Optimization of Stochastic Planning Models I
Concepts and Theory

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Abstract

This paper considers stochastic linear programming models for pvoduttkmpla^ing where cost coefficient and RHS term uncertainties arc represented by finite discrete probability distribution functions. The sedation of the two-stage fixed recourse problem is considered, for which a sensitivity-based successive disaggregatkm algorithm is outlined. The bounding properties of the aggregate sub-problems are examined in the rontext of the disaggregatton algorithm. Illustrative examples of the two-stage algorithm are presented.

Keywords: production planning, stochastic programming, linear programming, aggregate models, successive disaggregation algorithm.

1 Introduction

Planning involves making optimal decisions about future events based on current information and future projections. The general problem may be stated as follows: *given* a model of the process (i.e., set of constraints), knowledge of the current state and future events, and an objective function which reflects the cost/risk preferences,./?**/ the solution which minimizes the objective function without violating the constraints. While current information *may* be certain, future events are inevitably stochastic. A simple production network is shown in Figure 1. Material balance nodes are represented as circles and processing units as rectangles. Raw materials and intermediates are purchased on the markets subject to prevailing prices, contracts, and availability. Products are sold cm the market based on demands, prevailing prices, contracts, and production capacity.

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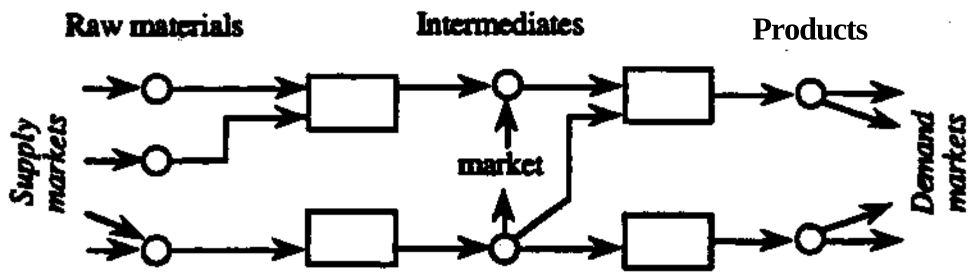

Figure 1. Simple production network with multiple feeds, intermediates, and products, all or part of which depend on uncertain market conditions governing supply, demand, aod pricing.

Uncertainty occurs in both the objective function and the constraints. The objective function includes uncertainty in future costs, prices, and actions. The process model constraints can include both endogenous (e.g., yield coefficients) and exogenous (e.g., future product demands) stochastic parameters. Failure to account for the uncertainty of key parameters (e.g., future supplies and demands) in decision problems can lead to non-optimal and infeasible decisions (e.g., see Birgc, 1993, 1982a, 1992). However, deterministic optimization approaches to uncertainty are predominant in chemical engineering today (e.g., Grossmann and Straub, 1991). While considerable theoretical work has been done in formulation and solution of stochastic optimization problems (see reviews by Dempster, 1980; Wets, 1989), current methods are practically limited to single- or two-stage problems due to the computational complexity of including the probability dimension. A real need exists for improved methods for solving (planning) optimization problems with uncertainties.

In this paper we consider the two-stage problem with the uncertainties of future supplies, demands, and prices characterized by finite, discrete probability distribution functions. In § 2 the production planning model is presented and in § 3 the two-stage stochastic linear programming (LP) formulations are developed. In § 4 we present a small motivating example to provide some insight into the nature of this problem. In § 5 we examine general issues related to the optimality and feasibility of the optimal solution and in § 6 the solution methods for these problems are reviewed. To overcome the large dimensionality of the problem and the associated computational expense, we propose in § 7 a successive disaggregation algorithm for the solution to the two-stage stochastic LP. In § 8 we examine the theoretical bounding properties of the aggregate sub-problems in the context of the disaggregation algorithm that is outlined in § 9. Illustrative examples of the method are included in § 10. Part I of this paper considers conceptual and theoretical issues of the proposed method, while the detailed implementation of the algorithm using a sensitivity-based approach for rcpartitioning is given in "Part IT (Clay and Grossmann, 1994).

2 Problem Statement and Model

We consider the optimization of two-stage stochastic linear planning models. The stochastic parameters (e.g., costs, demands, and supplies) are characterized by discrete probability distributions defined over a finite probability space. We formulate the mathematical problem as a two-stage stochastic linear programming problem with fixed recourse. In this section, we describe the production planning model and its formulation.

In formulating the production planning model we follow the notation of Sahinidis and Grossmann (1992), adding a stochastic dimension to the problem. We define the following notation in order to develop the mathematical modeL

Index sets:

i process; (i = 1,..., N_F).

j chemical; (j = 1,..., N_c).

l market; (l = 1, ..., N_M).

t time period (stage); (t = 1,..., N_T).

Parameters:

a^L_{jlt}, a^u_{jlt} lower/upper bounds for purchases of chemical j from market l during stage t.

d^L_{jlt}, d^U_{jlt} lower/upper bounds for sales of chemical j to market l during stage t.

N_c number of chemicals in the network.

N_M number of markets.

N_p number of processes in the network.

N_T number of time periods (stages) considered

V^u_{jt} upper bound on the inventory of chemical j at stage t.

W^u_i upper bound on the operating level of process i.

F_{jlt} purchase price of chemical j in market l during stage t.

γ_{jlt} sales price of chemical j in market l during stage t.

δ_{it} unit operating cost for process i during stage t.

μ_{ij} material balance coefficients for process i and input chemical j.

ν_{ij} material balance coefficients for process i and output chemical j.

Variables:

I_{ijt} amount of chemical j consumed (input) by process i during stage t.

00,	amount of chemical *j* produced (output) by process i during stage r.
z	expected cost (min formulation) or profit (max formulation).
P_{jlt}	amount of chemical; purchased from market / at die beginning of stage r.
S_{jU}	amount of chemical ý sold to market / at die beginning of stage *u*
Vj,	inventory of chemical y at stager.
W_u	iterating level of process i during stager.

