Applications of forcing to definability problems in the arithmetical hierarchy

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Abstract

Forcing arguments are used to obtain generalizations of some well-known theorems about the degrees of unsolvability with the jump operator.
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Introduction. It has been remarked by Addison [1] and Hinman [6] that applications of forcing techniques (Cohen [2]; Feferman [3]) often allow results about recursiveness to be extended to higher levels of the arithmetical (and hyperarithmetical) hierarchy. In this paper we present a description of the forcing method, and then use this technique to obtain generalizations of some well-known theorems about the degrees of unsolvability. We prove:

(1) a generalized Spector's theorem [15],

\[ V \exists b \exists c [a^{(n)} = b^{(n)} = c^{(n)} - b \lor c], \quad n < \Omega \]

and

\[ \exists a \exists b \forall (tB) \equiv \exists (v(U) \equiv a \lor b = O^x(\omega)) \];

(2) a generalized Friedberg's theorem [4],

\[ \forall a \forall b [c^{(n)} = c \lor a^{(n)} = b \lor a^{(n)}], \quad n < \Omega \]; and

(3) a generalized Kleene-Post theorem [11],

\[ 3A3B[A/D_n B \land B/E_n A \land d(A) \leq O^{(n)}] \land \quad d(B) \leq O^{(n)}], \quad n < \Omega \].
Each of these theorems concern the existence of sets (characteristic functions) of natural numbers. Our proofs will involve the construction of a total function as the union of a chain of initial segments. This general approach to degree problems was initiated by Kleene and Post in [11]. In the original Kleene-Post construction one is presented with a sequence of recursive conditions, and then defines a function (or functions) to satisfy these conditions by successively choosing greater initial segments in order to meet each condition one by one. In substance, we do the same. Forcing however, allows us to handle sequences of prescribed arithmetical conditions that are not necessarily recursive.

1. Preliminaries.

The purpose of this section is to present notation and set forth some definitions. Much of the contents are standard and refer mainly to [9] and [13].

**Prime number factorization.** Let the prime numbers in order of magnitude be \( P_0 \cdot P_1 \cdot \ldots \cdot P_i \cdot \ldots \). Let \( a \) be an arbitrary natural number. By the fundamental theorem of arithmetic there is a unique representation of \( a \), if \( a > 0 \), of the form

\[
(1) \quad a = p_0^{a_0} \cdot p_1^{a_1} \cdot \ldots \cdot p_i^{a_i} \cdot \ldots
\]

As shown by Kleene in [9], the following functions are all primitive recursive:

\[ p_i = \text{the } i+1\text{-th prime number}; \]
(a) = \begin{cases} 
\text{the exponent } a_{j} \text{ of } p_{j} \text{ in (1), if } a \neq 0; \\
0, \text{ if } a = 0; \\
\text{the number of non-zero exponents in (1), if } a > 0? \\
0, \text{ if } a = 0.
\end{cases}

We can represent any finite sequence \( a_{0}, \ldots, a_{s} \) of natural numbers by the number \( a = p_{0}^{a_{0}} \cdots p_{s}^{a_{s}} \); then \( \text{lh}(a) \) is the length \( s + 1 \) of the sequence represented by \( a \).

A \textbf{sequence number} is a number \( a = p_{0}^{a_{0}} \cdots p_{s}^{a_{s}} \) so that for all \( i \leq s, \ a_{i} > 0 \). For any two sequence numbers \( a \) and \( p \), define \( a > p \) if and only if \( \text{lh}(a) \leq \text{lh}(p) \) and \( (p)_{j} = (a)_{j}, \text{ for all } i < \text{lh}(p) \).

Let \( f \) be any partial function whose domain includes the set \( \{0, 1, 2, \ldots, n\} \). Define \( f(n+1) = \prod_{i < n} p_{i}^{a_{i}} \). \( f(n+1) \) is a sequence number. Moreover, if \( a \) is any sequence number, and if a partial function \( f \) is defined by \( f(i) = (a)_{i} - 1, \text{ for all } i < \text{lh}(a) \), then \( a = "f(\text{lh}(a))" \).

\textbf{Arithmetical properties.} Let \( \gamma \) be a one-place function variable ranging over number theoretic functions.

\textbf{Definition 1.} A predicate \( A(x, y_{1}, \ldots, y_{k}) \), \( k \geq 0 \), is an \textbf{arithmetical property} if and only if it is expressible in the form

\[ Q_{1}y_{1}, \ldots, Q_{j}y_{j} R(\bar{r}(y_{j}), x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{j-1}), \]
where \( j \not\leq 1 \), for each \( i \leq j \), \( Q^i \) is \( 3y_i \) or \( Vy^i \) and \( R(w, x_1, \ldots, x_k, y_1, \ldots, y_j) \) is a recursive predicate.

Observe that by Definition 1, the negation of an arithmetical property is an arithmetical property.

**Lemma 1.** Suppose \( R(w, x_1, \ldots, x_k) \) is any recursive predicate.
Define a new predicate \( R^* (w, x_1, \ldots, x_k) \) by \( R^* (w, x_1, \ldots, x_k) = 3v[lh(v) \leq lh(w) \& \forall i < lh(v) \((v)_i = (w)_i) \& R(v, x_1, \ldots, x_k)] \).

Then,

(i) \( R^* (w, x_1, \ldots, x_k) \) is a recursive predicate;

(ii) \( 3y R^* (T(y), x_1, \ldots, x_k) = 3y R(r''(y) ^ x^* \ldots, x^*) \); and

(iii) if \( a \) and \( p \) are two sequence numbers and \( a < p \), then \( R^* (a, x^* \ldots, x^*) \) implies \( R^* (p, x_1, \ldots, x_k) \).

**Lemma 2.** Suppose \( R(w, x_1, \ldots, x_k) \) is any recursive predicate.
Define a new predicate \( R^! (w, x_1, \ldots, x_k) \) by \( R^! (w, x_1, \ldots, a_i) = Vv[lh(v) \leq lh(w) \& \forall i < ih(v) \((v)_i = (w)_i) \rightarrow R(v, x_1, \ldots, x_k)] \).

Then,

(i) \( R^! (w, x_1, \ldots, x_k) \) is a recursive predicate;

(ii) \( Vy R^! (T(y), x_k, \ldots, x^* j) = Vy R(T''(y), x_L, \ldots, x_j) \); and

(iii) if \( a \) and \( p \) are two sequence numbers and \( a > 3 \), then \( R^! (a, x_1, \ldots, 3^*) \) implies \( R^! (p, x_L, \ldots, x^*) \).

The proofs are immediate. A recursive predicate that satisfies clause (iii) of Lemma 1 will be called **monotonic increasing**, and a recursive predicate that satisfies clause (iii) of Lemma 2
will be called monotonic decreasing. We will assume, without loss of generality, that every arithmetical property is expressed in the form

\[ Q_1 y_1 \cdots Q_j y_j R(\tau(y_j), x_1, \ldots, x_k, y_1, \ldots, y_{j-1}) \]

with \( R(w_3 x, \ldots, x_k, y_1, \ldots, y_{j-1}) \) monotonic — monotonic increasing if \( Q_j y_j \) is \( 3y_j \) and monotonic decreasing if \( Q_j y^* \) is \( V y_j \).

In particular we will use the starred and primed versions respectively of \( T_n \) and \( T_n \) as defined in [D]. \( T_n \) as defined in pO] enables the normal form and enumeration theorems to be written using \( f \) instead of \( \bar{f} \).

