$ and $S(5, (R, s)) = \varnothing$. Thus, $\psi^*(R, s) = (e, b, c, d, e)$.

Before proving Proposition 1, we make a simple observation. The following property says that if an agent's assignment does not change when her preference changes, nobody else's does either.

Nonbossiness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$, then $\varphi(R) = \varphi(R'_i, R_{-i})$.

Proposition 4 The null rule is the only rule that satisfies strategy-proofness, equal treatment of equals, and nonbossiness.

Proposition 3 suggests that no interesting rule satisfies *equal treatment of equals, strategy-proofness, nonbossiness.*

Proof of Proposition 3: It is obvious that Z satisfies the three properties. Let $\varphi \neq Z$ be another rule satisfying them. Then there are $R \in \mathcal{R}^N$, $i \in N$, $a \in X$, such that $\varphi_i(R) = a$. We claim that for each $j \in N$, $\varphi_j(R) \neq 0$. Indeed, if there is $j \in N \backslash \{i\}$ such that $\varphi_j(R) = 0$, then simply letting $R'_j = R_i$, we have $\varphi_i(R'_j, R_{-j}) = \varphi_j(R'_j, R_{-j}) = \varphi_j(R) = 0$, contradicting *nonbossiness*. Next, let $j \in N \backslash \{i\}$ and $\varphi_j(R) \neq a$. Let $b \equiv \varphi_j(R)$. Since φ satisfies *strategy-proofness* and *equal treatment of equals,* by Claim 1 in the proof of Proposition 1, $0 \; R_i \; b$. Let R'_i be such that $a \; R'_i \; b \; R'_i \; 0 \; R'_i \; x$ for each $x \in X \backslash \{a, b\}$. By *strategy-proofness,* $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$. But, since $b \; R'_i \; 0$ and φ satisfies *strategy-proofness* and *equal treatment of equals,* $\varphi_j(R) \neq \varphi_j(R'_i, R_{-i})$, contradicting *nonbossiness*.

Q.E.D.

Proof of Proposition 1: Suppose φ^* is not *strategy-proof.* Then there are $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $\varphi^*_i(R'_i, R_{-i}) \; P_i \; \varphi^*_i(R)$. Let $a \equiv \varphi^*_i(R'_i, R_{-i})$. Since $a \; P_i \; \varphi^*_i(R)$, this means there is $j \in N\backslash\{i\}$ such that $a \in U_0(R_j)$. But then, $\varphi^*_i(R'_i, R_{-i}) \neq a$. Suppose φ^* is not *envy-free.* Then there are $R \in \mathcal{R}^N$and $i, j \in N$ such that $\varphi^*_j(R)\ P_i\ \varphi^*_i(R)$. Let $a \equiv \varphi^*_j(R)$ and $b \equiv \varphi^*_i(R)$. By definition of φ^*, $a \in (\{0\} \cup L_0(R_i))$. This implies that $0\ P_i\ b$. Thus, $\varphi^*_i(R) \neq b$. Claim 1 will be useful in proving the other statement in Proposition 1.

Claim 1: *Let φ be a rule satisfying strategy-proofness and no-envy. Then for each $R \in \mathcal{R}^N$ and each $i \in N$, $\varphi_i(R) \cap (\cup_{j\neq i} U_0(R_j)) = \varnothing$.*

Proof of Claim 1: Suppose there are $R \in \mathcal{R}^N$ and $i \in N$ such that $\varphi_i(R) \cap (\cup_{j\neq i} U_0(R_j)) \neq \varnothing$. Let $a \equiv \varphi_i(R)$. Note that $a \neq 0$. Let $j \in N\backslash\{i\}$ be such that $a \in U_0(R_j)$. Let $R' \in \mathcal{R}^N$ be such that $R'_k = R_k$ for each $k \in N\backslash\{i\}$ and $R'_i = R_j$. By *no-envy,* $\varphi_i(R') = 0$. If $\varphi_i(R) = a\ P_i\ 0$, i gains by reporting R_i instead of R'_i when her true preference is R'_i, contradicting *strategy-proofness.* If $0\ P_i\ \varphi_i(R) = a$, i gains by reporting R'_i instead of R_i when her true preference is R_i, contradicting *strategy-proofness.*

Now since, for each $R \in \mathcal{R}^N$ and each $i \in N$, $\varphi^*_i(R) = f(R_i, \widetilde{X} \backslash \cup_{j\neq i} U_0(R_j))$, by Claim 1, φ^* Pareto dominates any other rule satisfying the two properties.

Q.E.D.

Proof of Proposition 2: We first prove the first statement of Proposition 2. Suppose there exist a problem (R, s) and agents $i, j \in N$ such that $UF_j(R, s)\ P_i\ UF_i(R, s)$. This means that at the final application of the procedure, object $UF_j(R, s)$ is available. Since agent i does not demand it, this means that $UF_i(R, s)\ P_i\ UF_j(R, s)$.

To prove the second statement, suppose there exist another envy-free rule φ, a problem (R, s), and an agent $i \in N$ such that $\varphi_i(R, s)\ P_i\ UF_i(R, s)$. Note that at the last application of the UF procedure to (R, s), each agent is assigned her favorite object among the available ones. This means that object $\varphi_i(R, s)$ is removed from the set of available objects at some application of the procedure. If this happens when the procedure is applied for the first time, then φ is not envy-free. If it happens when the UF procedure is applied for the second time, this means that for φ to be envy-free, φ assigns some agent an object that was removed from the set of available objects when the UF procedure is applied for the first time. But this is not possible by the former statement. Continuing in this fashion, we conclude that φ cannot assign any object that is removed from the set of available objects at some application of the UF procedure.

Q.E.D.