We formulate the production planning model as a multi-stage stochastic linear programming (MSLP) problem, although this paper is restricted to the solution to the two-stage model (i.e., Af_r = 2 instances). Parameter uncertainties for limits in supplies (a), demands (</), arid cost coefficients (T, y, *8)* in future stages are characterized by probability distributions. These distributions can be considered functions representing our degree of belief that a parameter takes on a specific value. We seek the action to take at the current time which minimizes some expected utility function while ensuring feasibility in the future stages. Using the expected value cost function, the multi-stage LP model for production planning is as follows.

nun 2= £*.,»......, \tl(V, - rA)+X W} (1a)
1*1 (jm\ /«1 $m\ J

s.t. /^«TJijW, *ViJ.t* (lb)

$O \wedge v_9 W_k$ Vi\y,r (lc)

W^W? *Vi.t* (Id)

$$V_{jt-1}+\sum_{i=1}^{N_p}O_{ijt}-\sum_{i=1}^{N_p}I_{ijt}+\sum_{l=1}^{N_M}P_{jlt}=V_{jt}+\sum_{l=1}^{N_M}S_{jlt} \qquad \text{Y/,r} \qquad \text{(le)}$$

V/rSVj Vy\r (If)

*af^PfiZa^* Vy,/,r (lg)

$df_u £ S_{Jlt} £ d\wedge$ Vy,/,r (lh)

$$I_{ijt},O_{ijt},V_{jt},W_{it}\geq 0 \qquad \forall i,j,l,t, \qquad \text{(1i)}$$

where *0* is the stochastic parameter vector which can include cost coefficients and supply / demand limits (i.e., RHS terms).

The objective function (la) is the minimization of the expected cost, determined by the costs for purchases and operations minus the net sales revenues. The conversion material balances for each process *i* are given by (lb-c); constraint (Id) determines the maximum capacity for each process. The overall material balances for each chemical species *j* are given by (le); constraint (If) represents an upper limit for the inventory of chemical;. The purchasing and sales limits are given by (lg) and (lh), respectively. The non-negativity constraints are given in (li).

Several observations on the structure and nature of MSLP (1) follow. When uncertainties exist in the future stage parameters then the activities of all future stages are stochastic variables. We generally (although not necessarily) assume that the current parameters are known with certainty. The decision cycle is such that as the future unfolds we expect ta reoptimize the problem according to previous actions and parameter realizations. This leads to a recourse formulation, explicitly accounting for our expected iroptimizatkm in future stoges in accordance with past decisions parameter realizations. By fixing the process yield coefficient vectors, ffy and i>#9 the stochastic parameters exist strictly as RHS or objective function terms. Hence the transition matrices are fixed, implying a fixed recourse problem.

3 Two-stage Stochastic LP Model

In this work we consider the solution of the two-stage planning problem with stochastic cost coefficients and RHS terms. For the two-stage case (i.e., N_T » 2) planning problem (1) can be formulated as the fixed recourse stochastic LP given by (2) (see § 5 for discussion of adding slacks). We will refer to this formulation since the notation is more concise, and we are restricting our attention to the solution of the two-stage problem in this paper.

$$\begin{array}{lll} \min & Z \wedge J \wedge + E \wedge C \wedge \} & \\ \text{s.t.} & A_l x_l = b_x & \\ & B_1 x_1 + A_2 x_2 = b_2 & \\ & 0 \leq x_t \leq U\wedge & \text{Vr} = 1, \ldots, 7\backslash \end{array} \qquad (2)$$

where 0_2 is the stochastic parameter vector defined on the probability space *(0%7,P)*, and $b_2(0_2)$ and $c_2(0_2)$ are stochastic linear functions. The triple *(0,7,P)* defining the probability space is composed of the (non-empty) event space (*0*), the G-field in *0* (J), and the probability measure on *J* *(P)*.

Applying the *certainty equivalent transformation* (GET) to problem (2) (see Dantzig, 1987) yields the deterministic equivalent problem as follows:

(PO)	min	z~clx_l-¥^$p_{2k}cl_tx_{2k}$ (kmK)		(3a)
	s.t.	i4x, = l\		(3b)
		$B_1x_1 + A_2x_{2k}$ =*hk	$VkeK$	(3c)
		$0 \le x_1 \le U_{x_1}$		(3d)
		$0 \le x_{2k} \le U_{x_2}$	$\forall k \in K$,	(3e)

where AT denotes the set of possible stage-2 events defined on the finite, discrete probability space *(G,f,P)*. Note that problem (PO) is an exact approximation to problem (2) for the case of discrete probabilities. In general its drawback is that its size grows exponentially and hence ft may become too large to be solved explicitly. As an example, a problem with 20 uncertain parameters with 6 (independent) discrete values for each parameter would give rise to 6 * s $3.7 \cdot 10^{15}$ events.

4 Motivating Example

In order to provide some insight into the nature of problem (2), we consider a simplified planning problem with uncertain demands. Example problem (EX1) is a two-stage stochastic LP with fixed recourse. The objective, z, is the sum of stage-1 production, JC„ and the expected value of combined stage-2 production (denoted by y, and y_2). The production levels are represented as continuous non-negative real variables with cost coefficients equal to 1. Stage-2 product 1 and 2 demands are denoted by d, and dj, respectively. The demand uncertainties are represented by discrete probability distributions. Each demand has three independent states with corresponding probabilities. The discrete probability space is given by the union of the demand 1 and 2 independent spaces, with joint probability denoted by p_v. The stage-2 stochastic parameter vector is denoted as 0_2 = {d,,^}. The problem can be formulated as a fixed recourse stochastic LP as follows:

min *z=^ + E^+yi)* (EX1)

s.t. y,£ 34-di

$y_2 \ge -d_1 + 3d_2$

$y_1 + y_2 - x_1 \le d_1 + d_2$

$x_1 \ge 0$

$y_1, y_2 \ge 0,$

where ^ « {1,1.5,2}, p_u = {0.2,0.6,0.2},

^ = {1,1.5,2}, p_{2j} = {0.1,0.7,0.2},

$$p_{ij} = p_{1i} \cdot p_{2j},$$
$$x_1 \in \Re^1,\ y_1 \in \Re^1,\ y_2 \in \Re^1.$$

There are numerous possible approaches to solving problem (EX1). A common practice is to use the mean values (in place of the probabilistic expansion) of the stochastic parameters. This aggregate (mean-value) model approach is equivalent to using ones best guess as to the value of the demands, an intuitively appealing approach. However, solution to the mean-value problem gives *T* « 9.15 and ^ • 3-05, an infeasible solution to the original problem (EX1). The infeasibility can be seen by substituting the point (4 = 2, ^2 = 2) into the constraint set to give the following results:

$$y_1 \geq 3d_1 - d_2 = 4$$
$$y_2 \geq -d_1 + 3d_2 = 4$$
$$y_1 + y_2 - x_1 \leq d_1 + d_2 = 4$$
x,=3.05.

The first two constraints imply that $y_x + y_2$ £ 8, while the third and fourth constraints imply that $y_x + y_2$ £ 7.OS. Hence, $x_x = 3.05$ is an infeasible solution for this particular future outcome. Similarly, examples can be readily constructed which show that the mean-value solution is non-optimal (or both non-optimal and infeasible) with respect to the original problem. Thus, this approach leaves much to be desired, and does not in general provide a valid optimal solution to the problem.