2. Foreing.

As briefly explained in the introduction of this paper, the forcing method will be applied to construct functions \( f \) as unions of chains of initial segments \( f^*, f^*_1, \ldots, f^*_n \). We desire relativized forms of such constructions. Therefore, at the \( n \)-th stage of a construction, \( f^*_{n+1} \) is chosen from some admissible subset of the set of all initial segments. Accordingly, forcing is defined relative to a notion of admissibility.

Definition 2. A characteristic sequence number is a sequence number \( a = \prod p \) so that \( a \in \{1, 2\} \), for all \( i < n + 1 \).
**Definition 3.** Let adm(oc) be a unary relation defined on the set of all characteristic sequence numbers. For any two characteristic sequence numbers a and \( \beta \), define \( a >_p \beta \) if and only if \( \text{adm} a >_p \beta \), adm(a), and adm(\( \beta \)). Let \( n_1, \ldots, n_k \) be arbitrary numerals. Then, the relation a adm-forces \( A(T, n_1, \ldots, n_k) \), in symbols \( a \vdash_{\text{adm}} A(T, n_1, \ldots, n_k) \), is defined inductively for arbitrary sequence numbers a and arithmetical properties \( A(T, x_1, \ldots, x_k) \), as follows:

(i) \( a \vdash_{\text{adm}} \exists y R(T(y), x_1, \ldots, x_k) \), where \( R(w, x_1, \ldots, x_k) \) is recursive, if adm(a) and \( R(\alpha, n_1, \ldots, n_k) \);

(ii) \( a \vdash_{\text{adm}} \forall y R(T(y), x_1, \ldots, x_k) \), where \( R(w, x_1, \ldots, x_k) \) is recursive, if adm(a) and for each \( p >_p a \), \( R(\beta, n_1, \ldots, n_k) \);

(iii) \( a \vdash_{\text{adm}} \exists y A(T, n_1, \ldots, n_k, y) \), if for some \( \alpha \in \omega \), \( a \vdash_{\text{adm}} A(T, n_1, \ldots, n_k, n) \);

(iv) \( a \vdash_{\text{adm}} \forall y A(T, n_1, \ldots, n_k, y) \), if adm(a), and for each \( 3 >_p a \) and for each \( n \in \omega \) there is some \( y >_p a \) so that \( y \vdash_{\text{adm}} A(r, n_1, \ldots, n_k) \).

**Lemma 3.** If \( a \vdash_{\text{adm}} A \) and \( p >_p \alpha \), then \( P \vdash_{\text{adm}} A \).

**Proof:** if \( a \vdash_{\text{adm}} \exists y R(r''y, n_1, \ldots, n_k) \), and \( R(w, x_1, \ldots, x_k) \) is recursive, then \( R(\alpha, n_1, \ldots, n_k) \). Thus \( p >_p a \) implies \( R(p, n_1, \ldots, n_k) \), because \( R(w, x_1, \ldots, x_k) \) is monotonic increasing.
If $a \models_{adm} \forall y R(T(y), n_{x}, \ldots, x^\lambda)$ and $Rf w^\lambda, \ldots, x^\lambda$ is recursive, then $p >_{adm} a$ implies $R(P, x_{1}, \ldots, 3^\lambda)$ by definition.

Since $>_a d m$ is transitive, $p \models_{adm} \forall v R(T(y), J^\lambda \cdot \cdot \cdot J^\lambda) *$ for each $p >_{adm} a$.

Suppose $a \models_{adm} 3y A(T, n_{1}, \ldots, ^\lambda n^\lambda y)$ and $A(r, x_{1}, \ldots, x^\lambda y)$ is an arithmetical property. Then, for some $n \in \mathbb{N}$, $\exists x \models_{adm} A(T, n_{1}, \ldots, n_{K}, n)$. Assume as induction hypothesis that $p \models_{adm} A(T, n_{1}, \ldots, n_{K}, n)$, for every $p >_{adm} a$. Then, by definition, for each $0 >_{adm} a$, $p \models_{adm} 3y A(r, n_{r} \ldots, 1^\lambda, 7)$.

Suppose $a \models_{adm} \forall y A(r, n_{1}, \ldots, ^\lambda n^\lambda y)$ and $A(T, 11^\lambda \ldots, x^\lambda, y)$ is an arithmetical property. For each $y >_{adm} a$ and $n \in \mathbb{N}$ there is $6 >_{adm} y$ so that $6 \models_{adm} A(r, n_{1}, \ldots, n_{K}, n)$. Let $p >_{adm} a$.

$>_a d m$ is a transitive relation, therefore for each $y >_{adm} p$ and $n \in \mathbb{N}$ there is a $6 >_{adm} y$ so that $6 \models_{adm} A(r, n_{1}, \ldots, n_{K})$.

Thus, $p \models_{adm} y y A(T, n_{1}, \ldots, ^\lambda n^\lambda y)$.

Lemma 4. For each $a$ so that $adm(a)$, numerals $n_{1}, \ldots, n_{K}$ and arithmetical property $A(T, x_{1}, \ldots, x_{K})$, there is some $P$ so that $p >_{adm} a$ and either $P \models_{adm} A(T, n_{1}, \ldots, n_{K})$ or $P \models h_{adm} \overline{A(r, n_{r} \ldots, n_{K})}$.

Proof: The proof is by induction on the number of quantifiers, $j$, under which $A(r, x_{1}, \ldots, x_{K})$ is expressible in the form given in Definition 1.

Case $j = 1$. For some recursive predicate $R(w, x_{1}, \ldots, x^\lambda)$, $A(r, x_{1}, \ldots, x_{fc})$ is expressible in the form $3y R(T(y), x_{1}, \ldots, x_{K})$.
or \( \forall y \, R(\overline{T}(y), x_1, \ldots, x_k) \). It follows from Definition 3 that

- either \( 3p >_{\text{adm}} a \, P \models_{\text{adm}} 3y \, R(\overline{y}, n, \ldots, 1) \) or
- \( a \models_{\text{adm}} \forall y \, T^*(T^*(y), n, \ldots, 1) \). Thus, if \( A(T, x, \ldots, x_k) \equiv 3y \, R(T(y), x, \ldots, x_k) \), then there is some \( p >_{\text{adm}} a \) so that \( P \models_{\text{adm}} A(T, x, \ldots, x_k) \).

And, if \( A(T, x_1, \ldots, x_k) \equiv \forall y \, R(T(y), x_1, \ldots, x_k) \), then a \( \models_{\text{adm}} A(T, x_1, \ldots, x_k) \) or there is some \( g >_{\text{adm}} a \) so that \( P \models_{\text{adm}} A(T, x_1, \ldots, x_k) \).

Case \( j > 1 \). Assume as induction hypothesis that Lemma 4 is true for each arithmetical property expressible in the form given in Definition 1 with fewer than \( j \) quantifiers. Let \( A(T, x, \ldots, x_k) \) be expressible in the form with \( j \) quantifiers. Then, there is an arithmetical property \( B(T, x, \ldots, x_k) \) so that \( A(T, x, \ldots, x_k) \equiv 3y \, B(T, x_1, \ldots, x_k, y) \) or so that \( A(T, x, \ldots, x_k) \equiv \forall y \, B(T, x_1, \ldots, x_k, y) \) and so that \( B(T, x, \ldots, x_k, y) \) is expressible in the form with fewer than \( j \) quantifiers. Suppose a \( \models_{\text{adm}} B(T, x_1, \ldots, x_k, y) \). Then, there exists \( P >_{\text{adm}} a \) and \( n \in a \) so that for each \( Y >_{\text{adm}} B, \quad Y \models_{\text{adm}} B(T, x_1, \ldots, x_k, y) \). Therefore, by induction hypothesis \( \exists y >_{\text{adm}} B \, \models_{\text{adm}} B(T, n, x_1, \ldots, x_k) \). Since \( \exists y >_{\text{adm}} B, \) and \( B >_{\text{adm}} a, \) \( Y >_{\text{adm}} a. \) Thus, for some \( Y \models_{\text{adm}} B(T, n, x_1, \ldots, x_k) \).