The following remarks will be repeatedly used in the following lemmata. Remark 1 says that if an object a is scarce for an agent in a (sub)problem then each object that she prefers to a is scarce for her in the same (sub)problem. Let $\mathcal{P}_{(R,s)}$ be the set consisting of all subproblems of (R, s) and itself, i.e., $\mathcal{P}_{(R,s)} \equiv \underset{N' \subset N}{\cup} (R_{-N'}, s - |N'| * \mathbf{1})$.

Remark 2 *Let a problem $(R, s), I \in \mathcal{P}_{(R,s)}$, $a \in X$ and $i \in N$ be given. Then $a \in S(i, I)$ iff $(U_a(R_i) \cup \{a\}) \subseteq S(i, I)$.*

The next remark says that if an object is eliminated by UF in a subproblem I of a problem, then it is also eliminated by UF in any subproblem $I'$of I.

Remark 3 *Let a problem (R, s) be given and let $a \in X$. Let $N' \subset N$. If a is eliminated by UF in $(R_{-N'}, s - |N'| * \mathbf{1})$ then for each $N'' \subseteq N\backslash N'$, a is eliminated by UF in $(R_{-(N' \cup N'')}, s - |N' \cup N''| * \mathbf{1})$.*

Lemma 2 says that if an object a is scarce for an agent i in a problem then a is also scarce for i in any of its subproblems that is obtained by removing an agent (other than i) and reducing the supply of each object by 1.

Lemma 1 Let a problem (R, s) be given and $a \in X$. Let $a \in S(i, (R, s))$. Then, for each $k \neq i$, $a \in S(i,(R_{-k}, s - \mathbf{1}))$.

Proof : We will first show that Lemma 2 holds for two agent problems. Let $N \equiv \{i, k\}$, a problem (R, s) and $a \in X$ be given. Let $a \in S(i, (R, s))$. We will show that $a \in S(i, (R_{-k}, s - \mathbf{1}))$. Since $a \in S(i, (R, s))$, each $x \in U_a(R_i) \cup \{a\}$ is

either (i) eliminated by UF in $(R_{-i}, s - \mathbf{1})$
or (ii) scarce for agent k in $(R_{-i}, s - \mathbf{1})$

Let $x \in U_a(R_i) \cup \{a\}$.

Case 1: (i) holds for x.
By Remark 2, x is eliminated by UF in $(R_{-ik}, s - 2 * \mathbf{1})$.

Case 2: (ii) holds for x.
Since $x \in S(k, (R_{-i}, s - \mathbf{1}))$ and there is no remaining agent in $(R_{-ik}, s - 2 * \mathbf{1})$ for whom x may be scarce in $(R_{-ik}, s - 2 * \mathbf{1})$, x is eliminated by UF in $(R_{-ik}, s - 2 * \mathbf{1})$.

Thus, in each case, x satisfies (i) of the definition of scarcity of x for agent i in $(R_{-k}, s - \mathbf{1})$. Since x is arbitrary, this is true for each object in $U_a(R_i) \cup \{a\}$. Hence, $x \in S(i, (R_{-k}, s - \mathbf{1}))$. Assume by induction that Lemma 2 holds for problems with an agent set of cardinality 3, ..., $|N| - 1$. We will show that it also holds for problems with an agent set of cardinality $|N|$.
Let a problem (R, s), $a \in$and $i, k \in N$, $i \neq k$ be given. Let $a \in S(i, (R, s))$. Since $a \in S(i, (R, s))$, each $x \in U_a(R_i) \cup \{a\}$ is

either (i) eliminated by UF in $(R_{-i}, s - \mathbf{1})$
or (ii) scarce for some agent l in $(R_{-i}, s - \mathbf{1})$.

We will show that $a \in S(i, (R_{-k}, s - \mathbf{1}))$. For this purpose, we show that each $x \in U_a(R_i) \cup \{a\}$ is

either (i) eliminated by UF in $(R_{-ik}, s - 2 * \mathbf{1})$.
or (ii) scarce for some agent m in $(R_{-ik}, s - 2 * \mathbf{1})$.

Let $x \in U_a(R_i) \cup \{a\}$.

Case 1: Object x is eliminated by UF in $(R_{-i}, s - \mathbf{1})$.
By Remark 2, x is eliminated by UF in $(R_{-ik}, s - 2 * \mathbf{1})$.

Case 2: Object x is scarce for some agent l in $(R_{-i}, s - \mathbf{1})$.

Subcase 1: $k = l$.
By definition of scarcity of an object for an agent in a problem, x is

either (i) eliminated by UF in $(R_{-ik}, s - 2 * \mathbf{1})$
or (ii) scarce for some agent j in $(R_{-ik}, s - 2 * \mathbf{1})$

If (ii) holds, let $m \equiv j$.

Subcase 2: $k \neq l$.
By the induction hypothesis, $x \in S(l, (R_{-ik}, s - 2 * \mathbf{1}))$.
Let $m \equiv l$.

In each case, x is
either (i) eliminated by UF in $(R_{-ik}, s - 2 * \mathbf{1})$

or (ii) scarce for some agent m in $(R_{-ik}, s-2*\mathbf{1})$. Thus, x satisfies either (i) or (ii) of the definition of scarcity of a for agent i in $(R_{-k}, s-\mathbf{1})$. Since x is arbitrary, this is true for each object in $U_a(R_i)\cup\{a\}$. Hence, $a\in S(i,(R_{-k}, s-\mathbf{1}))$.

Q.E.D.

The next lemma is that rule ψ^* is well-defined.

Lemma 2 *For each problem (R,s), ψ^* chooses an allocation feasible for (X,s).*

Proof: For each problem (R,s), we need to show that no object a is assigned to more than s_a agents in (R,s).