Another approach commonly used in industry is scenario analysis, whereby only a subset of all possible outcomes are considered. As with the mean-value approach, it is not difficult to construct examples which show similar shortcomings when a scenario analysis solution approach is used. We can conclude in general that any solution to (EX1) which does not include a complete representation of die probabilities is subject to be non-optimal and/or infeasible. This observation is a primary motivation for using the stochastic programming formulation, predicated on explicitly accounting for the parameter uncertainties via probability distribution representations.

Another approach to solving our example problem is to reformulate (EX1) into the certainty equivalent problem below denoted (EXICET)- In this reformulation the stage-2 variables and constraints are expanded over the probability dimension, which in this case includes nine possible outcomes (i.e., all *ij* pairs).

$$\min \quad z = x_1 + \sum_{ij \in i \times j} p_{ij}(y_{1ij} + y_{2ij}) \qquad (EX1_{CET})$$

s.t. *y^ZSdti-dtj* Vi/eixy

$$y_{2ij} \ge -d_{1i} + 3d_{2j}$$ Vi/€ixy

$$y_{1ij} + y_{2ij} - x_1 \le d_{1i} + d_{2j}$$ Vyeixy

$$x_1 \ge 0$$

$$y_{1ij}, y_{2ij} \ge 0 \qquad \forall ij \in i \times j.$$

where

$$x_1 \in \Re^1,$$

$$y_{1ij} \in \Re^1,\ y_{2ij} \in \Re^1 \qquad \forall ij \in i \times j.$$

Here y_w and y^ are continuous stage-2 variables expanded over the discrete probability space. Note that the size of this LP is 19 variables and 27 inequalities versus 3 variables and 3 inequalities for the case where the problem is deterministic.

Since the demand uncertainties are represented by discrete finite probability distributions, there is no error of discretization. Consequently, solving (EXICET) gives the exact optimal solution to (EX1) of 1^* = 10.1 and *xf* = 4, feasible for all nine outcomes. While this approach is appealing since it is guaranteed to give the exact solution (if a bounded feasible solution exists), the exponential growth in constraints and variables in the deterministic equivalent can quickly lead to unmanageably large problems (see Dantzig, 1987). Our proposed solution method is based on solving aggregate models formed by partitioning the probability space so as to represent the influences on the stage-1 activities using die smallest possible expansion of variables and constraints. In § 10 we consider the solution to (EX1) via the proposed successive disaggregation algorithm (see "Part IT for detailed description of the algorithm).

5 Optimality and Feasibility

Optimally

Having presented the small motivating example we now examine in general terms the optimality and feasibility conditions associated with the stochastic problem. We examine the optimality conditions for the two-stage case of the stochastic linear program, based on the certainty equivalent formulation (P0). To simplify the discussion we omit subscripts referring to stage-2 terms when it is assumed to be apparent The set *K* refers to the stage-2 event space defined on the probability space *(0%f,P)*. Hence, if *0* is a discrete space then AT is a direct mapping, and if *0* is continuous then AT is a discrete approximation thereof. The dual variables induced by (PO) are listed in Table 1.

The (KKT) optimally conditions for (PO) **are** as follows,

(statkmarity) $\partial\mathcal{L}$ s O = c, + i4,rA* + 21 £ A 4 - 0 i + « i / (4)

$$\frac{\partial \mathcal{L}}{\partial x_{2k}} = 0 = p_{2k}c_{2k} + A_{2k}^T\lambda_k - \rho_{Lk} + \rho_{Uk} \qquad \forall k \in K \qquad (5)$$

(complementarity) o£e, = 0, O^(J^ - U_{Xi}) = 0 (6a,b)

$$\rho_{Lk} x_{2k} = 0, \ \rho_{Uk}(x_{2k} - U_{x_2}) = 0 \qquad \forall k \in K, \qquad \textbf{(6c4)}$$

where £ is the Lagrange function and $a_L, o_v\%p_u$, and p_{Uk} are non-negative dual variables.

Table 1. Two-stage MSLP certainty equivalent (PO) dual variables.

Constraints	Dual Variables	Dual Dimension
(3b)	A*	m,xl
(3c)	*K*	(mjXDxlATl
(3d)	***0_L>0_v***	n,xl, A, xl
(3e)	***Pu>P\n.***	(/ijXOxlATl, (WjXDxlATl

In solving problem (PO), our primary interest is in finding the best action to take at the current time (i.e., JÇ*). Since the size (m,) of the stage-1 activity vector (x_x) is typically much less than that of the expanded stage-2 counterpart (see Dantzig, 1987), we might expect that not each term of stage-2 is significant with respect to the optimal stage-1 activity solution. However, inspection of (4) shows that in general this is not the case, since each torn of the stage-2 expansion is represented in the stationarity condition for x_x. Intuitively, we would expect that while potentially all stage-2 influences may be relevant (to die stage-1 solution), they do not all act completely independently. This concept is the cornerstone for the aggregation / disaggregation solution method, whereby the goal is to combine stage-2 influences in such a manner as to reflect the net influence on stage-1 with the minimum expansion over the stage-2 probability space.

Feasibility

Strictly speaking, the recourse formulation (PO) of the stochastic planning problem (1) requires feasibility for all activities for any optimal solution. That is, if any constraint is violated, then the problem is infeasible. This is particularly relevant for selecting the stage-1 activities, since in general we wish to avoid any action which is anticipated to lead to some operational infeasibility in the future. This restriction may in some instances be viewed as too inflexible. Imagine a future event with an extremely small probability which is forcing the stage-1 activity to move from what would otherwise be a better optimum point There are several ways to address this situation.

Before solving the certainty equivalent problem (PO) we can include slacks in the stochastic constraints. Introducing slack variables has the following results: (0 inequality constraints are replaced with equality constraints, *(ii)* (numerical) feasibility of the stochastic constraints can be insured for all events, and *(Hi)* penalties for feasibility violations can be added to the objective function. Since a probability can be assigned to each realization of the stochastic parameter vector (i.e«, each event), we can measure the probability of feasible operation. Assigning penalties to the feasibility slack activities in the objective function is similar to the "discrepancy cost*[9] approach suggested by Dempster (1965). Using Dempster's approach one assigns a cost to the violation of any of the constraint conditions. In the production planning context, one example would be to add a slack for producing less than the minimum demand for a product, and then penalizing this slack based on the cost of purchasing this makeup product on the outside market. Conversely, omitting penalty functions and (feasibility) slacks enforces strict feasibility for all projected future outcomes. The formulation we use for problem (PO) is applicable with or without the addition of slacks to insure numerical feasibility for all parameter realizations.