It follows that for some \( Y >_{\text{adm}} a \) \( \models_{\text{adm}} B(T, n, x_1, \ldots, x_k) \).

Thus, if \( A(r, x, \ldots, x_k) \equiv 3y \, B(T, x_1, \ldots, x_k, y) \), then there is some \( Y >_{\text{adm}} 0 \) so that \( \models_{\text{adm}} A(r, n_1, \ldots, n_k) \).
And, if \( A(T, x_1, \ldots, x_k) = \forall y B(T, x_1, \ldots, x_k, y) \), then there is some \( y > \text{adm} a \) so that \( y \vdash \text{adm}^V V \lor a \vdash \text{adm} A(\tau, n_1, \ldots, n_k) \).

Definition 4. If \( f \) is a number theoretic function, define \( \text{adm}(f) \) if for every natural number \( n \), \( \text{adm}(f(n)) \). If \( A(T, x_1, \ldots, x^n) \) is an arithmetical property and \( n_1, \ldots, n_k \) are numerals, the relation \( f \vdash \text{adm} A(T, n_1, \ldots, x^n) \) is defined by \( f \vdash \text{adm} A(\tau, n_1, \ldots, n_k) \) if and only if \( \text{adm}(f) \) and there is some \( n \) so that \( \text{adm} A(\tau, n_1, \ldots, n_k) \).

Definition 5. A set \( G \) of arithmetical properties is closed if:

(i) each arithmetical property in \( G \) is expressible without free number variables (if \( A \) is an arithmetical property, then \( T \) is free in \( A \));

(ii) for arbitrary numerals \( n_1, \ldots, n_k \) and recursive predicate \( R(w, x_1, \ldots, x_k) \), if \( \exists y R(T(y), n_1, \ldots, n_k) \) belongs to \( G \), then \( \forall y T^V(T(y), n_1, \ldots, n_k) \) belongs to \( G \) and conversely;

(iii) for arbitrary numerals \( n_1, \ldots, n_k \) and arithmetical property \( \exists y A(T, x_1, \ldots, x_k, y) \), if \( \exists y A(r, n^\ldots, n^y) \) belongs to \( G \), where \( A(T, x_1, \ldots, x_k, y) \) is also an arithmetical property, then \( \forall y A(r, x_1, \ldots, x_k, y) \) belongs to \( G \) and for each new, \( A(T, n_1, \ldots, n_k, n) \) belongs to \( G \);
(iv) if \( \forall y A(T, n_1, \ldots, x_k, y) \) belongs to \( G \), where
\( A(T, x_1, \ldots, x_k, y) \) is also an arithmetical property,
then \( \exists y \overline{A}(r, n_1, \ldots, y) \) belongs to \( G \), and for
each \( \forall y \overline{A}(T, n_1, \ldots, n_k, n) \) belongs to \( G \).

Lemma 5. Let \( G \) be a closed set of arithmetical properties and
let \( f \) be a number theoretic function so that \( \text{adm}(f) \). If for
each \( A \in G \), \( f \models A \) or \( f \models \overline{A} \), then for each \( A \in G \), \( A(f) \)
if and only if \( f \models \text{adm} \).

Proof: Suppose first that \( R(w, x_1, \ldots, x_k, y) \) is recursive and
\( 3y R(T^n(y), n_1, \ldots, n_k) \in G \). \( f \models \text{adm} \) \( 3y R(T^n(y), n_1, \ldots, n_k) \) \( \iff \) for
some \( n \), \( T(n) \models \text{adm} \) \( A(T, y, n_1, \ldots, n_k) \) \( \iff \) for some \( n \),
\( R(T(n), n_1, \ldots, n_k) \models \overline{A}(T, n_1, \ldots, n_k) \).

Suppose \( \forall y R(T^n(y), n_1, \ldots, n_k) \in G \). By Lemma 3, not both
\( f \models \text{adm} \) \( \forall y R(T(y), n_1, \ldots, n_k) \) and \( f \models \text{adm} \) \( 3y R(T^n(y), n_1, \ldots, n_k) \).
Therefore, \( f \models \text{adm} \) \( \forall y R(T(y), n_1, \ldots, n_k) \) \( \iff \) \( f \models \text{adm} \) \( 3y R(T^n(y), n_1, \ldots, n_k) \) \( \iff \) there is no \( n \) so that \( R(T^n(n), n_1, \ldots, n_k) \) \( \iff \) for each \( n \),
\( R(T^n(n), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \). \( \iff \) for some \( n \), \( T(n) \) \( \models \text{adm} \) \( A(T, n_1, \ldots, n_k, y) \) \( \iff \) for some \( n \), \( T(n) \) \( \models \text{adm} \) \( A(T, n_1, \ldots, n_k, y) \) \( \iff \) given \( n \), there is \( m \) so that \( T^n(n) \) \( \models \text{adm} \) \( A(T, n_1, \ldots, n_k, y) \) \( \iff \) for each \( n \),
\( R(T^n(n), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \).

Suppose \( 3y A(T, n_1, \ldots, n_k, y) \in G \). \( f \models \text{adm} \) \( A(T, n_1, \ldots, n_k, y) \) \( \iff \) for some \( n \), \( T(n) \) \( \models \text{adm} \) \( A(T, n_1, \ldots, n_k, y) \) \( \iff \) given \( n \), there is \( m \) so that \( T^n(n) \) \( \models \text{adm} \) \( A(T, n_1, \ldots, n_k, y) \) \( \iff \) for each \( n \),
\( R(T^n(n), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \). By induction hypothesis this is equiva-

lent to: \( \forall y R(T(y), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \) \( \iff \) \( \forall y R(T(y), n_1, \ldots, n_k) \). Thus,
\( f \models \text{adm} \) \( 3y A(T, n_1, \ldots, n_k, y) \) \( \iff \) \( 3y A(T, n_1, \ldots, n_k, y) \).
Suppose \( \forall y A(r, n_1, \ldots, n_k, y) \in G \) \( f \). If \( \text{adm} \) \( f \) \( f \) \( \forall y A(f, n_1, \ldots, 1^k, 7) \leftarrow f \) \( \exists y \ A(T, n_1; L, \ldots >^n_X n) \) \( \forall \text{adm} \) \( \exists y A(T, n_1, \ldots, n_k, n) \), by hypothesis, 44 for all \( n \), \( f \) \( \forall \text{adm} \) \( A(f, n_1, \ldots, n_k, n) \), by induction hypothesis, \( \forall v \ A(f, n_1, \ldots, n_k) \).

3. Theorems.

For each \( k > 0 \), let \( p^k \) be a recursive one-one mapping of \( \{1, \ldots, k\} \) onto \( \{1, \ldots, k\} \) with recursive inverse functions \( I_1 \), \( I_2 \), \( \ldots, I_k \). That is, for all \( z \), \( p^k (I_1 z), \ldots, I_k z) = z \). (Explicit examples are given in [13, p. 64].) We abbreviate \( p(x_1, \ldots, x_k) \) as \( <x_1, \ldots, x_k> \).