It is easy to see that ψ^* assigns each agent at most one object. We prove the second statement. Suppose, by contradiction there are a problem (R,s), $a\in\widetilde{X}$ and $K\subseteq N$ such that $K\equiv\{i\in N:\psi_i^*(R,s)=a\}$ and $|K|\geq s+1$. Let $k\in K$. Hence, $\psi_k^*(R,s)=a$ and $a\notin S(k,(R,s))$. Let $T\equiv\{i\in N\backslash\{k\}: a\ P_i\ \varnothing\}$. Note that $K\subseteq T\cup\{k\}$. This, together with $|K|\geq s+1$ imply that $|T|\geq s$.
Let $T_1, T_2\subseteq T$.
Let $T_1\equiv\{i\in T:$ there is $x\in U_a(R_i)$ such that $x\notin S(i,(R_{-k}, s-\mathbf{1}))\}$ and
$T_2\equiv\{i\in T: U_a(R_i)\subseteq S(i,(R_{-k}, s-\mathbf{1}))\}$. Note that $T_1\cup T_2=T$.

Let $i\in T_1$. By definition of T_1, there is $x\in U_a(R_i)$ such that $x\notin S(i,(R_{-k}, s-\mathbf{1}))$.
By Lemma 2, $x\notin S(i,(R,s))$. This, together with $x\in U_a(R_i)$ imply that $\psi_i^*(R,s)\neq a$. Since i is arbitrary, for each $i\in T_1$, $\psi_i^*(R,s)\neq a$. Then $K\subseteq T_2\cup\{k\}$.
Let $t\equiv|T_2|$. Note that $t\geq s$. By definition of T_2,

$$\text{for each } i\in T_2, U_a(R_i)\subseteq S(i,(R_{-k}, s-\mathbf{1})). \tag{1}$$

Let $j_1\in T_2$. By (1) and Lemma 2,

$$\text{for each } i\in T_2\backslash\{j_1\}, U_a(R_i)\subseteq S(i,(R_{-kj_1}, s-2*\mathbf{1})). \tag{2}$$

$\vdots$

Let $j_{t-1}\in T_2\backslash\{j_1, j_2, ..., j_{t-2}\}$. By $(t-1)$ and Lemma 2,

$$U_a(R_{j_t})\subseteq S(j_t,(R_{-kj_1j_2\dots j_{t-1}}, s-t*\mathbf{1})). \tag{t}$$

Since $s\leq t$, $s-t-1\leq -1$. Thus, a is eliminated by UF in $(R_{-kj_1j_2\dots j_{t-1}j_t}, s-(t+1)*\mathbf{1})$. Note that a satisfies (i) of the definition of scarcity of a for j_t in $(R_{-kj_1j_2\dots j_{t-1}}, s-t*\mathbf{1})$. This together with (t) imply that

$$a\in S(j_t,(R_{-kj_1\dots.j_{t-1}}, s-t*\mathbf{1}). \tag{1*}$$

(1*) and $(t-1)$ imply that

$$a\in S(j_{t-1},(R_{-kj_1\dots.j_{t-2}}, s-(t-1)*\mathbf{1})). \tag{2*}$$

$\vdots$

$(t-1^*)$ and (1) imply that

$$a \in S(j_1, (R_{-k}, s - \mathbf{1})). \tag{t^*}$$

Note that (t^*) implies that a satisfies (ii) of the definition of scarcity of a for agent k in (R, s). Note that $a \notin S(k, (R, s))$. The last two statements, together with the definition of an object being scarce for an agent in a problem imply that there is $x \in U_a(R_k)$ that satisfies neither (i) nor (ii) of the definition of scarcity of a for k in (R, s) but then $x \notin S(k, (R, s))$.

This, together with $x \in U_a(R_k)$ imply that $\psi^*_k(R, s) \neq a$. A contradiction.

Q.E.D.

Proof of Theorem 1: We first prove that ψ^* is *strategy-proof.* Suppose not. There are $i \in N$, a problem (R, s) and $R'_i \in \mathcal{R}$ such that
$\psi^*_i((R'_i, R_{-i}), s) = a$ and $a \; P_i \; \psi^*_i(R, s)$. Then, $a \in S(i, (R, s))$. Thus, a is
either (i) eliminated by UF in $(R_{-i}, s - \mathbf{1})$
or (ii) scarce for some agent j in $(R_{-i}, s - \mathbf{1})$.

Since this is true regardless of agent i's preference, a also satisfies either (i) or (ii) of the definition of scarcity of a for agent i in $((R'_i, R_{-i}), s)$. Note that $\psi^*_i((R'_i, R_{-i}), s) = a$. Thus, $a \notin S(i, ((R'_i, R_{-i}), s))$. The last three statements imply that there is $x \in U_a(R'_i)$ which is
neither (i) eliminated by UF in $(R_{-i}, s - \mathbf{1})$
nor (ii) scarce for any agent in $(R_{-i}, s - \mathbf{1})$ but then $x \notin S(i, ((R'_i, R_{-i}), s))$. The fact that in each problem ψ^* assigns each agent her favorite object among the ones that are not scarce for him together with $x \notin S(i, ((R'_i, R_{-i}), s))$ and $x \in U_a(R'_i)$ imply that $\psi^*_i((R'_i, R_{-i}), s) \neq a$. A contradiction.

We prove the *envy-freeness* of ψ^* through Lemma 4 which makes use of Lemma 2. Lemma 4 states that if an object a is scarce for an agent i in a problem then a satisfies (ii) of the definition of scarcity of a for agents different from i in the same problem.

Lemma 3 Let $a \in S(i, (R, s))$. Then, for each $k \in N$, $k \neq i$, a satisfies (ii) of the definition of scarcity of a for agent k in (R, s)[12].

Proof : Let $i, k \in N$, $i \neq k$ be given. Let $a \in S(i, (R, s))$. By Lemma 2, $a \in S(i, (R_{-k}, s - \mathbf{1}))$. When checking (ii) of the definition of scarcity of a for k in (R, s) for object a, let j in the definition be i. Thus for object a, (ii) of the definition of scarcity of a for agent k in (R, s) is satisfied.

Q.E.D.