Another approach is to relax the requirement of absolute feasibility for all outcomes and instead formulate the problem as a chance-constrained programming problem (Charncs and Cooper, 1959; Thompson, Cooper and Charncs, 1963; Taha, 1987 pp. 802-804). This formulation requires the constraints to be met within a predefined probability limit. For example, one might require that the constraints are met at least 95 per cent of the time, according to the cumulative probability distribution functions specifying the uncertainties of the system. A major drawback to this formulation is the introduction of nonlinearities in the constraints stemming from the distribution functions which are in general nonlinear functions. The recourse formulation can be viewed as a sub-set of the chance-constrained formulation, when the constraints are required to be met 100 per cent of the time. One could even formulate die recourse problem with a constraint on the probability of feasible solution by accounting for (probability-weighted) constraint violations, although doing so might necessitate introducing integer variables.

6 Review of Solution Methods

The study of the theory and solution of the multi-stage stochastic LP (MSLP) has paralleled the development of deterministic LP methods. Early references included seminal work on the formulation and problem structure (Dantzig, 1955,1963; Madansky, 1963; Rosen, 1963; Dempster, 1965; Wets, 1966), but left questions concerning the solution to the general problem largely unanswered. Since the certainty equivalent LP (P0), expanded to multi-stage as needed, is intractably large for all but the smallest problems (see Dantzig, 1987 for discussion of exponential expansion), current solution methods use Benders-based decomposition strategies (Van Slyke and Wets, 1969; Benders, 1962; Geoffrion, 1972). See Dantzig (1987) or Birge (1982b) for a discussion of the general multi-stage stochastic LP formulations. Comprehensive reviews of theory and solution practices are provided in the collections edited by Dempster (1980) and Ermoliev and Wets (1988). Of particular interest to our proposed disaggregation algorithm, is the review by Rogers et al. (1991) of methods based on the use of aggregate models.

Spurred in part by the expansion in computing power, recent progress has been made in solving the two-stage linear and mixed-integer stochastic programming problem using Benders-based schemes (see e.g., Dantzig and Glynn, 1989; Infanger, 1991; Wets, 1983, 1989; Gassmann, 1990; Bicnstock and Shapiro, 1985). Key extensions over previous methods include the utilization of parallel computers to solve the independent sub-problems resulting from Benders decomposition (Dantzig, 1987), and using importance sampling methods to bypass the exhaustive computations corresponding to the full probability model. However, exact representations of the problem are typically precluded by realistic expansion limits on current computing platforms. Since the recourse formulation is exponential in nature, we anticipate that approximation algorithms will continue to be required even as massively parallel computers become commonplace.

Extension to multi-stage problems via nested decomposition methods is conceptually straightforward. The multi-stage problem however remains intractable due to computational expense, arising from the nested structure of the problem and resultant exponential growth in the number of sub-problems (see Dantzig, 1987; Gassmann, 1990;

Louveaux, 1986; Birge, 1982b; Dempster, 1980). While a few specialized problems have been addressed (see Dantzig, 1987; Beale et al, 1980; Bicnstock and Shapiro, 1985; Karreman, 1963), general multi-stage linear problems remain computationally intractable. Multi-stage solution methods generally rely on nested decomposition strategies which involve solving series of two-stage sub-problems (Gassmann, 1990; Birge, 1982b; Ermoliev and Wets, 1988). Hence, advances in the solution to two-stage models are applicable toward improving multi-stage solution methods.

7 Aggregate Model Definition

At the cotter of the proposed solution method for die stochastic LP (P0) is the aggregate model. Conceptually, an aggregate model is defined by a partitioning of the probability space along with the partition-normalized means for the stochastic parameters (i.e., RHS terms and cost coefficients). Thus, when a single partition is used to define the probability space, the solution to the aggregate model is simply the mean-value solution to the stochastic LP. A precise mathematical definition is now provided, followed by a discussion of the theoretical properties relevant to the successive disaggregation algorithm.

The following notation will be used in defining the aggregate model and in the subsequent discussion of its properties. Additionally, any parameter or variable with a "bar" (e.g., $\bar{r}$) refers to either the mean value taken over the appropriate probability space (or sub-space) or to terms associated with the aggregate model solution. We restrict our definition and discussion to the two-stage case.

Index sets:

$k \in K$ set of stage-2 events; $N_K = |K|$.

$q \in Q$ set of disjoint partitions K_q whose union comprises the entire stage-2 event space K; $N_Q = |Q| \leq |K|$.

Parameters and variables:

b_1 stage-1 primal LP row RHS, $b_1 \in \Re^{m_1}$.

b_{2k} stage-2 primal LP row RHS expanded over event space, $b_{2k} \in \Re^{m_2}$.

$\bar{b}_{2q}$ stage-2 primal LP aggregate row RHS expanded over partition space, $\bar{b}_{2q} \in \Re^{m_2}$.

c_1 stage-1 primal LP cost, $c_1 \in \Re^{n_1}$.

c_{2k} stage-2 primal LP cost expanded over event space, $c_{2k} \in \Re^{n_2}$.

$\bar{c}_{2q}$ stage-2 primal LP aggregate cost expanded over partition space, $\bar{c}_{2q} \in \Re^{n_2}$.

p_{2k} stage-2 probability of discrete event $k \in K$, $p_{2k} \in \Re^1$.

$\bar{p}_{2q}$ stage-2 probability of aggregate event $q \in Q$, $\bar{p}_{2q} \in \Re^1$.

JC, stage-1 primal LP activity, x, $\in \Re^{n_1}$.

x_u stage-2 primal LP activity expanded over event space, x^ € 91"*.

x^ stage-2 primal LP aggregate activity expanded over partition space defined with respect to partitioning C. x^$eX*^1$.

z primal LP objective function to be minimized, z € 9t\

$\bar{z}_c$ primal aggregate LP objective function to be minimized defined with respect to partitioning Q, $?_0$ cSt^1.

$ stochastic parameter vector, 0e9t^w*.

With the notation preliminaries aside, and using the fixed recourse formulation as per problem (PO) we define the aggregate model for a given set of partitions G as follows:

(PA) min ^ t f a + Xp^cJjCj, (7a)

s.t. AX, =l\ (7b)

$$B_1x_1 + A_2x_{2q} = \bar{b}_{2q} \qquad \forall q \in Q \tag{7c}$$

$$0 \le x_1 \le U_{x_1} \tag{7d}$$

$$0 \le x_{2q} \le U_{x_2} \qquad \forall q \in Q, \tag{7e}$$

where p2f» £/>2t : ?€fi;*, CAT, (8)

$$\bigcup_{q \in Q} K_q = K, \tag{9}$$

$$\bar{b}_{2q} = \sum_{k \in K_q} p_{2k} b_{2k} \Big/ p2,. \qquad 00)$$

$$\bar{c}_{2q} = \sum_{*\bullet*,} p_{2k} c_{2k} \; / \; {}_{2q}. \tag{11}$$

The optimality conditions for (PA) are similar to those presented above for the deterministic equivalent problem (PO), except that the expansion now takes place over partitions q e Q as opposed to events *ksK.* Note that l£|g|£|Ar| when all partitions are required to be non-empty. Furthermore, it is assumed that no zero-probability events are included in AT, and hence $p_{2f} > 0$ *VqeQ.*

