1971

Arithmetical reducibilities.

Alan L. Selman
Carnegie Mellon University
NOTICE WARNING CONCERNING COPYRIGHT RESTRICTIONS:
The copyright law of the United States (title 17, U.S. Code) governs the making of photocopies or other reproductions of copyrighted material. Any copying of this document without permission of its author may be prohibited by law.
ARITHMETICAL REDUCIBILITY, II

by

Alan L. Selman

Report 71-15

February 1971
ARITHMETICAL REDUCIBILITIES, II

by

Alan L. Selman

Abstract

Certain reducibilities which generalize many-one reducibility are studied. Let $\leq_{\text{rm}}$ be the result of eliminating the bounded quantifier in the definition of $\leq_1$. It is shown that $\leq_1$ differs from the reducibility $\leq_{\text{rm}}$ on sets of the same Kleene-Post degree. Also, a characterization of "\( \Sigma \) in" is given, which for \( n = 1 \) enables us to make more precise the difference between "\( A \in \Sigma_1^B \)" and "\( A \leq_1 B \)."
Arithmetical Reducibilities, II

by

Alan L. Selman

Introduction.

Concepts and notation present in this paper refer to our paper [3]. For brevity, Theorem x.y of [3] will be cited here as Theorem I.x.y. For the convenience of the reader we repeat here the following two definitions.

Definition 1. If $\mathcal{R}$ and $\mathcal{I}$ are binary relations defined on the set of all subsets of $\omega$, then $\mathcal{R}$ is an $\mathcal{I}$-reducibility relation, if $\mathcal{R}$ is reflexive, $\mathcal{R}$ is transitive, and for all sets $A$ and $B$, if $A \mathcal{R} B$, then $A \mathcal{I} B$.

Definition 2. $A \subseteq_n B \iff \forall X[B \in \Sigma^X_n \rightarrow A \in \Sigma^X_n]$, $n \geq 1$.

$A \subseteq_n B \iff \forall X[B \in \Pi^X_n \rightarrow A \in \Pi^X_n]$, $n \geq 1$. $A \subseteq_n B \iff$ there exist recursive functions $f$ and $g$ so that $\forall x(x \in A \iff \exists y \exists z f(y) \in B)$.

The $\Sigma_n$-reducibilities $\subseteq_n$, $n \geq 1$, (and to a lesser extent the $\Pi_n$-reducibilities $\subseteq_n$, $n \geq 1$), were studied in Chapter 2 of [3]. Also, citing Theorem I.2.8, $A \subseteq_n B \leftrightarrow A \subseteq_n B$ for all sets $A$ and $B$ so that $B \neq \emptyset$ and $B \neq \omega$. It was shown that none of the reducibilities $\subseteq_n$ generalize relative recursion, but it is an immediate consequence of Theorem I.2.8 and the hierarchy theorem, Theorem I.2.3, that each $\subseteq_n$ does generalize many-one reducibility.
One aim of the present paper is to make clearer the difference between $\Sigma_1$ and "\( \Sigma_1 \) in". The first two sections are largely devoted to this end. Central to this discussion is the concept of a positive reducibility to be introduced in section 1. Also, this concept will enable us to elaborate on the principal open questions raised in [3].

Another aim of this paper is to study certain other reducibilities which also generalize many-one reducibility. In this direction, our attention is restricted to certain $\Sigma_1$-reducibilities which arise naturally from our considerations of the sequence $\mathbb{S}_n$, $n \geq 1$. This study will be taken up in sections 3 and 4. In section 3 we study a reducibility, $\leq_{\text{rm}}$, which is the result of eliminating the bounded quantifier in the definition of $\mathbb{S}_1$. It is proved in this section that $\mathbb{S}_1$ differs from $\leq_{\text{rm}}$ on sets of the same Kleene-Post degree. In section 4 we study the reducibility $\mathbb{S}_1 \cap \mathbb{P}_1$. As is easily seen (Theorem 6), $\mathbb{S}_1 \cap \mathbb{P}_1$ is a proper subrecursive reducibility.

1. Positive Reducibilities.

Definition 3. Let $A$ and $B$ be any two sets. If $A \in \Sigma_n^B$, then $A \in \Sigma_n^B$ in a positive sense if there is a predicate $\exists y S(x,y)$ which satisfies the following two properties:

(i) $\forall x (x \in A \rightarrow \exists y S(x,y))$; and

(ii) $S$ is constructed using the propositional connectives $\land$ and $\lor$, together with bounded quantifiers, from predicates $P_1, \ldots, P_k, P_i \in \Sigma_n$, $i = 1, \ldots, k$, $k \geq 1$, and from predicates $f(x,y) \in B$ and $f(x,y, x_1, \ldots, x_n) \in B$, $f$ recursive, $x_1, \ldots, x_n$ not free in $S$, $n \geq 1$. 

Definition 4. A $\Sigma_n^*$-reducibility relation $\mathcal{R}$ is positive if for each set $A$ and $B$ so that $A \mathcal{R} B$, $A \in \Sigma_n^*$ in a positive sense.

Theorem 1. If $\mathcal{R}$ is a positive $\Sigma_n^*$-reducibility, then $\mathcal{R} \subseteq \Sigma_n^*$.

Proof. The proof consists of an easy induction argument.

Essentially, if $A \mathcal{R} B$ and $B \in \Sigma_n^*$, then there is a predicate $S(x,y)$ which satisfies properties (i) and (ii) of Definition 3, and there is a predicate $R^C$ which is recursive in $C$ so that $x \in B \iff \exists z \forall z \ldots Q_n R^C(x, z_1, \ldots, z_n)$. If all occurrences of $B$ in $S$ are replaced by $\exists z_1 \forall z_2 \ldots Q_n R^C(x, z_1, \ldots, z_n)$, then, because $S$ contains no occurrences of $\sim$ and no occurrences of unbounded quantifiers, the resulting predicate can be put into prenex normal form $\Pi M$, where the prefix $\Pi$ consists of $n$-alternating quantifiers, and the matrix $M$ is recursive in $C$. Thus $A \in \Sigma_n^*$.