We now prove that ψ^* is *envy-free.* Suppose not. There are agents $i, k \in N$, $i \neq k$ such that $\psi^*_k(R, s) = a$ and $a \; P_i \; \psi^*_i(R, s)$. Then, $a \in S(i, (R, s))$. By Lemma 4, a satisfies (ii) of the definition of scarcity of a for k in (R, s). Since $\psi^*_k(R, s) = a$, $a \notin S(k, (R, s))$. This, together with the fact that a satisfies (ii) of the definition of scarcity of a for agent k in (R, s), imply that there is $x \in U_a(R_k)$ such that both (i) and (ii) of the definition of scarcity of a for i in (R, s) fail.

[12] Note that the lemma does not say that a is scarce for unit k in (X_H, R, s_H). For this to be true for each $x \in U_a(R_k) \cup \{a\}$, either (i) or (ii) of the definition of scarcity of a should be satisfied. The lemma says that the above statement holds for object a and doesn't say anything about objects in $U_a(R_k)$.

But this implies that $x \notin S(k,(R,s))$. This, together with $x \in U_a(R_k)$ imply that $\psi_k^*(R,s) \neq a$. A contradiction.

Q.E.D.

For the remaining of the proofs, it is worthwhile to observe that if one ever concludes that an object a is scarce for an agent i in a (sub)problem, then part (i) of the scarcity definition should hold for a so that the test of the definition terminates. In other words, when checking for the scarcity of a for i in a (sub)problem I, we should always end up in a subproblem of I, say I', in which a is eliminated by UF in I'.

Remark 4 *Let a problem (R,s), $I \in \mathcal{P}_{(R,s)}, a \in X$ and $i \in N$ be given. Let $a \in S(i,I)$. Then for each $x \in U_a(R_i) \cup \{a\}$, there is a subproblem I' of I such that i is not in the agent set of I' and x is eliminated by UF in I'.*

Lemma 5 is key to prove that ψ^* is Pareto dominant in the class of *strategy-proof* and *envy-free* rules. Lemma 5 states that no *envy-free* and *strategy-proof* rule ever assigns an agent an object that is scarce for her.

Lemma 4 No envy free and strategy-proof rule assigns an agent an object that is scarce for her.

Sketch of the proof: Before formally proving Lemma 5, we present a sketch of the proof for a particular problem and a particular envy-free and strategy-proof rule.

Step 1: Consider the problem in Example 3. Following the same steps in Example 3 we obtain that $a \in S(1,(R,s))$. Suppose, by contradiction that there exists an *envy-free* and *strategy-proof* rule φ such that $\varphi_1(R,s) = a$.

Let R_1' be as shown below. Let $x_0 = a$. By *strategy-proofness*, $z_0 \equiv \varphi_1((R_1', R_{-1}), s) = a$. Hence, $s_a \geq 1$. Since $a \in S(1,(R,s))$, a is either eliminated by UF or scarce for some agent in $(R_{-1}, s - \mathbf{1})$. Hence, a satisfies either (i) or (ii) of the definition of scarcity of a in $((R_1', R_{-1}), s)$. In Example 3 we determined that $a \in S(2,(R_{-1}, s-\mathbf{1}))$. Since a is the favorite object of 1 under R_1', $a \in S(1,((R_1', R_{-1}), s))$. The preference profile is as follows:

R_1'	R_2	R_3	R_4	R_5
$\boxed{x_0}$	$\boldsymbol{b}$	c	d	e
x_1	$\boldsymbol{a}$	b	c	d
x_2	0	0	0	b
$\vdots$				0

We change the next agent's preference. We noted above that $z_0 = a \in S(2,(R_{-1}, s-\mathbf{1}))$. By *no-envy*, $z^1 \equiv \varphi_2((R_1', R_{-1}), s) \in U_a(R_2) \cup \{a\} = \{a,b\}$. Suppose $z^1 = b$. (One can come up with the contradiction easily in the other case). Let $x_1 = z^1$. Let $N_1' \equiv N_0' \cup \{2\} = \{1,2\}$ and $X_1 \equiv X_0 \cup \{x_1\} = \{a,b\}$. Note that $|N_1'| = 2$. Let $R_2' = R_1'$. By *strategy-proofness*, $z_1 \equiv \varphi_2((R_{N_1'}', R_{-N_1'}), s) \in U_b(R_2') \cup \{b\} = X_1 = \{a,b\}$.
By *no-envy*, $\varphi_l((R_{N_1'}', R_{-N_1'}), s) = z_1$ for each $l \in N_1'$. Hence, $s_{z_1} \geq 2$. Suppose $z_1 = b$. (Otherwise if we assume $z_1 = a$, the contradiction is immediate). The preference profile is as follows:

R'_1	R'_2	R_3	R_4	R_5
a	a	$\boldsymbol{c}$	d	e
$\boxed{b}$	$\boxed{b}$	$\boldsymbol{b}$	c	d
x_2	x_2	0	0	b
$\vdots$	$\vdots$			0

Since $z_0 = a \in S(2, (R_{-1}, s - \mathbf{1}))$, by Remark 1,
$(U_a(R_2) \cup \{a\}) = \{a, b\} \subseteq S(2, (R_{-1}, s - \mathbf{1}))$. Hence, each of objects a and b is either eliminated by UF or scarce for some agent in $(R_{-12}, s - 2 * \mathbf{1})$. Thus $\{a, b\} \subseteq S(2, ((R'_2, R_{-12}), s))$. Indeed in Example 3 we determined that a is eliminated by UF in $(R_{-12}, s - 2 * \mathbf{1})$ and $b \in S(3, (R_{-12}, s - 2 * \mathbf{1}))$.