The feasibility properties of the aggregate problem (PA) are essentially the same as those for the deterministic equivalent problem (PO), within the context of the aggregate model. That is, the fixed recourse formulation is retained in (PA), requiring the stage-1 decision to be feasible for all stage-2 (aggregate) outcomes. Problem (PA) is formed as a linear combination of the constraints and objective function terms from (PO), and as such is a relaxation of the full-space problem (see proof of Theorem 1 later in paper). Hence, a feasible solution to problem (PA) is not necessarily feasible for problem (PO), and particular care must be taken when acting upon the aggregate solution to ensure feasibility for (PO). The reader may refer to Appendix A for further discussion on the feasibility of problem (PA) solution with respect to problem (PO).

8 Aggregate Model Bounding Properties

Depending on the structure of the stochastic LP, the aggregate model has bounding properties which are of special interest with respect to the successive disaggregation solution method In this section we consider three cases of fixed recourse two-stage stochastic LP's and the bounding properties provided by the corresponding aggregate models. Rogers et al. (1991) discuss aggregate model bounding properties, and include an extensive list of related references.

Stochastic production planning models can be delineated according to die sources of uncertainty in the model. The bounding properties correspond to the convexity or concavity of the recourse sub-problem, which depends on the types of uncertain parameters in the model. A summary of the three cases, along with the key bounding characteristics is presented in Table 2. We now consider the bounding properties for each case individually.

Table 2. Two-stage MSLP aggregate model bounding characteristics.

MSLP case	RHS terms	Cost coefficients	Recourse problem	Bounding property
1	stochastic	fixed	convex	•
2	fixed	stochastic	concave	?Zz •
3	stochastic	stochastic	convex/concave	none

Case 1: Fixed Costs / Stochastic RHS

The convexity of case-1 problem instances follows directly from the definition of (PA), which can be viewed as a standard LP known to be convex. We are interested in the bounding properties of (PA) with respect to the exact solution obtained from (PO). Furthermore, we are concerned with the behavior of the bounds as (PA) is successively disaggregated (or aggregated).

Lemma 1. Consider problems (SI) and (S2) defined as follows:

$$\min \; z\text{-}f(x) \qquad \text{(SI)}$$
$$\text{s.t.} \; h_k(x) = O \qquad VkeK$$
$$xZO,$$

and

$$\min \; z'\text{-}f(x) \qquad \text{(S2)}$$
$$\text{s.t.} \; \underset{*€«,}{\underline{X}}{}^{a}A(*)\text{~}O \quad V\text{^}efi$$
$$x \geq 0,$$

where $a_4 > 0$, *fix)* is convex, and A»(x) is linear. Then z* £ 2" provided that (SI) is bounded.

Proof (S1) and (S2) can be written, respectively, as:

$$n\dot{a}n \; z\text{-}f(x):xeF \qquad \text{(Sla)}$$

$$\dot{m}inz'=f(x):xeF'. \qquad \text{(S2a)}$$

Assume that z* < z'\ Then is follows that *F'cF*. Then 3 i such that $A_k(\hat{Jc}) = O$ V* € AT and $I_{4€ir<}\, a_t\text{^}(\hat{x}) \# 0$: ? e Q. Since $a_k > 0$, this is a contradiction, and hence F C F*. and $z^{\#}$ ^V*. // Q.E.D.

Theorem 1. Given any partitioning *Q* which conforms to (9) in (PA), then *Z'ZZ'Q*, where z* and z'_Q denote optimal solutions to (case-1) problems (PO) and (PA), respectively.

Proof Multiplying the stage-2 constraints of (PO) by p_{2k}, summing over keK_b, and dividing by the partition probabilities, p_{u}, yields:

$$\min \; \hat{z}_c = c_1^r x_1 + £Xft{*}c_2{}^I{>}2» \qquad (\hat{S})$$

$$\text{s.t.} \quad J4,X, \qquad S \text{^}$$
$$B_1 x_1 + A_2 \sum_{k \in K_q} p_{2k} x_{2k} / p_{2q} = \bar{b}_{2q} \qquad \forall q \in Q$$
$$x_1 \geq 0$$
$$x_{2k} \geq 0 \qquad \forall k \in K,$$

where

$$\bar{b}_{2q} = \sum_{k \in K_q} p_{2k} b_{2k} \Big/ p_{2q}.$$

From Lemma 1, we know that $z^{*} \leq V_Q$.

1x1

%.f*2ftA»/Paf, (12)

which then yields:

$$\sum Z \ldots \leq \ldots p_{2q} c_2^T x^* . \tag{13}$$

which can be used to replace the objective function in $(\hat{S})$. Similarly, $x^\wedge \geq 0$: $k \in K_n$ can be replaced by $z^\wedge \geq 0$: $\$ \in Q$, yielding problem (PA). Substituting the $x^\wedge$ terms for the $x^\wedge$ terms constitutes a weakening of the $(\hat{P})$ bounds via further application of Lemma 1. Consequently, $V_Q \leq Z^*_Q$ and hence $z^* \geq z^*_Q$. // Q.E.D.

Corollary 1. For case-1 problems $z^\wedge \leq z^\wedge$, where partitioning $CL_1 : |Q_{i+1}| = i+1$ is formed by disaggregating (2,.

Proof The pnx>f follows directly by normalizing the partition probabilities and leapplying Lemma 1 over the appropriate partition subspace. Thus, any disaggregated partition is a sub-problem whose contribution to the objective must be greater than or equal to that of the aggregate over the same subspace. //Q.ED.

Therefore, for case-1 problem instances disaggregation will increase or leave unchanged the value of the objection function. And, successive disaggregation must monotonically increase the lower bound on the objective, z, such that:

$$z| \ldots \leq \ldots z_I \leq z. \leq z \, l_{K \, i} = z \setminus \tag{14}$$

Case 2: Stochastic Costs / Fixed RHS

When the cost coefficients are stochastic and the RHS terms fixed (i.e., case-2 problems), the recourse function is concave. An analysis similar in spirit to that above leads to the result that disaggregation monotonically decreases the upper bound (given by the solution to the aggregate problem) on the objective, $\bar{z}$. However, an alternative and in this case preferable analysis is to show that the case-1 and case-2 problems are structurally similar, and that case-2 problems can be converted through dual transformation to case-1 problems.