For two degrees of unsolvability \( a \) and \( b \), \( a \in E \) \( \sim \) \( b \) will mean \( \sim n \), that there is an \( A \) in \( a \) and a \( B \) in \( b \), so that \( A \in E^B_n \).

References [13] and [14] are cited as standard references to the fundamental concepts in the study of degrees, \( \leq_r \) will denote relative recursiveness. The following Theorem 1 for the case \( n = 1 \) without the additional properties \( b / X \) and \( a \in E^T \) is due to Spector [16]. The technique used to prove \( b \in E \) and \( a \in E^T \) is due to Shoenfield [15].

Theorem 1. \( \forall a \exists b \forall c [a^{(n)} = b^{(n)} = c^{(n)} = b V c \& b / c \& c \in E^T] \).

Proof. Let \( h \) be a function with degree \( a \). Two functions \( f \) and \( g \) will be defined so that:

(i) \( f \leq h^{(n)} \) \( \& \) \( g \leq h^{(n)} \);

(ii) \( f^{(n)} \leq f V g \) \& \( g^{(n)} \leq f V g \);
(iii) $h \uparrow f \& h \uparrow g$; and

(iv) $d(f)$ differs from the degree of every set which is

$\sim_n$ in $h$, $d(g)$ differs from the degree of every set

which is $\sim_n$ in $h$.

Define $\text{adm}(a) \Rightarrow \forall x \left[ 2x < h(a) \Rightarrow (\forall z \sim x = Mx) \right]$. (iii)

will be satisfied if $f$ and $g$ are defined so that $\text{adm}(f)$ and $\text{adm}(g)$ — for in that case, for each $x$, $h(x) = f(2x) = g(2x)$.

Let $C_{n,e}$ denote the characteristic function of the $e$-th set $f \sim_n$ in $h$. Let $\phi_z$, for any function $f$, denote the $z$-th function recursive in $f$. (iv) will be satisfied if $f$ and $g$ are defined so that:

($iv'$) $\forall x \left[ \forall y \left[ \forall z \left[ f \downarrow z \lor (\forall n \phi_z x) \right] \right] \right] \land \left[ (\forall z \left[ g \downarrow z \lor (\forall n \phi_z x) \right] \right] \right] \lor \left[ (\forall z \left[ f \downarrow z \lor (\forall n \phi_z x) \right] \right] \right]

Let $B_n(r''(x_n), e, e, x_1, \ldots, x_{n-1})$ denote $\tilde{z}_n((x_n)^e \uparrow e \uparrow \cdots \uparrow x_{n-1})$ if $n$ is odd, and denote $T_{n-1}^{m,n} e, e, x_1, \ldots, x_{n-1}$, if $n$ is even. Let $Q_{x_1}$ denote $ax_1$ if $i$ is odd, and denote $Vx_{x_1}$ if $i$ is even. Let $n$ be fixed. Let $e, x_1, \ldots, x_{n-k}$ be constants,

where $1 \leq k \leq n$, and let $m = <e,x_1,\ldots,x_{n-k}>$. To the arithmetical property $Q_{x_{n-(k-1)}\ldots} Q_{x_n} B^\uparrow(T(x_n), e, e, x_1 \ldots, x^e, x_{n-(k-1)} \ldots, x^a)$ we associate the index number $n \cdot m + k$. Define $[n^m + k]$ to be the arithmetical property with index number $n^m + k$. Clearly, to each integer $I$, $I^1$, there exist unique $m$ and $k = 1, \ldots, n$ so that $I = n^m + k$. Thus with $n$ fixed this indexing is unambiguous.
For ease of notation, we will write \(-i[I]\) for the negation of the arithmetical property \([t]\), rather than \(\overline{[I]}\). The set of all arithmetical properties of the form \([I]\) and \(-i[\ell]\), \(I > 1\) is a closed set of arithmetical properties.

Construction of \(f\) and \(g\):

**Stage 0.** Define \(f_0 = g_0 = 1\). Since \(lh(f_0) = lh(g_0) = 0\), \(adm(f_0)\) and \(adm(g_0)\).

**Stage \(4l + 1\).** By induction hypothesis \(f_{4l}\) and \(g_{4l}\) are defined; \(adm(f_{4l})\), \(adm(g^\ell)\), and \(lh(f_{4l}) = lh(g^\ell)\).

**Case 1.** \(3m, k[I + 1 = n*m + k & 0 < k < n]\). By Lemma 4, there is an \(a\) so that \(a > f_\ell\) and either \(a \notsim adm^{-1}\), \([I + 1]\) or \(\mathcal{A}^\ell \notsim adm^{-1}\). Define \(f_{4l+1}\) to be the least such \(a\). Define

\[
g_{4l+1} = g_4 \cdot \prod_{lh(f_{4l}) \leq i < lh(f_{4l+1})} p_i^{(f_{4l+1})_i}.
\]

**Case 2.** \(3m[^\ell + 1 = n*m + n]\).

If \(3a \notsim adm fA [L + 1] > \) then let \(P = \mbox{adm} f_4 A \notsim \mathcal{A}^\ell \notsim adm^{-1}\).

\(lh(a)\) is odd \& \(a \notsim adm [l + 1]\). Define \(f^\ell + 1 = P - p_{lh(\ell)}\), and define

\[
g_{4l+1} = g_4 \cdot \prod_{lh(f_{4l}) \leq i < lh(\ell)} p_i^{(f_\ell+1)_i} \cdot p_{lh(\ell)}^{(b)_i}.
\]

Otherwise, let \(3 = ua (a > \), \(f_\ell\), \& \(lh(\ell)\) is odd \& \(a \notsim adm^{-1}[l + 1]\).
In this case, define $\gamma_{4t+1} = \frac{1}{\alpha_i, r}$ and define

$$g_{4t+1} = g_{4t} \cdot \prod_{1h(f_{4t}) \leq i < 1h(\beta)} p_i \cdot p_{1h(\beta)}.$$

Stage $4t + 2$. $f$ and $g$ are to be defined as in stage $4t + 1$, but with $f$ and $g$ interchanged.

Stage $4t + 3$. By induction hypothesis $f$ and $g$ are already defined, $\text{adm}(f_{4t+2})$, $\text{adm}(g_{4t+2})$, and $1h(f_{4t+2}) = 1h(g_{4t+2})$. Let $I = \langle x, y, z \rangle$. $f_{A9, 0}$ shall be constructed at this stage so that for all admissible extensions $F$ of $f_{4t+2}^A$ either, $\langle y \rangle / C_n, x$ or $F' \not\preceq_{D, x}^n$. Let $\bar{f} = f_{4t+2} \cdot p_{1h(f_{4t+2})} \cdot p_{1h(f_{4t+2})+1}$ and

$$f^1 = f_{4t+2} \cdot p_{1h(f_{4t+2})} \cdot p_{1h(f_{4t+2})+1}, \text{ if } 1h(f_{4t+2}) \text{ is even.}$$

If $1h(f_{4t+2})$ is odd, then let $f^0 = f_{4t+2} \cdot 1h(f_{4t+2})$ and

$$f^1 = f_{4t+2} \cdot p_{1h(f_{4t+2})}.$$

Case 1. There do not exist characteristic sequence numbers $a$ and $p$ so that $\text{adm}(a)$, $f_{A9, 0} \lhd a$, $3 < a$, and $f^0 \lhd \beta$. In this case define $f_{4t+3} = f^0$.