We now continue to change the next agent's preference. By *no-envy*,
$z^2 \equiv \varphi_3((R'_{N'_1}, R_{-N'_1}), s) \in U_b(R_3) \cup \{b\} = \{b, c\}$. Assume $z^2 = c$. (One can come up with the contradiction easily in the other case). Let $x_2 = z^2$. Let $N'_2 \equiv N'_1 \cup \{3\} = \{1, 2, 3\}$ and $X_2 \equiv X_1 \cup \{c\} = \{a, b, c\}$. Note that $|N'_2| = 3$. Let $R'_3 = R'_2$. By *strategy-proofness*, $z_2 \equiv \varphi_3((R'_{N'_2}, R_{-N'_2}), s) \in U_c(R'_3) \cup \{c\} = X_2 = \{a, b, c\}$.
By *no-envy*, $\varphi_l((R'_{N'_2}, R_{-N'_2}), s) = z_2$ for each $l \in N'_2$. Hence, $s_{z_2} \geq 3$. Suppose $z_2 = c$. (Contradiction is immediate for the other cases). The preference profile is as follows:

R'_1	R'_2	R'_3	R_4	R_5
a	a	a	$\boldsymbol{d}$	e
b	b	b	$\boldsymbol{c}$	d
$\boxed{c}$	$\boxed{c}$	$\boxed{c}$	0	b
x_3	x_3	x_3		0
$\vdots$	$\vdots$	$\vdots$		

Since $z_1 = b \in S(3, (R_{-12}, s - 2 * \mathbf{1}))$, by Remark 1,
$(U_b(R_3) \cup \{b\}) = \{b, c\} \subseteq S(3, (R_{-12}, s - 2 * \mathbf{1}))$. Hence, $z_2 = c$ is either eliminated by UF or scarce for some agent in $(R_{-123}, s - 3 * \mathbf{1})$. But then c satisfies either (i) or (ii) of the definition of scarcity of c for 3 in $((R'_3, R_{-123},), s - 2 * \mathbf{1})$. Indeed in Example 3, we determined that c is eliminated by UF in $(R_{-123}, s - 3 * \mathbf{1})$. To conclude the argument we need to show that each object in X_2 is either eliminated by UF or scarce for some agent in $(R_{-123}, s - 3 * \mathbf{1})$. (Because $\varphi_3((R'_{N'_2}, R_{-N'_2}), s)$ could be any object in X_2). Since $\{a, b\} \subseteq S(2, ((R'_2, R_{-12}), s - \mathbf{1}))$, by Lemma 2, $\{a, b\} \subseteq S(2, ((R'_2, R_{-123}), s - 2 * \mathbf{1}))$. By $R'_2 = R'_3$, $\{a, b\} \subseteq S(3, ((R'_3, R_{-123}), s - 2 * \mathbf{1}))$. Hence each of a and b is either eliminated by UF or scarce for some agent in $(R_{-123}, s - 3 * \mathbf{1}))$. Indeed in Example 3, we determined that a and b are eliminated by UF in $(R_{-123}, s - 3 * \mathbf{1})$.

Step 1 ends because each object in X_2 is eliminated by UF in $(R_{-123}, s - 3 * \mathbf{1})$.

Step 2: Let Z be the set of all objects that are eliminated by UF in $(R_{-123}, s - 3 * \mathbf{1})$. We have $Z = \{a, b, c, d\}$. Note also that $X_2 \subset Z$.

We next determine the set of agents $J \subseteq N \backslash N'_2 = \{4, 5\}$ each of which prefers some object x in Z to null under R, and when UF is applied to $(R_{-123}, s - 3 * \mathbf{1})$ gets an object that is less preferred to x. We have $c \ P_4 \ 0$, $UF_4(R_{-123}, s - 3 * \mathbf{1}) = 0 \in L_c(R_4)$ thus $4 \in J$. Objects b and d are the only objects that are in Z and preferred by 5 to null; $b \ P_5 \ 0$, $d \ P_5 \ 0$, $UF_5(R_{-123}, s - 3 * \mathbf{1}) = e$, $e \notin L_b(R_5)$ and $e \notin L_d(R_5)$. Thus $5 \notin J$. Hence, $J = \{3\}$. We now complete the preferences of

1, 2 and 3 by letting all objects in $Z \backslash X_2$ be more preferred to 0 and less preferred to those in X_2 and letting 0 be more preferred to $X \backslash Z$. The preference profile is as follows:

R'_1	R'_2	R'_3	R_4	R_5
a	a	a	$\boldsymbol{d}$	e
b	b	b	$\boldsymbol{c}$	d
$\boxed{c}$	$\boxed{c}$	$\boxed{c}$	0	b
d	d	d		0
0	0	0		

Since $z_2 = c$ is eliminated by UF in $(R_{-123}, s - 3 * \mathbf{1})$ and $s_c - 3 \geq 0$ there is at least one agent in J that prefers c to null under R and is assigned by UF in $(R_{-123}, s - 3 * \mathbf{1})$ an object less preferred to c. Agent 4 is the only candidate. By *no-envy*, $z^3 \equiv \varphi_4((R'_{N'_2}, R_{-N'_2}), s) \in U_c(R_4) \cup \{c\} = \{c, d\}$. Assume $z^3 = d$. (The argument applies for the other case). Let $N'_3 \equiv N'_2 \cup \{4\} = \{1, 2, 3, 4\}$. Note that $|N'_3| = 4$. Let $R'_4 = R'_3$. Since 4 prefers c to null under R_4 and UF assigns 4 in $(R_{-123}, s - 3 * \mathbf{1})$ an object that is less preferred to c, each object in $U_c(R_4) \cup \{c\}$ is eliminated by UF in $(R_{-123}, s - 3 * \mathbf{1})$. Thus $z^3 \in Z$ and $z^3 \; P'_4 \; 0$. By *strategy-proofness*, $z_3 \equiv \varphi_4((R'_{N'_3}, R_{-N'_3}), s)$ P'_4 0. Thus $z_3 \in Z$. By *no envy*, $\varphi_l((R'_{N'_3}, R_{-N'_3}), s) = z_3$ for each $l \in N'_3$. Thus $s_{z_3} \geq 4$. Since $z_3 \in Z$, it is eliminated by UF in $(R_{-123}, s - 3 * \mathbf{1})$. This, together with $s_{z_3} \geq 4$ imply that there are at least 2 agents in J that prefer z_3 to null under R and are assigned by UF in $(R_{-123}, s - 3 * \mathbf{1})$ an object less preferred to z_3. Hence $|J| \geq 2$. A contradiction.