To simplify the discussion, consider a canonical representation of the stochastic LP stated as follows:

$$\min \quad c^T x: \ AxZb, \ \ x*0. \tag{15}$$

Taking (IS) as the primal problem, the dual is formulated as:

$$\max \quad b^T n \ : \ A^T K£C. \quad **0, \tag{16}$$

which can be rewritten as:

$$\min \quad -b^T n \ : \ -A^T xZ\text{-}c_9 \ \ KZO. \tag{17}$$

Thus (17) represents a stochastic program with uncertain cost coefficients. Problems (IS) and (17) give identical values for the objective function (given a finite feasible solution to each). Therefore, case-1 and case-2 problems can be considered interchangeable via the dual transformation.

Corollary 2. For case-2 problems ^^z\

Proof Consider (IS) with fixed RHS terms and uncertain cost coefficients. The dual is given by (16) or equivalently (17), which has uncertain RHS terms and fixed cost coefficients. From case-1 problem analysis (Theorem 1) we know that -z^ £-z*, and hence £**•. //QJE.D.

Corollary 3. For case-2 problems z^* = z^ £...£ z^ £ z^ £...≤ z^*_α.

Proof The proof follows that of Corollary 1 in light of Corollary 2 as applied to the case-2 problem. //Q.E.D.

Case 3: Stochastic Costs and RHS

For the general case when both cost coefficients and RHS terms are stochastic, no bound on (PO) is provided by the solution to the aggregate model (PA). Hence, we cannot use the aggregate model as a bound for case-3 problems. We can however, use the sensitivity information associated with the aggregate solution, as well as the upper bound provided by applying the aggregate optimal stage-1 activity vector in the full-space problem (as discussed in "Part IT).

9 Solution Method Outline

A high-level flowchart of the proposed successive disaggrcgation algorithm is shown in Figure 2. We present this overview in order to convey the basic concepts behind the solution method The reader may refer to "Part IF (day and Grossmann, 1994) for a detailed discussion of the proposed algorithm **along** with computational results on test problems.

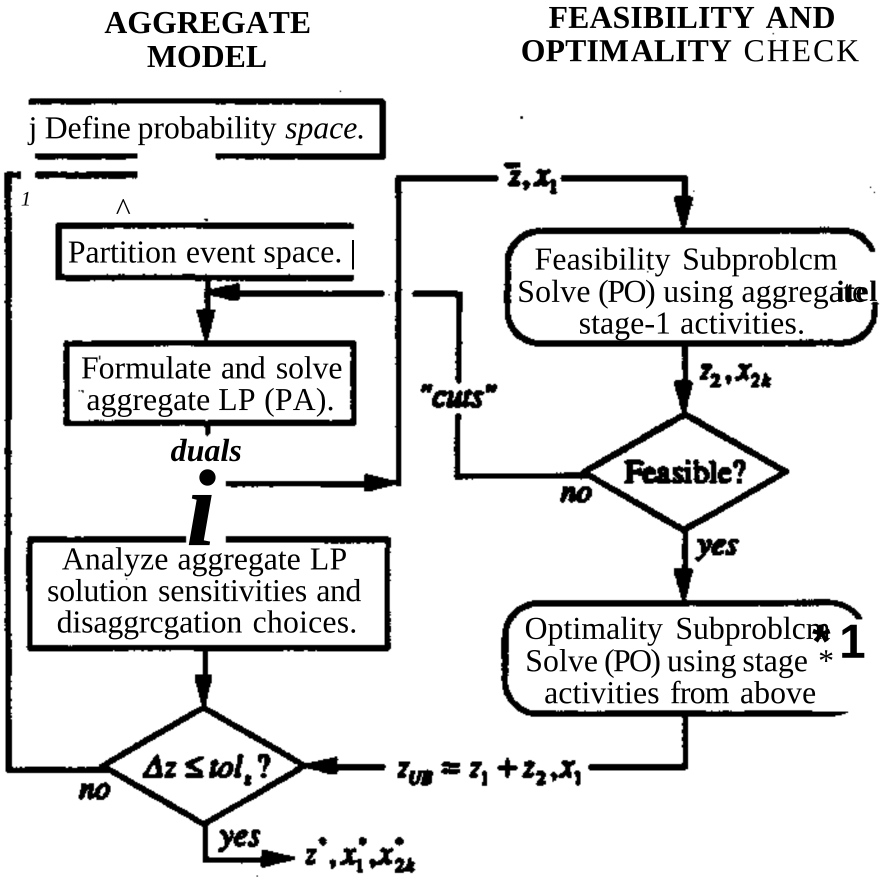

Figure 2. Overview of the two-stage successive disaggrcgation algorithm for non-concave objective, based on certainty equivalent problem (PO) and aggregate problem (PA).

The algorithm can be viewed as three basic steps: (i) problem definition, (ii) aggregate model formulation, solution, and rcpartitioning analysis, and *(Hi)* feasibility and optimality check. In step i the problem is defined by specifying the planning model constraints and objective along with all relevant parameter probability distribution information. Step ii is centered around the aggregate model definition, solution, and analysis. The aggregate model (defined in § 7) is formed by a probability-weighted linear combination of variables and/or constraints. These combinations are dependent upon the partitioning of the event space and the corresponding partition probabilities. For every partitioning defined, the resultant aggregate model is solved to optimality yielding an approximation of the optimal stage-1 activity vector, as well as the dual (sensitivity) information for the stage-2 partition-aggregated variables and/or constraints. Analysis of the sensitivities is used to determine whether further disaggregation of the event space is needed, and if so to project the'best* repartitioning.

Step iii is the feasibility and optimality check which provides the solution to the complete stage-2 activities (and duals). Tosimplify the discussicm of step wi, consider the case where the costs are fixed and the RHS terms are stochastic. In this case, the solution of the aggregate model is a lower bound on (PO). Furthermore, the lower bound monotonically increases with successively disaggregated partitionings. The stage-2 activities may be needed either for further analysis or as part of the solution to a multi-stage problem using a nested decomposition strategy (e.g., see Gassmann, 1990 or Birge, 1982b).

Starting with (PA) defined over a single partition, its solution $\bar{x}_1^*$ is tested for feasibility in (PO), simply by solving the stage-2 component of (PO) using the additional constraint x, * x̄j*. Considering x, a fixed input to (PO) allows us to solve for the stage-2 component of the problem via a collection of (summed) independent (and relatively small) LP sub-problems. If any one of these sub-problems is infeasible, the most violating (indexed by largest negative slack) constraints are identified, and the corresponding event k^ € K^ c K : x, = x,* is used to generate a feasibility "cut" in (PA). Such a cut is simply a reflection of the x, constraint and feasibility space with respect to the stage-2 solution x_{lk}. In the simplest case, a cut can be defined as the (PA) augmenting constraint set corresponding to the infeasible event k^ € K^ c K : x, = J*ι such that:

$$A_2 x_2^c = b_{2k} - B_1 x_1 \qquad \forall^* = k^\wedge \in K^\wedge \subset K,$$
$$0 \le x_2^c \le U_{x_2} \qquad \forall c \in C = \text{"cut" set.}$$

Initially, the solution to (PA) is repeatedly tested for (x_x) feasibility, adding cuts until (PA) maintains (x;) feasibility with respect to (PO), assuming a feasible solution x_x exists for (PO). If one does not exist, (PO) is infeasible and must be reformulated to produce a valid solution. In "Part IT a feasibility cut will be described which does not require the evaluation of all the stage-2 LP sub-problems.