Remark. It is clear that Theorem 1 will not hold if material implication and negation are used in the underlying propositional logic of Definition 3 (ii). (Also, see Theorem 3 and the discussion preceding Theorem 3). Moreover, suppose $\varphi$ is an arbitrary truth function of two arguments and suppose $\varphi$ is the binary connective whose truth-table is given by $\hat{\varphi}$. Direct examination of the sixteen distinct truth-functions of two arguments shows that at least one of the following holds:

(1) $\varphi$ is defined in the logic generated by $\{\land, \lor\}$;

(2) $\hat{\varphi}$ is a constant function;

(3) negation is definable in the propositional logic generated by $\{\varphi, \land, \lor\}$;
Theorem 2. $S_1$ is a positive $\Sigma_1$-reducibility.

Proof. The theorem is a corollary of Theorem 1.2.8 for all but the special cases. For the special cases, $B = \emptyset$ and $B = w$, observe that if $A \in \Sigma_1$, then $A \in \Sigma_1^B$ in a positive sense for all $B$.

Corollary 1. If $A \in \Sigma_1$ in a positive sense, $B \not= \emptyset$ and $B \not= w$, then there exist recursive functions $f$ and $g$ so that

$$
\forall x(x \in A \leftrightarrow \exists y \forall z \leq f(y) g(x, y, z) \in B).
$$

Corollary 1 is interesting, since Definition 3 allows for predicates $\exists yS$ of arbitrary finite length.

Is $S_n$, for $n > 1$, a maximal $\Sigma_n$-reducibility? Is there something analogous to Theorem 1.2.8 for $n > 1$? We conjecture that the converse of Theorem 1 is true. We state this in the following Conjecture 2.

An argument identical to the proof of Theorem 1 proves the following lemma.

Lemma 1. If $A \in \Sigma_n^B$ in a positive sense and $B \in \Sigma_n^C$ in a positive sense, then $A \in \Sigma_n^C$ in a positive sense.
Conjecture 1. $\mathcal{S}_n$ is a positive $\Sigma_n$-reducibility.

Conjecture 2. $A \in \Sigma_n^B$ in a positive sense $\iff \forall X [B \in \Sigma_n^X$ in a positive sense $\implies A \in \Sigma_n^X]$. 

By Lemma 1, the implication from left to right of Conjecture 2 is true. By Corollary I.2.1, Theorem 1, and Theorem 2, both Conjectures 1 and 2 are true for the case $n = 1$. Conjecture 2 implies both Conjecture 1 and the maximality of $\mathcal{S}_n$. In fact for $n > 1$, let $\mathcal{J}_n$ denote the relation defined by $A \mathcal{J}_n B \iff A \in \Sigma_n^B$ in a positive sense. (By Corollary 1, Theorem I.2.8, and Theorem I.2.2, if $B \neq \emptyset$ and $B \neq \omega$, then $A \mathcal{J}_1 B \iff A \in \Sigma_1^B$ in a positive sense.) Then, suppose $\mathcal{J}_n \subseteq \mathcal{R} \subseteq \Sigma_n$ in", and suppose Conjecture 2 is true. There exist sets $A$ and $B$ so that $A \mathcal{R} B$ and $A \nsubseteq \mathcal{J}_n B$. Thus $\exists X [B \mathcal{J}_n X \& A \in \Sigma_n^X]$. $A \mathcal{R} B$ and $B \nsubseteq \Sigma_n$, but $A \in \Sigma_n^X$. Therefore, $\mathcal{R}$ is not transitive. By Lemma 1, $\mathcal{J}_n$ is transitive. Hence $\mathcal{J}_n$ is a maximal $\Sigma_n$-reducibility relation. By Theorem 1, $\mathcal{J}_n \subseteq \mathcal{S}_n$. Hence $\mathcal{J}_n = \mathcal{S}_n$ and $\mathcal{S}_n$ is a maximal $\Sigma_n$-reducibility.

2. The Relations "$\Sigma_n$ in".

The following Theorem 3 gives a characterization of $A \in \Sigma_n^B$, $B \neq \emptyset$ and $B \neq \omega$. A comparison of this characterization for $n = 1$ with Corollary 1 pinpoints the difference between "$A \in \Sigma_1^B$" and "$A \in \Sigma_1^B$ in a positive sense".
Theorem 3. For all sets $A$ and $B$, $B \neq \emptyset$ and $B \neq \omega$, the following are equivalent:

1. $A \in \Sigma^B_n$;

2. there exists a recursive predicate $R$ and recursive functions $f, g, h$ so that if $n$ is odd, then

$$
\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x, x_1, \ldots, x_n) \\
& \land \forall y, y < f(x_n) (g(x_n, y) \in B \land h(x_n, y) \notin B)],
$$

and if $n$ is even, then

$$
\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n \forall x_{n-1} [R(x, x_1, \ldots, x_n) \\
& \land \exists y, y < f(x_n) (g(x_n, y, x_1, \ldots, x_n) \in B \lor h(x_n, y) \notin B)],
$$

3. there exist recursive functions $f, g, h$ so that if $n$ is odd, then

$$
\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n \forall y, y < f(x_n) \\
(g(x, y, x_1, \ldots, x_n) \in B \land h(x_n, y) \notin B)),
$$

and if $n$ is even, then

$$
\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n \exists y, y < f(x_n) \\
(g(x, y, x_1, \ldots, x_n) \in B \lor h(x_n, y) \notin B)).
$$

Proof. Suppose $A \in \Sigma^B_n$, $B \neq \emptyset$, $B \neq \omega$, and $n$ is odd.

Let $Ch(z) = z$ is characteristic sequence number. (See [3, Chapter 2, §1].) For some $e$, $\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n \exists y_n \exists x_{n-1} [h(x_n), e, x_1, \ldots, x_{n-1}]$, where $h$ is the characteristic function of the set $B$. 