Proof of Lemma 5: First, we prove the following claim.

Claim 2: *Let a problem (R, s) and $a \in X$ be given. Let $N' \subset N$. If a is eliminated by UF in $(R_{-N'}, s - |N'| * 1)$ and $s - |N'| \geq 0$ then there is an agent $i \in N \backslash N'$ such that $a \in S(i, (R_{-N'}, s - |N'| * 1))$.*

Proof: Let a problem (R, s) and $a \in X$ be given. Let $N' \subset N$. Let a be eliminated by UF in $(R_{-N'}, s - |N'| * \mathbf{1})$ and $s - |N'| \geq 0$. Then there are at least $s - |N'| + 1$ agents in $N \backslash N'$ that prefer a to *null* and are assigned by UF in $(R_{-N'}, s - |N'| * \mathbf{1})$ an object that a is preferred to. Note that $s - |N'| + 1 \geq 1$. Thus, there is at least one such agent in $N \backslash N'$. Let $i \in N \backslash N'$ be such an agent, i.e., $a \; P_i \; 0$ and $UF_i(R_{-N'}, s - |N'| * \mathbf{1}) \in L_a(R_i)$. Since $UF_i(R_{-N'}, s - |N'| * \mathbf{1}) \in L_a(R_i)$, each $x \in U_a(R_i) \cup \{a\}$ is eliminated by UF in $(R_{-N'}, s - |N'| * \mathbf{1})$ and this, together with Remark 2, imply that each $x \in U_a(R_i) \cup \{a\}$ is eliminated by UF in $(R_{-(N' \cup \{i\})}, s - (|N'| + 1) * \mathbf{1})$. Thus, each $x \in U_a(R_i) \cup \{a\}$ satisfies (i) of the definition of scarcity of a for i in $(R_{-N'}, s - |N'| * \mathbf{1})$.

Q.E.D.

Suppose by contradiction there exists an *envy-free* and *strategy-proof* rule φ, a problem (R, s), an object $a \in X$ and an agent $i \in N$ such that $\varphi_i(R, s) = a$ and $a \in S(i, (R, s))$.

For each $k \in N$, let R'_k be such that $x_0 \; R'_k \; x_1 \; R'_k \; x_2 \; R'_k .... x_p \; R'_k \; z \; R'_k \; 0 \; R'_k \; z'$ for each $z \in Z \backslash \bigcup_{s=0}^p x_s$ and each $z' \in X \backslash (\bigcup_{s=0}^p x_s \cup Z)$. Preferences R'_k will be constructed in steps 1&2 below, and therefore objects $(x_s)_{s=0}^p$ and the set Z will be determined throughout the proof. Let $x_0 = a$.

Step 1: Let $j_0 \equiv i$. By *strategy-proofness*, $z_0 \equiv \varphi_{j_0}((R'_{j_0}, R_{-j_0}), s) = x_0$. Hence, $s \geq 1$. Let $N'_0 \equiv \{j_0\}$ and $X_0 \equiv \{x_0\}$.

Since $z_0(= x_0) \in S(j_0, (R, s))$, it is either eliminated by UF or scarce for some agent in $(R_{-j_0}, s-\mathbf{1})$. Hence $z_0(= x_0)$ satisfies either (i) or (ii) of the definition of scarcity of $z_0(= x_0)$ in $((R'_{j_0}, R_{-j_0}), s)$. Since $z_0(= x_0)$ is the favorite object of j_0 under R'_{j_0}, the previous statement implies statement (0).

$$z_0(= x_0) \in S(j_0, ((R'_{j_0}, R_{-j_0}), s)). \tag{0}$$

By (0), $z_0(= x_0)$ is
either (i) eliminated by UF in $(R_{-j_0}, s - \mathbf{1})$.
or (ii) scarce for some agent j in $(R_{-j_0}, s - \mathbf{1})$.

Note that $s - 1 \geq 0$. If (i) holds then by Claim 2, there is an agent $j \in N \backslash N'_0$ such that $z_0(= x_0) \in S(j, (R_{-j_0}, s - \mathbf{1}))$, i.e., (ii) also holds.

Let $j_1 \in N \backslash N'_0$ be such that $z_0(= x_0) \in S(j_1, (R_{-j_0}, s - \mathbf{1}))$. By *no-envy*, $z^1 \equiv \varphi_{j_1}((R'_{j_0}, R_{-j_0}), s) \in U_{z_0}(R_{j_1}) \cup \{z_0\}$. Let $x_1 = z^1$. Note that $x_1 \in U_{z_0}(R_{j_1}) \cup \{z_0\}$. Let $N'_1 \equiv N'_0 \cup \{j_1\}$. Note that $|N'_1| = 2$. Let $X_1 \equiv X_0 \cup \{x_1\}$. By *strategy-proofness*, $z_1 \equiv \varphi_{j_1}((R'_{N'_1}, R_{-N'_1}), s) \in X_1$. By *no-envy*, $\varphi_l((R'_{N'_1}, R_{-N'_1}), s) = z_1$ for each $l \in N'_1$. Hence $s \geq 2$. By $z_0(= x_0) \in S(j_1, (R_{-j_0}, s-\mathbf{1}))$ and Remark 1, $(U_{z_0}(R_{j_1}) \cup \{z_0\}) \subseteq S(j_1, (R_{-j_0}, s-\mathbf{1}))$. Specifically, $x_1 \in S(j_1, (R_{-j_0}, s-\mathbf{1}))$. Hence, x_1 satisfies either (i) or (ii) of the definition of scarcity of x_1 for j_1 in $((R'_{j_1}, R_{-N'_1},), s - \mathbf{1})$. By (0), Lemma 2and $R'_{j_1} = R'_{j_0}$, $z_0(= x_0) \in S(j_1, ((R'_{j_1}, R_{-N'_1}), s - \mathbf{1}))$[13]. (Indeed, by (0) and Lemma 2, $z_0(= x_0) \in S(j_0, ((R'_{j_0}, R_{-N'_1}), s-\mathbf{1}))$. This, together with $R'_{j_1} = R'_{j_0}$ imply the conclusion.) This, together with the previous statement imply statement (1).