Case 2. There do exist characteristic sequence numbers $a$ and $p$ satisfying the hypotheses of case 1, but there do not exist characteristic sequence numbers $a$ and $p$ so that $f_{A9, 0} \lhd a$, $3 < a$, and $f^1 < c^p_{n, z}$. In this case define $f_{A9, 4t+3} = f$. 
Case 3. There exist characteristic sequence numbers $a, p, a^1$, and $p$ so that $f_{4l+2} \Vdash \text{adm} \Delta c, f_{4l+2} \Vdash \text{adm} \Delta c, p, p^0, p^1, f^0 < t_{p^0},$ and $f^1 < t_{p^1}$. Choose such $a, p, a^1,$ and $p$. Since $f^0$ and $f^1$ differ for some argument, $p^0$ and $p^1$ must disagree for some argument. Hence, either $p^0$ or $p^1$, disagrees with $C$ for that argument. If $p$ disagrees, define $f_{4l+3} = a$; otherwise define $f_{4l+3} = a^1$.

Define $g_{4l+3} = \Pi_{p^i} (f_{4l+3})^i, \text{lh}(f_{4l+2}) \leq \text{lh}(f_{4l+3})$.

Stage 41 + 4. $f_{4l+4}$ and $SN_{4l+4}$ are to be defined as in stage 46 + 3, but with $f$ and $g$ interchanged.

Define $f(y) = \text{lt}_{m_l y < \text{lh}(f_j), y - 1},$ and define $g(y) = (g_{\mu m}(y < \text{lh}(f_j)) y = 1.$

Claim i. $f <^e h^{(n)}$ and $g <^e h^{(n)}$.

Proof: It is only necessary to see that the conditions used to define $f$ and $g$ are at most $n$ in $h$. First, consider cases 1 and 2 of stage 4l + 1. If $I + 1 = n^* m + k, 0 < k < n,$ then $[I + 1]$ has fewer than $n$ alternating quantifiers. Thus, by Definition 3 of the forcing relation, $a \vDash_{\Delta \text{adm}} [I + 1]$ is $S_1$ in $h,$ for some $i < n$. Thus, $3a > f, (a \vDash_{\Delta} [I + 1] \vDash [I + 1] \nu a \vDash \text{adm} L_{\text{adm}} \vDash \text{adm} \text{adm} L^*_{\Delta \text{adm}}$ is at most $S_n$ in $h$. If $t + 1 = n^* m + n,$ then by Definition 3
of the forcing relation, a $I_{ad^m}^{1+1}$ is $8_n$ in $h$. Thus

3a > $I_{adm}^{1+1}$ (a $I_{adm}^{[I + 1]}$) is $f_n$ in $h$. Similarly, the hypotheses in stage $4l + 2$ are $f_n$ in $h$. It is easy to observe that the hypotheses of cases 1, 2, and 3 of stages $4l - 3$ and $4l+4$ are $f_n$ in $h$. Therefore, both $f$ and $g$ are recursive in predicates which are $f_n$ in $h$. That is, $f \triangleleft h^{(n)}$ and $g \triangleleft h^{(n)}$.

Claim ii. $f^{(n)} \triangleleft f \lor g$ and $g^{(n)} \triangleleft f \lor g$.

Proof; We prove $f^{(n)} \triangleleft f \lor g$, $g^{(n)} \triangleleft f \lor g$ is proved mutatis mutandis. For each $m$ and $k = 1, 2, \ldots, n$, $f \vert[-[n^m + k]$ or $f \vert[-[n^m + k]$. Therefore, by Lemma 5, $f \vert[-[n^m + k]$ if and only if $[n^m + k]$ $(f)$. In particular, since

3x, $T(x_n), e, e, x_{-1}, \ldots$ is $[n.e + n]$, $2x, T(x_n), e, e, x_{-1}, \ldots$ is $[n.e + n]$ if and only if $f \vert[-[n^m + k]$. We show that for each $e$,

$1 \triangleleft ad^m [n^e + n] \triangleleft f(\mathbf{n} - 1) \triangleleft ad^m [n^e + n]$, in fact, suppose $3a \triangleleft f_4(\mathbf{n} - 1) \triangleleft ad^m [n^e + n]$. Then, $f_4(fn^e + n); \mathbf{f} \triangleleft$ is an admissible extension of such an $a$. Therefore, by Lemma 3, $f_4(\mathbf{n} - 1) \triangleright ad^m fn^e + n]$. Thus, for some $m$, $\mathbf{f}(m) \triangleright ad^m fn^e + n]$, that is, $f \triangleright ad^m [n^e + n]$. Now, suppose there exists an $m$ so that $T(m) \triangleright [n^e + n]$. For such an $m$, $\triangleright ad^m f_4(\mathbf{n} - 1) \triangleright ad^m [n^e + n]$. 
If \( f > \alpha \) \( \text{adm} f^4((n \cdot e, n) - 1) \), then of course

\[ \exists \alpha > \text{adm} f^4((n \cdot e, n) - 1) \ast \text{adm} [n \cdot e + n]. \]

Therefore,

\[ f \models \text{adm} [n \cdot e + n] \models \exists x > \text{adm} f^4((n \cdot e, n) - 1) \ast \text{adm} [n \cdot e + n]. \]

Define a function \( K \) by

\[ K(0) = \lor [f(x) \land g(x)], \]

\[ K(x + 1) = \forall y > K(x) \land f(y) \land g(y). \]

If and only if \( f(K(2e)) = 1 \). Hence,

\[ \exists x_1 \ldots Q x_n \exists \frac{1}{n} \{ f(x_n), e, e, x_1, \ldots, x_{n-1} \} \]

if and only if \( f(K(2e)) = 1 \). Thus \( f^{(n)} \models f \lor g. \)

Claim iii. \( h \models f \) and \( h \models g \), since \( \text{adm}(f) \) and \( \text{adm}(g) \).

Claim iv. \( \forall x [V z f^4 P_n, x V V y c^f] \) and

\[ \begin{array}{c}
\forall x [V z g^4 P_n, x V V y c^f] \\
\end{array} \]

Proof: It will be shown for every \( x, y, \) and \( z \), that either

\[ C \models n, x / < y > \text{ or } f^4 n, x >. \]

For any \( x, y, \) and \( z \), let \( I = < x, y, z >. \) If \( \models 4l + 3 \) is constructed according to case 1 or case 2 of stage \( 4l + 3 \), and if
\[ C_{n,x} = \frac{f}{y} C, \text{ then } 4 \cdot 0 + 2^{-1} \cdot f^{n + x}. \text{ Hence, } f \not\models p^{n + x}. \text{ If } f_{4d + 1} \text{ is constructed according to case 3 of stage } 41 + 3, \text{ then} \]

\[ C = \frac{f}{4d + 2}. \text{ Hence, } C \models \neg f. \]

To complete the proof of Theorem 1, let \( \psi = \phi(f) \) and let \( c = \delta(c) \). By (i), (ii), and (iii), \( a^{(n)} \leq b^{(n)} \leq b \lor c \lor a^{(n)} \) and \( a^{(n)} \leq c^{(n)} \leq b \lor c \lor a^{(n)} \). By (iv), \( b \not\models \) and \( c \models T \).

In Theorem 1 an arbitrary number \( n \) is given, and then remains fixed throughout the entire proof. The idea of the following theorem is to force the set of all arithmetical properties and negations of arithmetical properties of the form \([n \cdot m + k]\), for all \( n, m \) and \( k = 1, 2, \ldots, n \). (Of course, our indexing must be altered since it is ambiguous if \( n \) is not fixed.) Also, the theorem will not be presented in a relativized form, so every sequence number \( a \) is admissible, and we will write \( \\| \), rather than \( \\| \text{adm} \).