\( \forall x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n T^n_n(h(x_n), e, x_1, \ldots, x_{n-1}) \)

\[ \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [Ch(x_n) \land \forall y < \ell h(x_n)((x_n)_y = 1 \land \forall y \in B) \land T^n_n(x_n, e, x_1, \ldots, x_{n-1})]. \]

Let \( R(x, x_1, \ldots, x_n) \equiv Ch(x_n) \land T^n_n(x_n, e, x_1, \ldots, x_{n-1}). \)

Then

\[ \forall x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x, x_1, \ldots, x_n) \land \forall y < \ell h(x_n)((x_n)_y = 1 \land y \in B)]. \]

\[ \forall y < \ell h(x_n)((x_n)_y = 1 \leftrightarrow y \in B) \leftrightarrow \forall y < \ell h(x_n)((x_n)_y = 1 \rightarrow y \in B) \land \forall y < \ell h(x_n)(y \in B \rightarrow (x_n)_y = 1). \]

Let \( a \in B \) and \( b \notin B \). Define

\[ g(x_n, y) = \begin{cases} \ y, (x_n)_y = 1 \\ a, \text{ otherwise.} \end{cases} \]

Define

\[ h(x_n, y) = \begin{cases} \ y, (x_n)_y = 2 \\ b, \text{ otherwise.} \end{cases} \]

\[ \forall y < \ell h(x_n)((x_n)_y = 1 \rightarrow y \in B) \leftrightarrow \forall y < \ell h(x_n)g(x_n, y) \in B. \]

Also,

\[ \forall y < \ell h(x_n)(y \in B \rightarrow (x_n)_y = 1) \leftrightarrow \forall y < \ell h(x_n)h(x_n, y) \notin B. \]
Thus, \( x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x,x_1,\ldots,x_n) \land \forall y < t h(x_n) g(x_n,y) \in B \land \forall y < t h(x_n) h(x_n,y) \in B] \). Let \( f(x) = t h(x_n) \). Then,

\( x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x,x_1,\ldots,x_n) \land \forall y < f(x_n) (g(x_n,y) \in B \land h(x_n,y) \not\in B)] \). Hence, for \( n \) odd, (1) implies (2).

Define

\[
g_1(x,y,x_1,\ldots,x_n) = \begin{cases} 
g(x_n,y), R(x,x_1,\ldots,x_n) \\
 b, \neg R(x,x_1,\ldots,x_n)
\end{cases}
\]

Then, \( R(x,x_1,\ldots,x_n) \land \forall x < f(x_n) g(x_n,y) \in B \leftrightarrow \forall x < f(x_n) g_1(x,y,x_1,\ldots,x_n) \in B \). Thus \( x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [\forall x < f(x_n) (g_1(x,y,x_1,\ldots,x_n) \in B \land h(x_n,y) \not\in B)] \). That is, (2) \( \rightarrow \) (3), for \( n \) odd.

It is clear that (3) \( \rightarrow \) (1).

Now, suppose \( n \) is even. \( A \in \sum_n^B \). Thus, for some \( e \),

\( \forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \bar{T}_n^n(h(x_n),e,x,x_1,\ldots,x_{n-1}) \) where \( h \) is the characteristic function of \( B \).

\[
x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \bar{T}_n^n(h(x_n),e,x,x_1,\ldots,x_{n-1}) \\
\quad \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n [Ch(x_n) \land \forall y < t h(x_n) ((x_n)_y = 1 \\
\quad \leftrightarrow y \in B \rightarrow \bar{T}_n^n(x_n,e,x_1,\ldots,x_{n-1})] \\
\quad \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \exists x_n [Ch(x_n) \land \forall y < t h(x_n) ((x_n)_y = 1 \\
\quad \leftrightarrow y \in B \land \bar{T}_n^n(x_n,e,x_1,\ldots,x_{n-1})].
\]

As for the case \( n \) odd, there exists a recursive predicate \( R(x,x_1,\ldots,x_n) \) and recursive functions \( g \) and \( h \) so that
\( x \in A \iff \exists x_1 \forall x_2 \ldots \exists x_{n-1} \exists x_n [R(x, x_1, \ldots, x_n) \land \forall y < \ell h(x_n) g(x_n, y) \in B \land \forall y < \ell h(x_n) h(x_n, y) \in B]. \) As before, let \( f(x_n) = \ell h(x_n). \) Then,

\[
\forall x \in A \iff \exists x_1 \forall x_2 \ldots \exists x_{n-1} \exists x_n [R(x, x_1, \ldots, x_n) \\
\lor \exists y < f(x_n) (g(x_n, y) \in B \lor h(x_n, y) \in B)].
\]

Interchanging \( g \) and \( h, \) and \( \bar{R} \) and \( R, \) we have \((1) \rightarrow (2). \) \((2) \rightarrow (3) \) is proved as in the case \( n \) odd. And, again, it is clear that \((3) \rightarrow (1). \) Thus, the proof of Theorem 3 is complete.

**Corollary 2.** For all sets \( A \) and \( B, B \neq \emptyset \) and \( B \neq \omega, A \in \Sigma^B_{n} \) if and only if:

1. if \( n \) odd, there exists a recursive predicate \( R \) and a recursive function \( f \) so that

\[
\forall x (x \in A \iff \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \left( y < f(x_n) \exists u \in B \exists v \in B \right) \bar{R}(x, x_1, \ldots, x_{n-1}, y, u, v));
\]

2. if \( n \) even, there exists a recursive predicate \( R \) and a recursive function \( f \) so that

\[
\forall x (x \in A \iff \exists x_1 \ldots \exists x_{n-1} \exists x_n \forall y < f(x_n) \exists u \in B \exists v \in B \right) \bar{R}(x, x_1, \ldots, x_{n-1}, y, u, v));
\]

For any two sets \( A \) and \( B, \) we have shown, in Theorems 1 and 2, that \( A \Sigma^B_1 \leftrightarrow A \in \Sigma^B_{\omega} \) in a positive sense. Moreover, by Theorem 1.2.8, Theorem 2, and Corollary 1, if \( B \neq \emptyset \) and \( B \neq \omega, \) then \( A \in \Sigma^B_{\omega} \) in a
positive sense if and only if there exist \( f, g, \) recursive so that \( \forall x(x \in A \iff \exists y \forall z < f(y)g(x, y, z) \in B) \). Compare this with the following Corollary 3.

