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October, 1970

GEOMETRIC PROGRAMS TREATED WITH SLACK VARIABLES*

by

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and

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Report 70-45

Abstract

Kochenberger and Woolsley have introduced slack variables into the constraints of a geometric program and have added their reciprocals to the objective function. They find this augmented program advantageous for numerical minimization[^] In this paper the augmented program is used to give a relatively simple proof of the "refined duality theory¹¹ of geometric programming. This proof also shows that the optimal solutions for the augmented program converge to the (desired) optimal solutions for the original program.

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1. <u>Introduction</u>.

This paper gives a new and somewhat simpler proof of the "refined duality theory¹¹ of geometric programming. The first proof was given by Duffin and Peterson [1,2], and another proof was given by Duffin [3]. Moreover, Rockafellar [5] has related geometric programming to his "generalized convex programming", Those three proofs are all different, and each gives different insight into the structure of geometric programs.

The present proof does not employ Subsidiary programs^T [1,2], linear programming [3], or convexity [5]. Only basic principles of the calculus are needed. Geometric programming includes linear programming as a special case, so the present paper also furnishes a new proof of the duality theory of linear programming.

Given a geometric program with posynomial functions, the treatment here begins by adding a slack variable to each constraint function. Also, the reciprocal of each slack variable is added to the objective function. Clearly, the program so augmented is also defined in terms of posynomial functions. Moreover, it is obvious that the constraints of this augmented program are tight at the minimum. This property makes the augmented program easier to analyze because equalities replace inequalities oFinally, the properties of the original program are deduced by carrying out certain limit operations.

The concept of the augmented program is due to Kochenberger [6], and it has been employed by Woolsey [7]. They are mainly concerned with numerical calculations in geometric programming, which are known to encounter certain difficulties when slack constraints are



present. Because the augmented program has no slack constraints those difficulties tend to be mollified.

Since the constraint inequalities of the augmented program may be assumed to be equalities it is possible to eliminate the slack variables. This results in an unconstrained program whose objective function is precisely the function introduced in the penalty methods of Carroll, Fiacco, and McCormick [4]_o

2. <u>Basic concepts</u>.

A <u>posynomial</u> g(t) is a function of positive variables $t_n, t_0, \hat{}, t_n$ expressed as a finite sum,

 $g(t) \perp \Sigma u_i(t)$,

where the terms $u_{\mathbf{i}}(t)$ have the form

 $u_i(t) \leq c_i t_1^{a_{i1}} t_2^{a_{i2}} \dots t_m^{a_{im}}$.

The exponents a_{ij} are arbitrary real constants/ but the coefficients c_i are positive constants. The primal geometric program to be considered is defined as follows.

Program A. Seek the minimum value of a posynomial $g_Q(t)$ subject to the constraints

 $f^{c}l > o'^{fc}2 > 0, \ldots, t_{m} > 0,$

and subject to the posynomial constraints

$$g_x(t) \notin i, g_2(t) \notin i, \dots, g_p(t) \leq 1$$
.

It is convenient to list all the terms as

^u1,^u2,...,^un

<u>and then let</u>

$$g_{0} \triangleq u_{1} + \cdots + u_{n_{0}}, \quad m_{0} \triangleq 1$$

$$g_{1} \triangleq u_{m_{1}} + \cdots + u_{n_{0}}, \quad ra_{x} \triangleq n_{0} + 1$$

$$g_{p} \triangleq u_{m}$$

If there is a point t which satisfies the constraints then program A is said to be <u>consistent</u>» The infimum of $g_Q(t)$ subject to the constraints of A is written as $M_A \stackrel{\text{onf}}{=} g_Q(t) \cdot \stackrel{\text{onf}}{=} \frac{M_A}{A}$ If $M_A > 0$ then program A is said to have a <u>finite infimum</u>«

Associated with the preceding minimization program is a maximization program termed the <u>geometric dual program</u>. This dual program B is defined as follows.

Program Bo Seek the maximum value of the product function

$$\begin{array}{c} n \text{ c. 6. } p \text{ A} \mathbf{k} \\ \mathbf{v}(6)^{\wedge} \Pi (-g^{\wedge})^{\perp} \Pi \mathbf{A}_{k} \mathbf{k} \\ i=1 \text{ i } k=0 \end{array}$$