**Theorem 2.** \( 3a3b[a^{m} = b^{m} = 0^{m} = a \lor b] \).

**Proof:** Two functions \( f \) and \( g \) will be defined so that:

(i) \( \delta(f) \leq 0^{(a)} \), \( \delta(g) < 0^{(u)} \) and
(ii) \( f^{(u)} \leq f \lor g \) and \( g^{(u)} < f \lor g \).

As before, let \( B(T(X), e, e, x, \ldots, X) \) denote

\[ T_{n}(T(X), e, e, x_{1}, \ldots, x_{n}), \text{ if } n \text{ is odd, and denote} \]

\[ T_{n}(T(X), e, e, x_{1}, \ldots, x_{n-1}), \text{ if } n \text{ is even. Let } Q_{x_{i}} \text{ denote } 3x_{i} \]

if \( i \) is odd, and denote \( Vx_{i} \), if \( i \) is even. For each natural
number \( I = <n, m>, n \geq 1, \) and \( m \geq 1, \) define \( [I] \) to be the arithmetical property

\[
\approx_{n-(k-l)} n \approx_{n-k'} n \approx_{(k-1)} n-1
\]

where \( m = n-q + k, \) \( 1 \leq k \leq n, \) and \( q = <e, x, \ldots, x> \). The set of all arithmetical properties \( [I] \) and \( T[\xi] \), for \( -t = <n, m>, n \geq 1, \) and \( m \geq 1, \) is a closed set of arithmetical properties.

Construction of \( f \) and \( g \):

**Stage 0.** Define \( f_0 = g_0 = 1. \)

**Stage 2b + 1.** By induction hypothesis \( f_x \) and \( g_x \) are defined and have the same length.

**Case 0.** There do not exist \( n \geq 1 \) and \( m \geq 1 \) so that \( I = <n, m>. \)

Define \( f_{2l+1} = f_{2l} \) and \( g_{2l+1} = g_{2l}. \)

**Case 1.** There exist integers \( n, m, q, \) and \( k \) so that \( n \geq 1, m \geq 1, \) \( I = <n, m>, m = n^q + k, \) and \( 0 < k < n. \)

Define

\[
f_{2l+1} = V > f_{2l} \quad [a \models [I] \text{ or } a \not\models \neg[I]]
\]

Define

\[
g_{2l+1} = g_{2l} \cdot \Pi_{1 \leq i < \text{lh}(f_{2l+1})} \left( \Pi_{1 \leq i < \text{lh}(f_{2l+1})} p_i \right).
\]

**Case 2.** There exist integers \( n, m, \) and \( q \) so that \( n \geq 1, m \geq 1, -t = <n, m>, \) and \( m = n^q + n. \)
If $3a > \ell_2$, $a \models [I]$, then let $0 = jua > f^\ell f a \models [*]$. Define

$$f_{2\ell+1} = \beta \cdot p_{1h(\beta)}^2,$$

and define

$$g_{2\ell+1} = g_{2\ell} \cdot \prod_{1h(f_{2\ell}) \leq i < 1h(\beta)} p_i^{(\beta)}.$$ 

Otherwise, let $p = jwu > f$, $(a \models 1[^\ell])$. In this case, define

$$f_{2\ell+1} = p_{1h(\beta)}^1,$$

and define

$$g_{2\ell+1} = g_{2\ell} \cdot \prod_{1h(f_{2\ell}) \leq i < 1h(\beta)} p_i^{(\beta)}.$$
The proof of this statement is identical to the proof presented for the similar statement in claim (ii) of Theorem 1.

Define a function $K$ by

$$K(l) = \exists x \exists n \exists m \left[ x = 2^{n} n m + n + 1 \text{ or } x = 2^{n} n m + n + 2 \right],$$

$$K(y+1) = \exists x \exists n \exists m \left[ x > K(y) \& (x = 2^{n} n m + n + 1 \text{ or } x = 2^{n} n m + n + 2) \right].$$

The $y$th argument $x$ for which $f(x) \neq g(x)$ is introduced at stage $K(y)$ of the construction of $f$ and $g$.

Define $p(n,m) = \exists y \left[ K(y) = 2^{n} n m + n + 1 \right]$. At stage $2^{n} n m + n + 1$, the $p(n,m)$th argument $x$ for which $f(x) \neq g(x)$ is introduced.

Define $h(l) = \exists x \left[ f(x) \neq g(x) \right]$, $h(y+1) = \exists x \left[ x > h(y) \& f(x) \neq g(x) \right]$.

$f$ and $g$ have been constructed so that

$$3a > f_{2}.<n,n-e,n> \neq 1^{n},n^{e} + n>$$

if and only if $f(h(p(n,e))) = 1$. Thus, for each $n$ and each $e$, $3x_{1} \ldots Q x_{n} B_{n}(f(x_{n}),e,e,x_{1},\ldots > x_{n-1})$ if and only if $f(h(p(n,e))); = 1.$
Therefore \( f'(n) \) is uniformly recursive in \( f \lor g \). By definition 
\(<x,y> \in f^{(a)} \Rightarrow x \in f'^{(y)} \). But, \( x \in f'^{(y)} \iff f(h(p(y,x))) = 1 \). Therefore, \( f'^{(w)} \leq f \lor g \).

To complete the proof of Theorem 2, let \( \sim = d(f) \) and let \( \sim = \varphi(g) \cdot \) By (i) and (ii), \( \sim \leq a \lor b \leq a \lor b \leq b(\omega) \).

The following Theorem 3 for the case \( n = 1 \) is a relativized version of Friedberg's characterization of the complete degrees [4].

**Theorem 3.** \( \forall a \forall b \exists c [c^{(n)} = c \lor a^{(n)} = b \lor a^{(n)}] \).

**Proof:** Let \( h \) be a characteristic function with degree \( \sim \).

Let \( g \) be a characteristic function with degree \( \sim \). A function \( f \) will be defined so that:

1. \( f^{(n)} \leq h^{(n)} \lor g; \)
2. \( g \leq f \lor h^{(n)}; \) and
3. \( h \leq f. \)

As in the proof of Theorem 1, define
\[
\text{adm}(a) \iff \forall x [2x < 1h(a) - 4 (a)_{2x} \sim 1 = h(x)].
\]
(iii) will be satisfied if \( f \) is defined so that \( \text{adm}(f) \).

Also let \( n \) be fixed, and let the arithmetical properties
\[
Qx_{n-1} \forall n^{(x)} \exists e,e,x_1 \ldots x_{n-1} \}
\]
be defined and indexed as in the proof of Theorem 1. Then, \( [n^{*m} + k], k = 1,2,\ldots,n, \) is

the arithmetical property \( Qx_{n-1} \forall n^{(x)} \exists e,e,x_1 \ldots x_{n-1} \}

where \( m = <e,x_{1'}, \ldots, x_{n-k}> \).
Construction of $f$:

Stage 0. Define $f_0 = 2^{h(0)} + 1$. $\text{adm}(f_0)$ and $\text{lh}(f_0)$ is odd.

Stage $I + 1$. By induction hypothesis $f^I$ is defined, $\text{adm}(f^I)$, and $\text{lh}(f^I) \equiv \text{odd}$. 

Case 1. $3m, k | x + 1 = n - m + k$ and $0 < k < n$. Define 

$$f_{l+1} = \mu \alpha \left[ \alpha > \text{adm} f^I \cdot p_{l+1} \cdot g(\alpha) \right] \text{ is odd} \quad \text{and} \quad (\alpha \vdash \text{adm} [l + 1])$$

By Lemmas 3 and 4, such an $\alpha$ exists.