**Corollary 3.** If \( B \neq \emptyset \) and \( B \neq \omega \), then \( A \in \Sigma_1^B \) if and only if there exist recursive functions \( f, g \) and \( h \) so that

\[
\forall x(x \in A \iff \exists y \forall z < f(y)(g(x, y, z) \in B \land h(y, z) \notin B)).
\]

3. **The \( \Sigma_1 \)-reducibility \( \leq_{rm} \).**

We consider in this section the effect of eliminating the bounded quantifier in the definition of \( \exists_1 \).

**Definition 5.** \( A \leq_{rm} B \iff \) there exists a recursive function \( f \) so that \( \forall x(x \in A \iff \exists y f(x, y) \in B) \).

**Theorem 4.**

1. \( \leq_{rm} \) is a \( \Sigma_1 \)-reducibility relation;
2. \( A \leq_m B \rightarrow A \leq_{rm} B \rightarrow A \leq_{\exists_1} B \);
3. \( (A \leq_{rm} \emptyset \rightarrow A = \emptyset) \land (A \leq_{rm} \omega \rightarrow A = \omega) \);
4. \( B \neq \emptyset \land B \neq \omega \rightarrow (A \in \Sigma_1 \rightarrow A \leq_{rm} B) \);
5. \( A \leq_{rm} B \land B \in \Sigma_1 \rightarrow A \in \Sigma_1 \);
6. \( \leq_r \not\leq_{rm} \).

**Proof.** The proofs follow immediately from the definition. We will present the proof of (4). Suppose \( A \in \Sigma_1 \land A \neq \emptyset \). Let \( a \in B \) and \( b \notin B \). Define
11.

\[
f(x,y) = \begin{cases} 
a, R(x,y) \\
b, \overline{R}(x,y),
\end{cases}
\]

where \( x \in A \leftrightarrow \exists y R(x,y) \). Then, \( x \in A \leftrightarrow \exists y f(x,y) \in B \). Suppose \( A = \emptyset \).

Choose \( b \not\in B \). Define \( f(x,y) = b \), all \( x \) and \( y \). Then, \( x \in A \leftrightarrow \exists y f(x,y) \in B \).

**Corollary 4.** \( A \leq_{rm} B \implies A \leq_m B \).

**Proof.** Let \( A \in \Sigma_1 \) so that \( A \) is not recursive. Then \( A \leq_m B \) only if \( B \) is not recursive. Thus, (4) above is not true for \( \leq_m \).

We show now that \( A \leq_{g_1} B \implies A \leq_{rm} B \). Thus, the bounded quantifier in the definition of \( \exists_1 \), Theorem 1.2.8 and Corollary 1 cannot be eliminated.

**Lemma 2.** Let \( f(x) = x^2 + 1 \) and \( g(x) = (x+1)^2 \). Then
\[
\forall x \forall y (x > 0 \land y > 0 \rightarrow f(x) \neq g(y)).
\]

**Proof.** If \( x^2 + 1 = (y+1)^2 \), then \( (y+1)^2 - x^2 = 1 \).

\[(y+1+x)(y+1-x) = 1. \] Thus, \( y + 1 + x = -1 \) and \( y + 1 - x = 1 \),
or \( y + 1 + x = 1 \) and \( y + 1 - x = 1 \). Thus \( y = -2 \) and \( x = 0 \),
or \( x = y = 0 \).

**Lemma 3.** There exist functions \( \alpha \) and \( \beta \) so that:

1. \( \forall x (\alpha(x) = 0 \lor \alpha(x) = 1), \forall x (\beta(x) = 0 \lor \beta(x) = 1) \);
2. \( \forall x (\alpha(x) = 0 \iff \beta(x^2 + 1) = \beta((x+1)^2) = 0) \),
\[ \forall x (\beta(x) = 0 \iff \alpha(x^2 + 1) = \alpha((x+1)^2) = 0) \);
(3) there is no partial recursive function $h$ so that
\[ \alpha(x) = 0 \iff \exists y \beta(h(x,y)) = 0; \]
(4) there is no partial recursive function $h$ so that
\[ \beta(x) = 0 \iff \exists y \alpha(h(x,y)) = 0. \]

**Proof.** Let $f(x) = x^2 + 1$ and $g(x) = (x+1)^2$. For each natural number, define $C(x)$ inductively by:

(i) $x \in C(x)$;
(ii) $y \in C(x) \rightarrow f(y) \in C(x) \land g(y) \in C(x)$;
(iii) $C(x)$ is the smallest set satisfying clauses (i) and (ii).

We define functions $\alpha$ and $\beta$ by induction. This construction differs from the constructions in [3] in that at stage $s+1$ not only are initial segments $\alpha_{s+1}$ and $\beta_{s+1}$ defined, but, for each $x < \ell h(\alpha_{s+1})$ so that $(\alpha_{s+1})_x = 1$, and for each $x < \ell h(\beta_{s+1})$ so that $(\beta_{s+1})_x = 1$, $\alpha$ and $\beta$ are defined on $C(x)$, so that (2) is satisfied, as follows: If $y \in C(x)$ and $\alpha(y) = 0$, then $\beta(f(y)) = \beta(g(y)) = 0$. If $y \in C(x)$ and $\beta(y) = 0$, then $\alpha(f(y)) = \alpha(g(y)) = 0$. Thus, at stage $s+1$, infinitely many values of $\alpha$ are defined.

Condition (3) is equivalent to the following (3'):
\[ (3') \forall e \exists x [ \alpha(x) = 0 \land \forall y ([e](x,y) \text{ defined } \rightarrow \beta([e](x,y)) = 1)] \]
or
\[ [\alpha(x) = 1 \land \exists y \beta([e](x,y)) = 0]. \]

Condition (4) is equivalent to the following (4'):
\[ (4') \forall e \exists x [ \beta(x) = 0 \land \forall y ([e](x,y) \text{ defined } \rightarrow \alpha([e](x,y)) = 1)] \]
or
\[ [\beta(x) = 1 \land \exists y \alpha([e](x,y)) = 0]. \]
Stage 0. Define $a_0 = \beta_0 = 1$.