Once the initial cycle of generating the (PA) feasibility cuts is complete, the stage-1 activities found by solving the aggregate model are tested for optimality in (PO). The feasibility sub-problem provides a proposed stage-2 solution, which is in turn used as a fixed input to solve for the stage-1 component of (PO), which we denote as the optimality sub-problem. Summing the stage-1 and -2 contributions to the objective from die two sub-problems gives an upper bound on (PO) as well as the optimal stage-1 solution for this counter problem. The upper bound and stage-1 (counter-) solution can then be tested for proximity to the lower bound and trial stage-1 (aggregate) solution, respectively.

A variant of die proposed method is to skip the evaluation of the upper bound which requires the solution of \K\ sub-problems ((PO) with fixed stage-1 activities). In this case feasibility cuts are generated with a special formulation as proposed in Tart IT to avoid the solution of the \K\ sub-problems. Similarly, a special formulation can be developed to evaluate the upper bound.

10 Illustrative Examples

In this section we present two relatively simple examples which convey the basic approach of the successive repartitioning algorithm. The specific details of the algorithm can be found in "Pan IT of this paper (Clay and Grossmann, 1994). The first example is problem (EX1), defined with stochastic RHS terms and fixed costs and introduced in § 4. This problem demonstrates the response of the algorithm when the aggregate solution is infeasible in the original problem (PO). The second example is a problem with stochastic costs and fixed RHS terms. The example demonstrates the method applied to case-2 problems.

Example 1 Revisited

In § 4 problem (EX1) was solved using the mean-value approximation (i.e., using formulation (PA) with a single partition including all events) to give $V = z_u = 9.15$ and Xj = 3.05. Solving (PO) (see Figure 2) with the added constraint (or fixed input) $x_x = 3.05$ gives an infeasible solution, indicating the mean value solution is infeasible. From the solution to the feasibility sub-problem we can identify the most violating constraint, which we then use as a feasibility "cut*[9] augmenting the constraints of (PA) for all future solutions. We note that the cut variables reflect the constraint and feasibility space for the most limiting event (corresponding to the most violating constraint). Furthermore, the cut variables do not appear in the objective function, and hence only serve to bound the JC, feasible space in (PA). In this case the most restricting event is when both demands are

maximal, corresponding to 17 = (3,3). We create a feasibility cut for this event, thus enforcing subsequent solutions to (PA) to be feasible with respect to that constraint, assuming a feasible solution does exist The feasibility cut is given by constraints:

$$y_1 \geq 4$$
$$y_2 \geq 4$$
$$y_1 + y_2 - x_1 \leq 4.$$

Again solving (PA), using a single partition plus the feasibility cut, gives z_u=10.1 and x, = 4. Solving (P0) with fixed x, =4 gives a stage-2 activity solution, Xj*. Solving (PO) with fixed x_u to check for optimality gives a feasible solution with z_m = 10.1 and *j = 4. Hence the objective function gap is 0.0, and the solution is complete (with both feasibility and optimality guaranteed). We have found the optimal solution by expanding over only one of nine events, and including one feasibility cut

Example 2

The second example (EX2) is a fixed recourse stochastic LP with fixed RHS's and stochastic costs (i.e., case-2 problem). We solve it using the dual transformed model (i.e., converting the problem to case-1 structure). Problem (EX2) certainty equivalent can be formulated as the following primal LP.

min z = x,+ £/ty(c£fj4vtc£yX2þ) (EX2)
ijmixj

s.t. x, + 2x^ + 3x£, £ d, Vïyeixy

x, + 3x£tf + Ixljj £ di Vi>' € 1 xj

x, SO

x^ ,Jx j ≥ 0 Vïyeixy,

where ^ = {1,2,3}, pi = {0.5,0.4,0.1},

^ = {1,2,100}, pi = {0.1,0.4,0.5},

$p_{2ij} = p^1_{2i} \cdot p^2_{2j}$,

$x_1 \in \Re^1$,

$x^1_{2ij} \in \Re^1$, $x^2_{2ij} \in$ 91[1] Viy € 1 xj.

Here x^- and x^ are continuous stage-2 variables (vector elements 1 and 2, respectively) expanded over the discrete probability space defined by crossing independent spaces indexed by 1 andy.

The solution to (EX2) via the proposed algorithm (see Figure 2) starts by forming the expected-value problem (PA) using a single partition (i.e., IQI= 1). The resultant aggregate problem (PA) is then dual transformed per § 8 to give:

min - ^ - X P * , ^ , * * ^) (EX2$_{PAdual}$)

Q

$$\begin{aligned}
\text{S.t.}\ & -1 * \wedge -2\pi^2_{2q} \geq -c_1 = -1 & \\
& -2 \quad -3\pi^2_{2q} \geq -p_{2q}\bar{c}^1_{2q} & \forall q \in Q \\
& -3\pi^1_{2q} - 2\pi^2_{2q} \geq -p_{2q}\bar{c}^2_{2q} & \forall q \in Q \\
& \pi^1_{2q}, \pi^2_{2q} \geq 0 & \forall q \in Q,
\end{aligned}$$

where $\pi^1_{2q} \in \Re^1,\ \pi^2_{2q} \in \Re^1 \qquad \forall q \in Q.$

We assume here that we use the variant of the algorithm that does not require evaluation of the upper bound (and hence we do not solve the IXI UP sub-problems). With lfil= 1, solution to (EX2pAduii) gives -I* =-1.6 and j£ =0. Solving (PO) with j^ =0 gives a feasible solution with $-z_{UB}$ = -1.426. Sensitivity and repartitioning analysis using the (PA) solution (see "Part 11") suggests splitting along probability dimension 2 between events 2/3 (see Figure 3) to maximize the increase in the lower bound $-\bar{z}$ \ Repeating the (PA) sensitivity analysis, repartitioning, dual transformation, and solution sequence until no further improvements in the lower bound are projected leads to exact solution (-y* SS-ZJJJ =-z") at six partitions (see Figures 3 and 4). Changing the sign of the objective values gives the correct (PO) value, and shows how the case-2 primal problem aggregate solution monotonically decreases with increased disaggregation. However, solving the dual transformed problem is preferable since the solution method provides both an upper and lower bound to check convergence.