<u>where</u>

$$X_{0} \stackrel{A}{=} 6_{1} + \ldots + 6^{n}, \quad m_{0} \stackrel{A}{=} 1$$

$$\stackrel{A_{\mu}}{\cdot} = \stackrel{A}{\cdot} \stackrel{m_{1}}{=} + \ldots + 6_{0}, \quad \stackrel{m}{=} \stackrel{A}{\cdot} \stackrel{n}{=} + 1, \quad \stackrel{n}{=} \stackrel{A}{\cdot} n.$$

$$\stackrel{P}{} \qquad P$$

$$\frac{h}{P} \qquad p$$
The variables $6_{1} \quad are subject to the linear constraints:$

$$6. 2 \stackrel{\circ}{>} i \stackrel{=}{=} 1^{3} \qquad (normality)$$

$$\stackrel{a}{}_{i=1} 6_{1} \stackrel{a}{=} 1^{3} \qquad (orthogonality)$$
Here i_{1} the constants c_{1} are the posynomial coefficients in program A,

and the constants a are the posynomial exponents in program A*

In evaluating the product function v(6) it is understood that $x^{x} = x^{-x} = 1$ for x = 0. This makes v(6) a continuous function in the otthant 6_{1} . 0 . Program B is said to be <u>consistent</u> if there is a point 6 which satisfies its constraints. The supremum of v(6) subject to the constraints of Program B is written as $M_{n} \xrightarrow{A}$ sup v(6). If $M_{-} < \infty$ then program B is said to have a B = B x

finite supremum.

The main goal of duality theory is to show that $M_A = 1VL$. Toward that end, the following lemma is needed. Lemma 1. Let u. and 6. be real numbers such that u. > 0 and 6^1 . J > 0 for $i = 1, \dots, N$; then $N \xrightarrow{x} N u. 6.$ $(Lu.)^A \wedge II (-ji) \xrightarrow{x} \times (Lu.)^A \wedge (L$

Moreover, this inecruality becomes an equality if, and only if, N 6.1 S u. = Au.., j = 1,2,...,N. J i ^x y

<u>Proofo</u> If all the 6_i are positive, let $4_{\underline{4}} 6^{A}$ and $U_{\underline{1}} \underline{A} u_{\underline{1}} / 6_{\underline{1}}$. Then the $e_{\underline{1}}$ are "weights¹¹, and the classical inequality stating that the weighted arithmetic mean of positive numbers $U_{\underline{1}} 2' \cdots N_{\underline{N}}$ is not less than the corresponding weighted geometric mean can be written as

> N N e. 1 ^{x x} 1 ^x

This is equivalent to the inequality of the lemma. Moreover, the classical inequality is an equality if, and only if,

 $^{U}l = ^{U}2$ " ••• = ^{U}N '

It is easy to see that this condition is equivalent to the condition stated in the lemma. The case where not all 6_i are positive is easily reduced to the case just treated, so the proof of Lemma 1 is complete.

Our first theorem shows that $M_{\mathbf{A}} J \ge M_{\mathbf{B}}$, and it also gives conditions that will ultimately help to prove that $MI_{\mathbf{A}} = M_{\mathbf{B}}$. <u>Theorem 1.</u> If t satisfies the constraints of program A and if 6 satisfies the constraints of program B, then

$$g_{o}(t) 2 v(6)$$

Moreover, this inequality is an equality if, and only if,

 $g_{0}(t) \delta_{i} = u_{i}(t)$, $i = 1, ..., n_{0}$,

and

<u>Proofo</u> By virtue of Lemma 1 we know that

$$(\mathbf{g})^{\lambda k} \wedge \mathbf{n}_{\mathbf{w}_{k}}^{\mathbf{n}_{k}} (\mathbf{\overline{b}})^{\mathbf{x}} \mathbf{x}_{k-1}^{\mathbf{k}_{k}} \mathbf{k} = \mathbf{0}, \mathbf{1}, \dots, \mathbf{p}.$$

Multiplying these p + 1 inequalities together gives the inequality

$$\overset{\mathbf{P}}{\mathbf{I}} \overset{\mathbf{K}}{\ast} > \overset{\mathbf{n}}{\mathbf{I}} \overset{\mathbf{V}}{\mathbf{T}} \overset{\mathbf{G}}{\mathbf{I}} \overset{\mathbf{P}}{\mathbf{K}} \overset{\mathbf{A}}{\mathbf{V}} \overset{\mathbf{v}}{\mathbf{I}} \overset{\mathbf{v}}{\mathbf{V}} \overset{\mathbf{O}}{\mathbf{I}} \overset{\mathbf{D}}{\mathbf{I}} \overset$$

where $D_{j} \uparrow 2^{n} 6^{n} a_{j}^{n}$. Since 6 satisfies the orthogonality conditions $D_{j} = 0^{n}$ it follows that

because $A_Q = 1$ and $g_k \uparrow 1$ for k = 1, ..., p. This proves the inequality of the theorem.