Stage $s + 1$. By induction hypothesis $a_s$ and $\beta_s$ are already defined. Also, the following conditions are satisfied:

1. \( \forall x [\alpha(x) \text{ defined } \& \alpha(x) = 0 \rightarrow \beta(f(x)) \text{ is defined and } \beta(g(x)) \text{ is defined } \& \beta(f(x)) = \beta(g(x)) = 0]. \)
2. \( \forall x [\beta(x) \text{ defined } \& \beta(x) = 0 \rightarrow \alpha(f(x)) \text{ is defined and } \alpha(g(x)) \text{ is defined and } \alpha(f(x)) = \alpha(g(x)) = 0]. \)
3. \( \forall x [\alpha(f(x)) \text{ defined } \& \alpha(f(x)) = 0 \rightarrow [\beta(x) \text{ defined } \& \alpha(g(x)) \text{ defined } \& (\beta(x) = 0 \leftrightarrow \alpha(g(x)) = 0)]. \)
4. \( \forall x [\beta(g(x)) \text{ defined } \& \beta(g(x)) = 0 \rightarrow [\alpha(x) \text{ defined } \& \beta(f(x)) \text{ defined } \& (\alpha(x) = 0 \leftrightarrow \beta(f(x)) = 0)]. \)

1. $s = 2^e$. $a_{2e+1}$ and $\beta_{2e+1}$ shall be defined at this stage so that $(3')$ is true at $e$ for all extensions of $a_{2e+1}$ and $\beta_{2e+1}$.

Case 1. $\exists x [(\alpha(x) \text{ has not been defined or } (\alpha(x) \text{ has been defined } \& \alpha(x) = 0)) \& \forall y [([e](x,y) \text{ defined } \rightarrow \beta([e](x,y) \text{ defined } \& \beta([e](x,y)) = 1)].\)

Note. $\alpha(x)$ defined includes both the case $x < \ell h(a_{2e})$ and $x \geq \ell h(a_{2e})$ where $\alpha(x)$ is defined at some stage $\leq 2e$. $\alpha(x) = 0$ includes the case $(a_{2e})_x = 1.$
Let \( a \) be the least \( x \) satisfying the hypothesis of case 1.

Suppose \( \alpha(a) \) is already defined and \( \alpha(a) = 0 \). Then \((3')\) is already satisfied at \( e \). If \( \alpha(\text{th}(2e)) \) is already defined, then define

\[
\alpha(\text{th}(2e)) + 1 = \alpha_{2e+1} = \alpha_{2e} \cdot \text{th}(2e)
\]

and \( \beta_{2e+1} = \beta_{2e} \). It is clear that \( \alpha_{2e+1} \) and \( \beta_{2e+1} \) satisfy the induction hypotheses \((5)-(10)\). If \( \alpha(\text{th}(2e)) \) is not defined, then define \( \alpha_{2e+1} = \alpha_{2e} \cdot \beta_{2e} \cdot \text{th}(2e) \) and \( \beta_{2e+1} = \beta_{2e} \cdot \alpha_{2e+1} \) and \( \beta_{2e+1} \) satisfy \((5)-(10)\).

Suppose \( \alpha(a) \) has not been defined. If \( a \neq f(b) \) and \( a \neq g(b) \), for any \( b \), then define

\[
\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{x \leq \alpha(2e)} \frac{h(x)}{p_x} \cdot \frac{1}{p_a},
\]

where \( h(x) = \alpha(x) + 1 \), if \( \alpha(x) \) is already defined, and \( h(x) = 2 \), otherwise. Define \( \beta_{2e+1} = \beta_{2e} \). Then, \((3')\) is satisfied at \( e \) by \( \alpha_{2e+1} \) and \( \beta_{2e+1} \). \( \alpha_{2e+1} \) is defined so that \( \alpha(a) = 0 \). Therefore, define values of \( \alpha \) and \( \beta \) on \( C(a) \) by the rules:

\[
y \in C(a) \land \alpha(y) = 0 \rightarrow \beta(f(y)) = \beta(g(y)) = 0, \quad \text{and} \quad y \in C(A) \land \beta(y) = 0 \rightarrow \alpha(f(y)) = \alpha(g(y)) = 0.
\]

Then, \( \alpha_{2e+1} \) and \( \beta_{2e+1} \) satisfy \((5)-(10)\).

Suppose \( \exists b a = g(b) \). By clause \((6)\), \( \beta(b) \) is not defined or, \( \beta(b) \) is defined and \( \beta(b) = 1 \). (In fact, if the latter, then \( b < \text{th}(2e) \).) Also, by \((7)\), \( \alpha(f(b)) \) is not defined or, \( \alpha(f(b)) \) is
defined and $\alpha(f(b)) = 1$. Define

$$\alpha_{2e + 1} = \alpha_{2e} \cdot \prod_{p_x \leq \alpha} \frac{h(x)}{p_x} \cdot \frac{1}{p_a},$$

where $h(x) = \alpha(x) + 1$, if $\alpha(x)$ is already defined, and $h(x) = 2$, otherwise. (Then, in particular, $\alpha(f(b)) = 1$, since $f(b) < g(b)$.)

If $\beta(b)$ is defined, define $\beta_{2e + 1} = \beta_{2e}$; if not define

$$\beta_{2e + 1} = \beta_{2e} \cdot \prod_{\beta_{2e} \leq x \leq b} \frac{h(x)}{p_x},$$

where $h(x) = \beta(x) + 1$, if $\beta(x)$ is already defined, and $h(x) = 2$, otherwise. Also, define values of $\alpha$ and $\beta$ on $C(a)$ as described above. Then (3') is satisfied at $e$ for $\alpha_{2e + 1}$ and $\beta_{2e + 1}$, and $\alpha_{2e + 1}$ and $\beta_{2e + 1}$ satisfy (5)-(10).