$$X_1 \subseteq S(j_1, ((R'_{j_1}, R_{-N'_1}), s - \mathbf{1})). \tag{1}$$

Note that $z_1 \in X_1$. By (1), z_1 is
either (i) eliminated by UF in $(R_{-N'_1}, s - 2 * \mathbf{1})$.
or (ii) scarce for some agent j in $(R_{-N'_1}, s - 2 * \mathbf{1})$.

Note that $s - 2 \geq 0$. If (i) holds then by Claim 2, there is an agent $j \in N \backslash N'_1$ such that $z_1 \in S(j, (R_{-N'_1}, s - 2 * \mathbf{1}))$, i.e., (ii) also holds.

We continue applying the same argument. In general, for each $v \in \mathbb{N} \backslash \{0\}$, let $j_v \in N \backslash N'_{v-1}$ be such that $z_{v-1} \in S(j_v, (R_{-N'_{v-1}}, s - v * \mathbf{1}))$. By *no-envy*, $z^v \equiv \varphi_{j_v}((R'_{N'_{v-1}}, R_{-N'_{v-1}}), s) \in U_{z_{v-1}}(R_{j_v}) \cup \{z_{v-1}\}$. If $z^v \notin X_{v-1}$, let $x_v = z^v$, otherwise let $x_v = x_{v-1}$. Let $N'_v \equiv N'_{v-1} \cup \{j_v\}$. Note that $|N'_v| = v + 1$. Let $X_v \equiv X_{v-1} \cup \{x_v\}$. Thus, if $z^v \notin X_{v-1}$, we have $X_v \backslash X_{v-1} = \{z^v\}$, otherwise $X_v \equiv X_{v-1}$. By *strategy-proofness*, $z_v \equiv \varphi_{j_v}((R'_{N'_v}, R_{-N'_v}), s) \in X_v$. By *no-envy*, $\varphi_l((R'_{N'_v}, R_{-N'_v}), s) = z_v$ for each $l \in N'_v$. Hence $s \geq v+1$. If $X_v \backslash X_{v-1} = \{z^v\}$, note that $x_v = z^v \in U_{z_{v-1}}(R_{j_v}) \cup \{z_{v-1}\}$. By $z_{v-1} \in S(j_v, (R_{-N'_{v-1}}, s - v * \mathbf{1}))$ and Remark 1, $(U_{z_{v-1}}(R_{j_v}) \cup \{z_{v-1}\}) \subseteq S(j_v, (R_{-N'_{v-1}}, s - v * \mathbf{1}))$. Specifically $x_v \in S(j_v, (R_{-N'_{v-1}}, s - v * \mathbf{1}))$. Hence x_v satisfies either (i) or (ii) of the definition of scarcity of x_v for j_v in $((R'_{j_v}, R_{-N'_v}), s - v * \mathbf{1})$. By $(v - 1)$, Lemma 2and $R'_{j_v} = R'_{j_{v-1}}$, $X_{v-1} \subseteq S(j_v, ((R'_{j_v}, R_{-N'_v}), s - v * \mathbf{1}))$. This, together with the previous statement imply statement (v).
If $X_v \equiv X_{v-1}$, then statement $(v - 1)$, Lemma 2and $R'_{j_v} = R'_{j_{v-1}}$ imply statement (v).

[13] This conclusion could be derived from $z_0(= x_0) \in S(j_1, (X_H, R_{-j_0}, s_H - \mathbf{1}))$ and z_0 being j_1's favorite object under R'_{j_1}. Nonetheless, the argument given in the proof is preferred to have symmetry between arguments as we continue changing units' preferences.

$$X_v \subseteq S(j_v, ((R'_{j_v}, R_{-N'_v}), s - v * \mathbf{1})). \qquad (v)$$

Note that $z_v \in X_v$. By (v), z_v is

either (i) eliminated by UF in $(R_{-N'_v}, s - (v+1) * \mathbf{1})$.

or (ii) scarce for some agent j in $(R_{-N'_v}, s - (v+1) * \mathbf{1})$.

Note that $s - (v+1) \geq 0$. If (i) holds then by Claim 2, there is some agent $j \in N \backslash N'_v$ such that $z_v \in S(j, (R_{-N'_v}, s - (v+1) * \mathbf{1}))$, i.e., (ii) also holds.

Let $v \in \mathbb{N}$. Since $X_v \subseteq S(j_v, ((R'_{j_v}, R_{-N'_v}), s - v * \mathbf{1}))$, by Remark 3, for each $x \in X_v$ there is a subproblem of $((R'_{j_v}, R_{-N'_v}), s - v * \mathbf{1})$, say I', such that j_v is not in the agent set of P' and x is eliminated by UF in I'. In other words, for each $x \in X_v$, the test of scarcity of x for j_v in $((R'_{j_v}, R_{-N'_v}), s - v * \mathbf{1})$, at some point terminates in part (i) of the scarcity definition.

By the finiteness of N, we will have $f \in \mathbb{N}$, $f \leq n$ such that for each $x \in X_f$, x is eliminated by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$. Since $z_f \in X_f$, z_f is eliminated by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$ where $z_f \equiv \varphi_{j_f}((R'_{N'_f}, R_{-N'_f}), s)$. By *no-envy*, $\varphi_l((R'_{N'_f}, R_{-N'_f}), s) = z_f$ for each $l \in N'_f$. Hence, $s - (f+1) \geq 0$. Let $p \equiv f$.