Case 2. $3m[l + 1 = n - m + n]$. If

$$\exists \alpha > \text{adm} f^I \cdot p_{l+1} \cdot g(\alpha) \text{ is odd} \quad \text{and} \quad (\alpha \vdash \text{adm} [l + 1])$$

then define

$$f_{l+1} = \mu \alpha \left[ \alpha > \text{adm} f^I \cdot p_{l+1} \cdot g(\alpha) \right] \text{ is odd} \quad \text{and} \quad (\alpha \vdash \text{adm} [l + 1])$$

Otherwise, define

$$f_{l+1} = \mu \alpha \left[ \alpha > \text{adm} f^I \cdot p_{l+1} \cdot g(\alpha) \right] \text{ is odd} \quad \text{and} \quad (\alpha \vdash \text{adm} [l + 1])$$

Note that for all $x$, $x < \text{lh}(f)$. Define $f(x) = (f^I) * 1$.

Define a function $K$ by $K(x) = f^I_x$. As in the proof of claim i of Theorem 1 and 2, it is easy to see that $K \preceq h^{(n)} V g$. Thus, it is proved that $f \preceq h^{(n)} V g$. We need to prove the stronger statement:
Claim i. \( f(n) \leq h(n) \) \( \forall g \).  

Proof: Again, as in the proof of Theorem 1,  

\[
\exists x_1 \ldots \exists x_n \subseteq B^i(I(x), e, e, x_{-1}, \ldots, x_{-k}) \iff f \leq \text{adm} [n - l + n].
\]

Suppose \( f \vdash x_1 \ldots x_n \leq [n - e + n] \). Then of course, for some \( m \),  

\[
f_{n + e + n} \leq \text{adm} [n - e + n].
\]

Conversely, suppose that for some \( m \),  

\[
f_{n + e + n} \leq \text{adm} [n - e + n].
\]

Thus,  

\[
\exists x_1 \ldots \exists x_n \subseteq B^i(T(x), e, e, x_{-1}, \ldots, x_{-k}) \iff f \leq \text{adm} [n - e + n].
\]

The right hand side is recursive in \( h(n) \) and \( g \). Thus,  

\[
f(n) \leq h(n) \forall g.
\]

Claim ii. \( g \leq f \forall h(n) \).  

Proof: \( g(x) = f(lh(K(x))) \), for all \( x \). Using the definition of \( K \), substitute \( f(lh(K(x))) \) for \( g(x) \) in the definition of \( K \), to obtain  

\[
K \leq f \forall h(n).
\]

Then, use \( g(x) = f(lh(K(x))V) \), to obtain  

\[
g \leq f \forall Vh(n).
\]
Claim iii. \( h \leq f \), since \( \text{adm}(f) \).

To complete the proof of Theorem 3, let \( c = \text{adm}(f) \). By (i), (ii), and (iii), 
\[
\sim^{(n)} f \sim^{(n)} \sim^{(n)} V b \sim^{(n)} c \sim^{(n)} \sim^{(n)} f \sim^{(n)} .
\]

Corollary 1. \( \forall a^{(n)} < b \iff 3c[a^{(n)} \rightarrow b \land c \rightarrow a^{(n)} ] \).

The proof is immediate.

Corollary 2. \( \forall a \forall b \forall c \sim^{(n+1)} = c \sim^{(n)} V a^{(n+1)} = b \sim^{(n)} a^{(n+1)} ] \).

Proof:
\[
\sim^{(n+1)} \land c \sim^{(n+1)} \leq: c \sim^{(n)} \sim^{(n+1)} = (c \sim^{(n+1)} \sim^{(n)}) V c \sim^{(n+1)} .
\]

Corresponding to the original Kleene-Post construction [11] of \( 1 \)-incomparable sets in \( \mathcal{F} \), we can now prove the existence of \( 1 \)-incomparable sets in \( A_{n+1} \). In the following theorem we incorporate ideas from Theorem 3 to get a stronger result. Peter Hinman [7] has proved, corresponding to the Friedberg-Muchnik theorem ([5] and [12]), that there exist \( A \),-incomparable sets in \( S \).

Theorem 4. \( \exists A \exists B \exists A_{\sim^{(n)}} \mathcal{E}(A)_{\sim^{(n)}} = 0_{\sim^{(n)}} = \mathcal{E}(B)_{\sim^{(n)}} ] \).

Proof: Two functions \( f \) and \( g \) will be defined so that

(i) \( \mathcal{E}(f^{(n)}) \leq \mathcal{E}(g^{(n)}) \) and \( \mathcal{E}(g^{(n)}) \leq \mathcal{E}(f^{(n)}) \); and

(ii) \( f/2^{(n)} \) and \( g^{(n)} \).

Let \( B^{(n)}_{\sim^{(n)}}(x, e, u, x_i) \) denote \( \sim^{(n)}(x, e, u, x_i) \), if \( n \) is odd, and denote \( \sim^{(n)}(x, e, u, x_i) \), if \( n \) is even. Let \( Qx_i \) denote \( 3x_i \), if \( i \) is odd, and denote \( Vx_i \), if \( i \) is even. Let \( e, x_i, \ldots, x_{n-k} \) be constants, where \( 1 \leq k \leq n \), and let
To the arithmetical property
\[ Qx_n \ldots Qx_n B^l(T(x), e, u, x_{n-1}, \ldots, x_1) \]
with one free number variable \( u \), we associate the index number \( n \cdot m + k \). Define \([n \cdot m + k](u)\) to be the arithmetical property with index number \( n \cdot m + k \). With \( n \) fixed this indexing is unambiguous.

Observe that \([n \cdot e + n](e)\) is the arithmetical property
\[ 3x \ldots Qx_n B^l(T(x), e, e, x_{n-1}, \ldots, x_1) \]
(It may be assumed that \(<x> \succ x\), for all \( x \).)

Construction of \( f \) and \( g \):

**Stage 0.** \( f_0 = g_0 \prec 1 \).

**Stage 64 + 1.** By induction hypothesis \( f_{6^l} \) and \( g_{6^l} \) are defined.

**Case 1.** \( 3x, a, m, k \{ 4 = <x, a> \& x \prec n \cdot m - f k \& 0 < k < n \} \).
In this case define \( f_{6^l+1} = f_{6^l} \cdot [a] \| T \cdot [x](a) \) or \( \cdot [x](a) \),
and define \( g_{6^l+1} = g_{6^l} \).

**Case 2.** \( \forall x, a, m, k \{ 4 \prec <x, a> \& x = n \cdot m + k \} \) (\( k = 0 \) or \( k = n \)).
Define \( f_{6^l+1} = f_{6^l} \) and \( g_{6^l+1} = g_{6^l} \).

**Stage 64 + 2.** \( f_{6^l+2} \) and \( g_{6^l+2} \) are to be defined as in stage
64 + 1, but with \( f \) and \( g \) interchanged.

**Stage 64 + 3.** By induction hypothesis \( f_{6^l+2} \) and \( g_{6^l+2} \) are defined.

**Case 1.** \( 3x [a \succ f_{6^l+2} \& a \| T \cdot [n \cdot 4 + n] (l h(g_{6^l+2}))] \).
Define
\[ f_{6t+3} = \mu \alpha > f_{6t+2} \alpha \vdash [n \cdot t + n](1h(g_{6t+2})) \].

Define
\[ g_{6t+3} = g_{6t+2}^{\#} p_1 h(g_{6t+2})^2 \].