Suppose $\alpha(a)$ is not defined and $\exists b a = f(b)$. By clause (6), $\beta(b)$ is not defined or, $b < \ell h(\beta_{2e})$ and $(\beta_{2e})_b = 2$. Also, by clause (8), $\alpha(g(b))$ is not defined or, $\alpha(g(b))$ is defined and $\alpha(g(b)) = 1$. Since $a = f(b) < g(b)$, $g(b)$ is not defined. Define

$$\alpha_{2e + 1} = \alpha_{2e} \cdot \prod_{\alpha \leq x \leq \ell h(\alpha_{2e})} \frac{h(x)}{p_x} \cdot \prod_{\alpha \leq x \leq g(b)} \frac{1}{p_x},$$

where $h(x)$ is defined as before. If $b < \ell h(\beta_{2e})$, define $\beta_{2e + 1} = \beta_{2e}$; otherwise define

$$\beta_{2e + 1} = \beta_{2e} \cdot \prod_{\beta_{2e} \leq x \leq b} \frac{h(x)}{p_x},$$
where, again, \( h(x) \) is defined as before. Define values \( \alpha \) and \( \beta \) on \( \mathcal{C}(a) \) in the usual manner. Then, \( \alpha_{2e+1} \) and \( \beta_{2e+1} \) satisfy \((3')\) at \((e)\), and satisfy \((5)-(10)\).

Case 1 of stage \( 2e+1 \) is now complete.

**Case 2.** \( \forall x [ (a(x) \text{ has not been defined or } a(x) = 0) \rightarrow \exists y ([e](x,y) \text{ defined } \& \ (\beta([e](x,y)) \text{ not defined or } \beta([e](x,y)) = 0)] ] \).

Let \( a \) be the least \( x \) so that, for all \( y \), \( a \neq f(y) \) and \( a \neq g(y) \), and so that \( a(a) \) is not yet defined. Let \( b \) be the least \( y \) satisfying the consequent of case 2 at \( x = a \).

Suppose \( \beta([e](a,b)) \) is defined and \( \beta([e](a,b)) = 0 \). Then, define

\[
\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{t \theta h(a_{2e}) \leq x \leq a} h(x),
\]

\( h(x) = a(x) + 1 \), if \( a(x) \) defined, \( h(x) = 2 \), otherwise. Define \( \beta_{2e+1} = \beta_{2e} \). Then, \((5)-(10)\) hold, and \((3')\) is satisfied at \( e \) for all extensions of \( \alpha_{2e+1} \) and \( \beta_{2e+1} \).

Suppose \( \beta([e](a,b)) \) is not defined. Also, suppose \( [e](a,b) \neq f(c) \) and \( [e](a,b) \neq g(c) \), for any \( c \). First, define

\[
\beta_{2e+1} = \beta_{2e} \cdot \prod_{t \theta h(\beta_{2e}) \leq x \leq [e](a,b)} h(x) \cdot \frac{1}{p_{x}} \cdot p_{[e](a,b)},
\]

where \( h(x) = \beta(x) + 1 \), if \( \beta(x) \) is already defined, and \( \beta(x) = 2 \), otherwise. Secondly, define
\[ \alpha_{2e+1} = \alpha_{2e} \prod_{\substack{\text{h}(\alpha_{2e}) \leq x \leq \alpha}} p^x(h(x)), \]

where \( h(x) = \alpha(x) + 1 \), if \( \alpha(x) \) is already defined, and \( h(x) = 2 \), otherwise. In particular, \( \alpha(a) = 1 \) and \( \beta([e](a,b)) = 0 \). Thus (3') is satisfied by \( \alpha_{2e+1} \) and \( \beta_{2e+1} \) at \( e \). Since \( \beta([e](a,b)) \) has been defined so that \( \beta([e](a,b)) = 0 \), define the necessary values of \( \alpha \) and \( \beta \) on \( C([e](a,b)) \) as before. That is, \( (y \in C([e](a,b)) \& \beta(y) = 0) \rightarrow \alpha(f(y)) = \alpha(g(y)) = 0 \), and \( (y \in C([e](a,b)) \& \alpha(y) = 0) \rightarrow \beta(f(y)) = \beta(g(y)) = 0 \). Then (5)-(10) are satisfied also.

Suppose \( \beta([e](a,b)) \) is not defined and \( \exists c C([e](a,b)) = g(c) \).

By clause (5), \( \alpha(c) \) is not defined or, \( \alpha(c) \) is defined and \( \alpha(c) = 1 \).

Also, by (9), \( \beta(f(c)) \) is not defined or, \( \beta(f(c)) \) is defined and \( \beta(f(c)) = 1 \). Firstly, define

\[ \beta_{2e+1} = \beta_{2e} \prod_{\substack{\text{h}(\beta_{2e}) \leq x \leq [e](a,b)}} p^x(h(x)) \prod_{\substack{\text{p}(e)(a,b)}} p^1 \]

where \( h(x) = \beta(x) + 1 \), if \( \beta(x) \) is already defined, and \( \beta(x) = 2 \), otherwise. (Then, in particular, \( \beta(f(c)) = 1 \), since \( f(c) \prec g(c) \).)

Secondly, define values of \( \alpha \) and \( \beta \) on \( C([e](a,b)) \) in the usual manner. Now we want to extend \( \alpha_{2e} \) so that \( \alpha(a) \) is defined, \( \alpha(a) = 1 \), \( \alpha(c) \) is defined, and \( \alpha(c) = 1 \). \( c \prec g(c) = [e](a,b) \).

Thus, \( c \notin C([e](a,b)) \). Hence \( \alpha(c) \) is still undefined, or \( \alpha(c) \) is defined and \( \alpha(c) = 1 \). \( a \) was chosen so that, for all \( x \), \( a \neq f(x) \) and \( a \neq g(x) \). Thus \( \alpha(a) \) is still undefined. Define
\[ \alpha_{2e+1} = \alpha_{2e} \cdot \prod_{h(\alpha_{2e}) \leq x < \max[a,b] \cdot h(x)} \]

where \( h(x) = \alpha(x) + 1 \), if \( \alpha(x) \) is defined, and \( \alpha(x) = 2 \), otherwise. \( \alpha(a) = 1 \) and \( \beta([e](a,b)) = 0 \), thus \((3')\) is satisfied at \( e \).