Step 2: Let Z be the set of all objects that are eliminated by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$. We know from step 1 that Z is nonempty. Indeed $X_f \subseteq Z$. Let J be the set of all agents in $N \backslash N'_f$ that prefer some object $x \in Z$ to *null* under R and are assigned by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$ an object that x is preferred to. Formally,

$$J \equiv \left\{ l \in N \backslash N'_f : \begin{array}{c} x \ P_l \ 0 \text{ and } UF_l(R_{-N'_f}, s - (f+1) * \mathbf{1}) \\ \in L_x(R_l) \text{ for some } x \in Z \end{array} \right\}$$

Let $g \equiv |J|$. Since z_f is eliminated by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$ and $s - (f+1) \geq 0$ there is at least one agent in J that prefers z_f to *null* under R and is assigned by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$ an object that z_f is preferred to. (Hence, J is nonempty.)

Let $j_{f+1} \in J$ be such that $z_f \ P_{j_{f+1}} 0$ and $UF_{j_{f+1}}(R_{-N'_f}, s - (f+1) * \mathbf{1}) \in L_{z_f}(R_{j_{f+1}})$. By *no-envy*, $z^{f+1} \equiv \varphi_{j_{f+1}}((R'_{N'_f}, R_{-N'_f}), s) \in U_{z_f}(R_{j_{f+1}}) \cup \{z_f\}$. Let $N'_{f+1} \equiv N'_f \cup \{j_{f+1}\}$. Note that $|N'_{f+1}| = f + 2$. Since $z^{f+1} \in U_{z_f}(R_{j_{f+1}}) \cup \{z_f\}$ and $UF_{j_{f+1}}(R_{-N'_f}, s - (f+1) * \mathbf{1}) \in L_{z_f}(R_{j_{f+1}})$, z^{f+1} is eliminated by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$. Hence, $z^{f+1} \in Z$. By *strategy-proofness*, $z_{f+1} \equiv \varphi_{j_{f+1}}((R'_{N'_{f+1}}, R_{-N'_{f+1}}), s) \ P'_{j_{f+1}} 0$. Thus $z_{f+1} \in Z$. By *no-envy*, $\varphi_l((R'_{N'_{f+1}}, R_{-N'_{f+1}}), s) = z_{f+1}$ for each $l \in N'_{f+1}$. Thus $s \geq f + 2$. Since $z_{f+1} \in Z$, it is eliminated by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$. This, together with $s \geq f+2$ imply that there are at least two agents in J that prefer z_{f+1} to *null* under R and are assigned by UF in $(R_{-N'_f}, s - (f+1) * \mathbf{1})$ an object that z_{f+1} is preferred to. Hence there is at least one such agent in $J \backslash \{j_{f+1}\}$.

In general, let $v \in \{1, .., g\}$. Let $j_{f+v} \in J \backslash \{j_{f+1}, .., j_{f+v-1}\}$ be such that $z_{f+v-1} P_{j_{f+v}} 0$ and $UF_{j_{f+v}}(R_{-N'_f}, s - (f+1) * \mathbf{1}) \in L_{z_{f+v-1}}(R_{j_{f+v}})$. By *no-envy*, $z^{f+v} \equiv \varphi_{j_{f+v}}((R'_{N'_{f+v-1}}, R_{-N'_{f+v-1}}), s) \in U_{z_f+v-1}(R_{j_{f+1}}) \cup \{z_{f+v-1}\}$. Let $N'_{f+v} \equiv N'_{f+v-1} \cup \{j_{f+v}\}$. Note that $|N'_{f+v}| = f + v + 1$. Since $z^{f+v} \in U_{z_{f+v-1}}(R_{j_{f+v}}) \cup \{z_{f+v-1}\}$ and

$UF_{j_{f+v}}(R_{-N'_f}, s-(f+1)*\mathbf{1}) \in L_{z_{f+v-1}}(R_{j_{f+v}})$, z^{f+v} is eliminated by UF in $(R_{-N'_f}, s-(f+1)*\mathbf{1})$. Hence, $z^{f+v} \in Z$. By *strategy-proofness*, $z_{f+v} \equiv \varphi_{j_{f+v}}((R'_{N'_{f+v}}, R_{-N'_{f+v}}), s)\ P'_{j_{f+v}} 0$. Thus $z_{f+v} \in Z$. By *no-envy*, $\varphi_l((R'_{N'_{f+v}}, R_{-N'_{f+v}}), s) = z_{f+v}$ for each $l \in N'_{f+v}$. Thus $s \geq f+v+1$. Since $z_{f+v} \in Z$, it is eliminated by UF in $(R_{-N'_f}, s-(f+1)*\mathbf{1})$. This, together with $s \geq f+v+1$ imply that there are at least $v+1$ agents in J that prefer z_{f+v} to *null* under R and are assigned by UF in $(R_{-N'_f}, s-(f+1)*\mathbf{1})$ an object that z_{f+v} is preferred to. Hence there is at least one such agent in $J \backslash \{j_{f+1}, .., j_{f+v}\}$. Hence, $|J| \geq v+1$.

Applying the same argument repeatedly one can show that when $v = g$, we have $|J| \geq g+1$. A contradiction.

Q.E.D.

We finally prove that ψ^* Pareto dominates any other *envy-free* and *strategy-proof* rule. Suppose by contradiction there exist another *envy-free* and *strategy proof* rule ψ, a problem (R, s), and an agent $i \in N$ such that $\psi_i(R, s) = z_0$ and $z_0\ P_i\ \psi^*_i(R, s)$. Then, $z_0 \in S(i, (R, s))$. By Lemma 5, $\psi_i(R, s) \neq z_0$. A contradiction.

Q.E.D.

7 References

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