**Case 2.** \( a > f_{6t+2} \rightarrow a \vdash [n \cdot t + n](1h(g_{6t+2})) \). By Lemma 4,

\[ 3a > f_6^{\#} + 2 \alpha \vdash i[n \cdot t + n](1h(g_{6t+2})) \].

Define
\[ f_{6t+3} = \mu \alpha > f_{6t+2} \alpha \vdash [n \cdot t + n](1h(g_{6t+2})) \].

Define
\[ g_{6t+3} = g_{6t+2}^{\#} p_1 h(g_{6t+2}) ^2 \].

**Stage \( Si + 4 \).** \( f \) and \( g \) are to be defined as in stage \( Dv44 \).

**Stage \( 6t + 3 \), but with \( f \) and \( g \) interchanged.**

**Stage \( 6t + 5 \).** By induction hypothesis \( f_{6t+4} \) and \( g_{6t+4} \) are defined.

If \( 3a > fg_6^{\#} + a \vdash [n^*t + n] \), then define
\[ f_{6t+5} = \mu \alpha > f_{6t+4} \alpha \vdash [n \cdot t + n](t), \] and define \( g_{6t+5} = g_{6t+4}^\#. \)

Otherwise, define \( f_{6t+5} = M^l > f_{6t+4}^a H -1 t^{n^*t + n}(t) \) and define \( g_{6t+5} = g_{6t+4}^\# \).

**Stage \( 6t + 6 \).** \( f \), \( g \) are to be defined as in stage \( Dv44 \).

**Stage \( 6t + 5 \), but with \( f \) and \( g \) interchanged.**
Define \( f(x) = (f^m, >_x)^m \) and define 
\[
g(x) = (g_{(\mu n(x<1h(g_n)))}^m)^x.1.
\]

Define \( A = \{x|f(x) = 1\} \), and \( B = \{x|g(x) = 1\} \).

**Claim i.** \( \Sigma(f^{(n)}) \leq \Sigma^{(n)} \) and \( \Sigma(g^{(n)}) \leq \Sigma^{(n)} / \Sigma \)

**Proof:** For each \( e \), the set \( G \) of all arithmetical properties and negations of arithmetical properties \( [n^m + k] (e) \), \( 1 \leq k \leq n \), where \( m = \langle e, x_1, \ldots, x_n \rangle \) is a closed set of arithmetical proper-

ies. Let \( e \) and \( x_1, \ldots, x_n \) be arbitrary constants, and let \( m = \langle e, x_1, \ldots, x_n \rangle \) . At stage \( 6e + 5 \), \( f^{e+n} \) is chosen so that \( f \vdash [n^e + n] (e) \) or \( f \vdash \neg [n^e + n] (e) \). Thus, given \( e \), for each arithmetical property \( A \) in \( G \), \( f \vdash \neg A \) or \( f \vdash A \). By Lemma 5, \( 3x^1 \ldots Qx^n B \vdash (T(x_1), e, e, x_1, \ldots, x_n) \) if and only if \( f \vdash [n^e + n] (e) \). \( f \vdash \neg [n^e + n] (e) \) if and only if \( f \vdash [n^e + n] (e) \). (This is easy to see, and has been argued previously.) Define a function \( K \)

by \( K(x) = f \) for all \( x \). \( d(K^{(n)}) \leq 0 \). (The argument is similar to

the proof of claim i of Theorems 1 and 2). It follows that \( 3x^1 \ldots Qx^n B \vdash (T(x_1), e, e, x_1, \ldots, x_n) \) if and only if \( K(6e + 5) \vdash [n^e + n] (e) \). The right hand side is recursive in \( 0^{(n)} \).

Similarly it may be proved that \( d(g^{(n)}) \leq 0^{(n)} \).

**Claim ii.** \( A \leq i^P \) and \( B \leq L_n^A \).
Proof: We will show that $B^\mathbb{R}$ is similar.

$B^\mathbb{E}$ if and only if there is some $e$ so that for all $a,$

$g(a) = 1$ if and only if $ax^\bot Q x^\bot B_n (f(x_j,e,a,x_{<L},...,x_{<n-1})$. 

For each $e,$ it will be shown that $9(\mathbb{E}_e) = 0$ if and only if $3x_1...Q x_n B^\bot (T(x_n),e,\mathbb{E}_e,x_{<1},...,x_{<n})$ -- from which it follows that $B^\mathbb{E}$. 

For each $e,$ the set $Q$ of all arithmetical properties and negations of arithmetical properties $[n \leq m + k] \mathbb{E}_e$ is a closed set of arithmetical properties. Given numerals $e$ and $x_1,...,x_n$,

$0 < k < n,$ let $m = <e,x_1,...,x_k>$ and let $I = <n-m+k, \mathbb{E}_e>.$$ At stage $6\ell + 1,$ $f_{6\ell+1}$ is chosen so that $f \models [n-m+k] \mathbb{E}_e$ or $f \models \mathbb{E}_e$. At stage $6\ell + 3,$ $f_{6\ell+3}$ is chosen so that $f \models [n*e + n] \mathbb{E}_e$ or $f \models \mathbb{E}_e$. 

Thus, for each arithmetical property $A$ in $Q,$ $f \models A$ or $f \models \neg A$. 

By Lemma 5, $3x_1...Q x_n B_n (T(x_n),e,\mathbb{E}_e,\mathbb{E}_e,x_{<1},...,x_{<n})$ if and only if $f \models [n*e + n] \mathbb{E}_e$. (It may be remarked that the necessity of stages $6\ell + 1$ is that for each $e,$ $\mathbb{E}_e$ is not known in advance.) Again $f \models [n*e + n] \mathbb{E}_e$ if and only if $3a [a > f_{6e+2} & a \models [n*e + n] \mathbb{E}_e]$. On the other hand, by definition of $9_{6e+3}, 3a [a > f_{6e+2} \forall \alpha [n*e + n] \mathbb{E}_e]$. If
and only if \( g(\text{lh}(q_{6e+2})) = 0 \). This completes the proof of claim ii.

The proof of Theorem 4 is now complete: \( A \in \mathcal{T}^n, B \in \mathcal{T}^n \), and by (i), \( d(A)^{(n)} = d(B)^{(n)} \).

**Remark.** Let \( \mathcal{F} \) be the set of all degrees. For each \( n \geq 1 \), let \( \mathcal{F}_n \) be the structure \( \langle \mathbb{N}, \leq, \rangle \). \( \langle \mathbb{N}, \leq, \rangle \) has been shown in this chapter that certain sentences which hold in \( \mathcal{F}_n \) hold in \( \mathcal{F} \) for all \( n \). Is \( \mathcal{F}_n \) elementarily equivalent to \( \mathcal{F}_{n'} \) for \( n, m \geq 1 \)? This question has been answered in the negative by C. G. Jockusch, Jr., in private communication.

Let \( G \) be the set of all degrees of arithmetical sets. The proof given by Jockusch uses the fact that \( G \) can be simultaneously first-order defined in \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) (A corollary to this fact, is Jockusch's result, announced in [8], that the structures \( \langle \mathbb{N}, \leq, \rangle \) and \( \langle G, \leq, \rangle \) are not elementarily equivalent.)

The method of proof leaves open two interesting questions. It is not known whether \( \mathcal{F}_n \) is elementarily equivalent to \( \mathcal{F}_{m'} \) for \( n \) and \( m \) both greater than one; and it is not known whether the structures \( \langle G, \leq, \rangle^{(n)} \) and \( \langle G, \leq, \rangle^{(m)} \) are elementarily equivalent, for \( n, m \geq 1 \).


FOOTNOTES

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