Also \((5)-(10)\) are satisfied by this \( \alpha_{2e+1} \) and \( \beta_{2e+1} \). (The only important clause in this case is \((9)\), which still holds, since \( \beta(g(c)) = 0 \), but \( \beta(f(c)) = \alpha(e) = 1 \).

Finally, suppose \( \beta([e](a,b)) \) is not defined and \( \exists c\{e\}(a,b) = f(c) \). By clause \((5)\), \( \alpha(c) \) is not defined or, \( \alpha(c) \) is defined and \( \alpha(c) = 1 \). Also, by \((10)\), \( \beta(g(c)) \) is not defined or, \( \beta(g(c)) \) is defined and \( \beta(g(c)) = 1 \). Since \( f(c) \) is not defined and \( f(c) < g(c) \), \( g(c) \) is not defined. Firstly, define

\[ \beta_{2e+1} = \beta_{2e} \cdot \prod_{h(\beta_{2e}) \leq x < \{e\}(a,b) \cdot h(x)} \]

where \( h(x) \) is defined as before. In particular \( \beta_{2e+1} \) is defined so that \( \beta([e](a,b)) = \beta(f(c)) = 0 \) and \( \beta(g(c)) = 1 \). Secondly, define the necessary values of \( \alpha \) and \( \beta \) on \( C([e](a,b)) \). Now we want to extend \( \alpha_{2e} \) so that \( \alpha(a) \) is defined, \( \alpha(a) = 1 \), \( \alpha(c) \) is defined, and \( \alpha(c) = 1 \). Proceed exactly as in the previous paragraph. Then \( \alpha_{2e+1} \) and \( \beta_{2e+1} \) are obtained so that \((3')\) at \( e \) and \((5)-(10)\) are satisfied.

Case 2 of stage \( 2e + 1 \) is now complete.
s = 2e + 1. α_{2e+2} and β_{2e+2} shall be defined at this stage so that (4') is true at e for all extensions of α_{2e+2} and β_{2e+2}.

Stage 2e + 2 is the same mutatis mutandis as stage 2e + 1.

Define α and β by α(x) = \(α_μ[μ\langle h(a) \rangle]')x^2 + 1\), and β(x) = \(β_μ[μ\langle h(β_a) \rangle]')x^2 + 1\).

Clearly, α and β satisfy (3') and (4') and therefore (3) and (4). By induction clauses (5) and (6), α(x) = 0 → β(x^2 + 1) = β((x + 1)^2) = 0, and β(x) = 0 → α(x^2 + 1) = α((x + 1)^2) = 0. By clauses (7)-(10), the converses are also true. Thus α and β satisfy clause (2).

The proof of Lemma 3 is complete.

**Theorem 5.** There exist sets A and B so that A ≤_r B, A ≤_s B, and A ≤_rm B. In fact, the \(S_1\)-degrees of A and B are identical and the \(rm\)-degrees of A and B are incomparable.

**Proof.** Apply Lemma 3 to obtain functions α and β. Let

A = \{x|α(x) = 0\} and B = \{x|β(x) = 0\}. Then, there exist recursive functions f and g so that \(∀x(x ∈ A ↔ f(x) ∈ B \& g(x) ∈ B)\), and \(∀x(x ∈ B ↔ f(x) ∈ A \& g(x) ∈ A)\). Thus A ≤_r B. (Also, B ≤_r A.) By the definition of \(S_1\), A ≤_s B and B ≤_s A. On the other hand, by Lemma 3, A ≤_rm B and B ≤_rm A.

It is also interesting to notice that for two sets A and B, the existence of recursive functions f and g so that

\(∀x(x ∈ A ↔ f(x) ∈ B \& g(x) ∈ B)\) does not imply A ≤_m B.
By Theorem I.2.2 (10), \( A \leq_r B \) does not imply \( A \lessdot_1 B \). Also, by Theorem 4 (6), \( A \leq_r B \) does not imply \( A \lessdot_{rm} B \). Theorem 5 gives an example of sets \( A \) and \( B \) so that \( d(A) = d(B) \), \( d_{g_1}(A) = d_{g_1}(B) \), and \( d_{rm}(A) = d_{rm}(B) \). Is there a set \( A \) so that \( d_{g_1}(A) = d_{g_1}(\overline{A}) \) and \( d_{rm}(A) = d_{rm}(\overline{A})? \) This question is open. Notice that by the following argument Lemma 3 cannot be used to obtain such a set \( A \). Suppose there exist recursive functions \( f \) and \( g \) so that \( x \in A \leftrightarrow f(x) \notin A \) & \( g(x) \notin A \) and \( x \in \overline{A} \leftrightarrow f(x) \in A \) & \( g(x) \in A \). Then, \( x \in A \rightarrow f(x) \notin A \). Also \( f(x) \notin A \rightarrow x \in A \), because \( x \in A \rightarrow f(x) \in A \). Thus \( A \leq_m B \), which implies \( A \leq_{rm} B \).

However, we have already established (Theorem I.2.9) the weaker result that there exists a set \( A \) so that \( A \) and \( \overline{A} \) are \( S_1 \)-incomparable, from which it follows that \( A \) and \( \overline{A} \) are also \( rm \)-incomparable.

4. The Reducibility \( S_1 \cap \mathcal{P}_1 \).

We consider in this final section the reducibility relation \( S_1 \cap \mathcal{P}_1 \). This reducibility is of some interest since it is easily defined and, as the next theorem shows, is between many-one reducibility and relative recursiveness.

Theorem 6.

1. \( S_1 \cap \mathcal{P}_1 \nsubseteq \{(A,B) | A \leq_r B \} \).

2. \( \{(A,B) | A \leq_m B \} \nsubseteq S_1 \cap \mathcal{P}_1 \).
Proof.

(1) Follows from Theorems I.2.5, I.2.6, and I.2.2(2).

(2) Clearly \( A \leq_{m} B \rightarrow A \leq_{g_{1}} B \ & A \leq_{\varphi_{1}} B \). Lemma 3 and Theorem 5 give us sets \( A \) and \( B \) so that \( x \in A \leftrightarrow f(x) \in B \) & \( g(x) \in B \). Thus \( A \leq_{g_{1}} B \) and \( A \leq_{\varphi_{1}} B \). On the other hand, \( A \) and \( B \) are constructed so that \( A \not\leq_{m} B \).