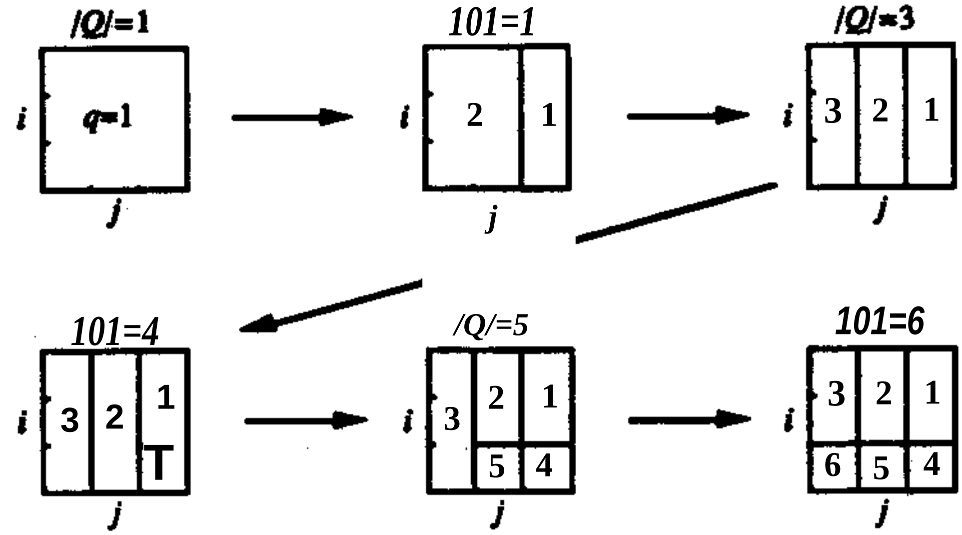

Figure 3. Disaggregation sequence for problem (EX2). Numbers in boxes indicate the partition index *q*.

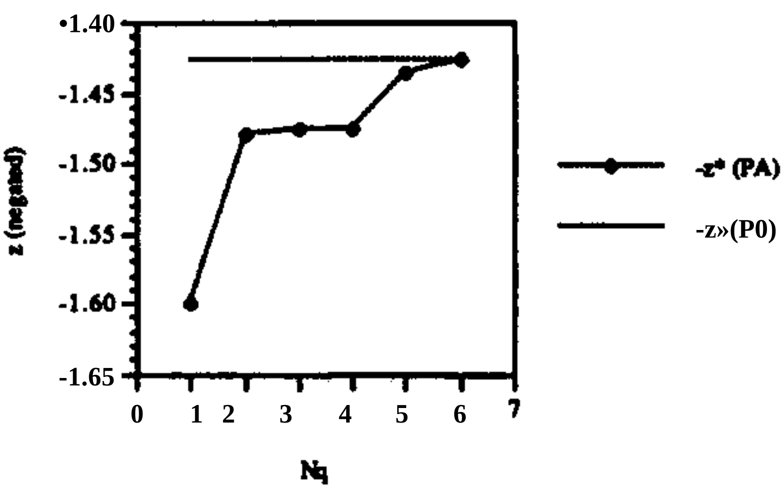

Figure 4. Aggregate bounds following disaggregation sequence for problem (EX2). Objective values negated to correspond to dual formulation of (PA). Nq«lgl.

Conclusions

In this paper we have presented general conceptual and theoretical issues that arise in the two-stage linear programming problem for production planning. In particular, it has been shown how aggregate models of fixed recourse stochastic (production planning) LFs provide approximate solutions and serve as the basis for a sensitivity-based successive disaggregation algorithm. Bounding properties have been established for particular problem structures to guide disaggregation of the aggregate probability space, using sensitivity analysis of the aggregate solutions along with repartitioning projection analysis. In "Part IT of this paper, we present the detailed two-stage successive disaggregation algorithm along with computational results and example problems.

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Appendix A
Feasibility of Two-stage Aggregate Stochastic LP

As shown in Theorem 1 problem (PA) is formed as a linear combination of the constraints and objective function terms from (PO), and as such is a relaxation of the full-space problem. Feasibility is of course strictly related to the constraints. It is perhaps easiest to visualize this constraint relaxation in the context of a simple example problem.

Consider the following example problem:

min z«s2x,+^x_u (Al)

s.t. x, +x_{2Jk}£ft_u VkeK

$0 \le x_1$

$0 \le x_{2k} \le 2$ $\forall k \in K$,

where ^ = { 1 3},

$p_{2k} = \{0.5\ 0.5\}$.

Letting die partitioning *Q* contain a single aggregate event gives:

$\bar{b}_{2q=1} = 2$,

which leads to the aggregate problem optimal solution as follows:

Xi=0;*^., =2; 1 = 2.

However, when x, = 0 is substituted back into (Al) the constraints are violated since:

$(2 \ge x_{22}) \ge (b_{22} = 3)$

is clearly infeasible.

This constraint relaxation is shown in Figure A.I, indicating that the aggregate model constraint space (i.e., feasible region) will always (for all problems) be larger than that of the deterministic equivalent problem. The line passing through points GG is the upper bound for die stage-2 activity. lines BF and AD represent die constraints for events 1 and 2, respectively, of (PO). line CE represents the aggregated constraint for (PA). The stage-1 activity for (PO) is limited by the feasible region for event 2 (defined by DAG), which is mote restrictive than the feasible region for event 1 (defined by DFBCXJ). The optimal value for the (PO) sub-problems (i.e., expanded constndnts) are points F and A for events 1 and 2, respectively, leading to a solution value of 3. The optimal value for (PA) is point C, leading to a solution value of 2, which is however infeasible for (PO).

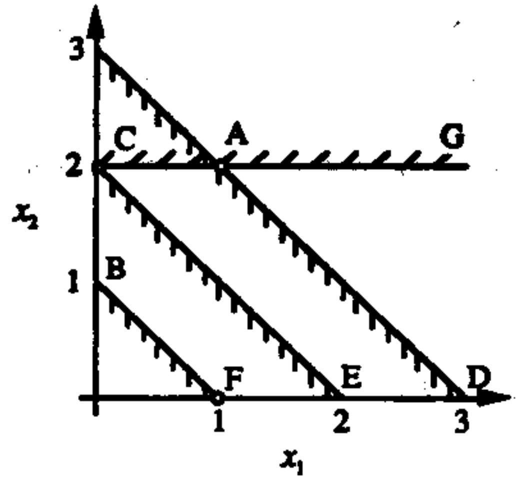

Figure A.I. Feasible regions for the deterministic equivalent (PO) and aggregate model (PA) problems for example problem (Al).