Let \( R^{X} \) denote a number theoretic predicate recursively uniformly in \( X \), where \( X \) is a set variable. By a theorem of Nerode [1, Theorem 11], \( A \) is truth-table reducible to \( B \) (\( A \leq_{tt} B \)) if and only if there exists such an \( R^{X} \) so that \( \forall x(x \in A \leftrightarrow R^{B}(x)) \). \( \leq_{m} \) and \( \leq_{1} \) can be expressed in this form. \( A \leq_{m} B \) if and only if \( \forall x(x \in A \leftrightarrow f(x) \in B) \) for some recursive function \( f \), and \( A \leq_{1} B \) if and only if \( \forall x(x \in A \leftrightarrow f(x) \in B) \) for some one-one recursive function \( f \). In either case, \( f(x) \in X \) is such an \( R^{X} \). We will say that a subrecursive reducibility \( \mathcal{R} \) is defined by predicates \( R^{X} \) if for all \( A \) and \( B \), \( A \mathcal{R} B \) is and only if there exists \( R^{X} \) so that \( \forall x(x \in A \leftrightarrow R^{B}(x)) \) and \( \forall c,d[\forall x(x \in C \leftrightarrow R^{D}(x)) \rightarrow C \mathcal{R} D] \).

Lemma 4. \( \exists A[A \leq_{g_{1}} \overline{A} \ & A \not\leq_{\varphi_{1}} \overline{A}] \).

Proof. Choose \( A \in \Sigma_{1} \) so that \( A \not\in \Pi_{1} \). Then \( A \leq_{g_{1}} B \), all \( B \). Thus, \( A \leq_{g_{1}} \overline{A} \). \( \overline{A} \in \Pi_{1} \), so \( A \not\leq_{\varphi_{1}} \overline{A} \). Thus, \( A \not\leq_{\varphi_{1}} \overline{A} \).

Theorem 7. \( A \leq_{tt} B \) does not imply \( A \leq_{g_{1} \cap \varphi_{1}} B \).
Proof. The proof follows from Lemma 4 since \( A \leq_{tt} \overline{A} \) for all \( A \).

Theorem 8. \( A \leq_{\mathbb{S}_1} B \) does not imply \( A \leq_{tt} B \).

Proof. There exist recursively enumerable sets \( A \) and \( B \) so that \( d(A) = d(B) \) and \( A \not\leq_{tt} B \) (see [2, §9.6]). \( A \in \mathbb{S}_1 \), hence \( A \leq_{\mathbb{S}_1} B \). Since \( B \in \mathbb{S}_1 \), \( B \in \mathbb{P}_{\mathbb{S}_1} \), \( B \leq_{\mathbb{S}_1 X} X \). \( B \leq_{\mathbb{S}_1} X \rightarrow A \leq_{\mathbb{S}_1} X \rightarrow A \in \mathbb{P}_{\mathbb{S}_1} X \). Thus, \( A \leq_{\mathbb{S}_1} B \). Therefore \( A \leq_{\mathbb{S}_1 \cap \mathbb{P}_{\mathbb{S}_1}} B \).

Definition 6. \( \mathbb{S}_1 = \{ R^X(x) | R^X(x) \text{ is uniformly recursive in } X \text{ and } \forall B \forall C [B \in \mathbb{S}_1 \rightarrow R^B C] \} \). \( \mathbb{S}_2 = \{ R^X(x,y) | R^X(x,y) \text{ is uniformly recursive in } X \text{ and } \forall B \forall C [B \in \mathbb{S}_1 \rightarrow R^B C] \} \).

Theorem 9. Suppose \( B \neq \emptyset \) and \( B \neq w \). Then \( A \leq_{\mathbb{S}_1} B \leftrightarrow \) there exists \( R^X(x,y) \in \mathbb{S}_2 \) so that \( \forall x (x \in A \rightarrow \exists y R^B(x,y)) \).

Proof. It is immediate from the definition of \( \mathbb{S}_1 \) that the right hand side implies the left hand side.

Suppose \( A \leq_{\mathbb{S}_1} B \). By Theorem 1.2.8, there are recursive functions \( f \) and \( g \) so that \( \forall x (x \in A \leftrightarrow \exists y \forall z \leq f(y) g(x,y,z) \in B) \). Define \( R^X(x,y) = \forall z \leq f(y) g(x,y,z) \in X \), \( R^X \in \mathbb{S}_2 \). This completes the proof.

Theorem 10. \( \mathbb{S}_1 = \{ R^X(x,x) | R^X(x,y) \in \mathbb{S}_2 \} \).

Proof. Obviously, \( R^X(x,y) \in \mathbb{S}_2 \) implies \( R^X(x,x) \in \mathbb{S}_1 \). If \( R^X(x) \in \mathbb{S}_1 \), define \( R^X(x,y) = R^X(x) \), then \( R^X(x,y) \in \mathbb{S}_2 \) and \( R^X(x) = R^X(x,x) \).
Open Questions.

1. By Theorem 8, $s_1 \cap p_1$ is not defined by predicates $R^X$ uniformly recursive in $X$. If $\forall x(x \in A \leftrightarrow R^B(x))$ and $R^X \in \Theta_1$, is $A \preceq_g p_1 B$? By definition of $s_1$, $\forall x(x \in A \leftrightarrow R^B(x))$ and $R^X \in \Theta_1$ implies $A \preceq_g p_1 B$, therefore it is sufficient to show $A \preceq p_1 B$.

2. Is $s_1 \cap p_1$ a maximal proper $\Sigma_0$-reducibility?

Remark. $s_2 \cap p_2 \not\subseteq \{(A, B) | A \preceq_{\Sigma_1} B\}$. Choose $A$ and $B$ so that $A \not\preceq_{\Sigma_1} B$ but so that for some recursive $R$, $\forall x(x \in A \leftrightarrow x \in B \& \forall z R(x, z))$. 
Bibliography