Clearly, $g_Q = v$ if, and only if, each of the p + 1applications of Lemma 1 gives an equality. But the p + 1conditions of the theorem are simply transcriptions of the equality condition of Lemma 1, so the proof of Theorem 1 is complete. 3_{\circ} The augmented program. Given program A, its augmented program A⁺ is defined as follows_o

Program A⁺. <u>Seek the minimum value of the posynomial</u>

$$\mathbf{G}_{o}(\mathbf{t},\mathbf{T}) \stackrel{\mathbf{A}}{=} \mathbf{g}_{\mathbf{t}}(\mathbf{t}) + \stackrel{\mathbf{F}}{\mathbf{L}} \mathbf{b} \mathbf{T}_{\mathbf{t}}^{\mathsf{T}^{1}},$$

 $\mathbf{k} = \mathbf{1}$

subject to the constraints

t. > 0, j = 1,...,m, T_k > 0, k = 1,...,p,

and subject to the posynomial constraints

 $G_k(t) \stackrel{f}{=} 9g_k(t) + bT_k f 1, k = 1, ..., p.$

<u>Tfoe constants</u> b <u>and</u> 8 <u>are positive, and</u> 0 < 1. Clearly, program A⁺ is in the standard form of a geometric program. Moreover, program A⁺ reduces to program A if 0 = 1 and b = 0.

To form the program B⁺ which is dual to program A⁺, it is necessary to add p additional dual variables $A_{\mathbf{k}}$ corresponding to the new terms $bT_{\mathbf{\tilde{k}}}^{\mathbf{1}}$ in the objective posynomial. Also, p more dual variables $A_{\mathbf{k}}^{t}$ are needed to correspond to the new terms $bT_{\mathbf{k}}^{t}$ in the constraint posynomials. The corresponding factors in the dual objective function are of the form $(b/A_{\mathbf{k}})^{-1}(b/A_{\mathbf{k}})^{-1}$. However, we might as well write this as $(b/A_{\mathbf{k}})^{\mathbf{2}_{\mathbf{k}}}\mathbf{k}$ because the orthogonality condition on the new variables is $-A_{\mathbf{k}}^{t} + A_{\mathbf{k}}^{t} = O_{0}$ Thus, the augmented dual program B⁺ can be defined as follows.

Program B⁺. Seek the maximum value of the product function

$$\mathbf{v}(\boldsymbol{\delta}, \boldsymbol{\Delta}) \triangleq {}^{\boldsymbol{\theta}}\mathbf{CT}\mathbf{\mu}(\boldsymbol{\delta}, \boldsymbol{\beta} \otimes \mathbf{k}_{1} \mathbf{k}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \otimes {}^{\boldsymbol{2}}\mathbf{A} = \mathbf{p}$$

<u>where</u>

$$a \underbrace{\underline{A}}_{\mathbf{k}}^{n} E \mathbf{6}_{\mathbf{i}},$$

$$\overset{m_{1}}{\overset{m_{1}}{\overset{m_{k}}{\overset{m_{k}}{=}}}} \mathbf{k}$$

$$\overset{\mathbf{A}_{\mathbf{k}}}{\overset{\mathbf{A}}{=}} \underbrace{\underline{A}_{\mathbf{k}}}_{\mathbf{k}} + S \mathbf{a}_{\mathbf{i}} > k = 1, \dots, p.$$

$$\overset{\mathbf{h}_{\mathbf{k}}}{\overset{\mathbf{h}}{=}} \underbrace{\mathbf{h}_{\mathbf{k}}}_{\mathbf{k}} + S \mathbf{a}_{\mathbf{i}} = 1, \dots, p.$$

The variables 6. and A_v are subject to the linear constraints

Here, the constants a_{ij} , b, c., and 9 are as given in program A^+ . Note that program B^+ reduces to program B if we set 6 = 1 and $A_k = 0$.

A geometric program is said to be <u>degenerate</u> if a term $XI_{\mathbf{h}}$ can be made to vanish without causing other terms to approach plus infinity. Otherwise, a program is said to be <u>canonical</u>. In what follows attention is restricted to canonical programs. The treatment of degenerate programs can then be reduced to that of canonical programs by deleting vanishing terms $u_{\mathbf{h}}$ (see Section VI.5 of[1]).

Without loss of generality it may be assumed that the matrix a_{11} is of rank m (see Section III₀3 of [1]). Then, the equations

$$log(u./c.) = \frac{f}{f} a_{i} log t_{i}, i = 1,...,n$$

i J- J^{IIII} XJ J

show that the variables $t_{\mathbf{j}}$ are uniquely determined by the terms u^. Consequently these equations show that if the terms $u_{\mathbf{i}}$ of

a canonical program are bounded away from plus infinity, say u^ f K, then the variables t_j are confined to a compact set in the interior of the first orthant. Thus, if program A is canonical and consistent there is a point t* such that inf $g_o(t) = g_o(t^*)$.

Theorem 2₀ If program A is canonical and consistent, then program B is consistent and

 $\min_{A^+} G_0(t,T) = \max_{B^+} V(6,A) \quad \underline{for} \quad 0 < b \text{ and } 0 < 0 < 1.$

<u>Proofo</u> Clearly, program A^+ is also a canonical consistent program. It follows that G_0 has a minimum for some values of the variables, say $t = t^*$ and $T = T^* \cdot$ For a fixed t the function G_0 is minimized by choosing the variables T_1, \ldots, T_p to take the slack out of the p constraints; so we can eliminate these slack variables to obtain the relation

$$6_{Q}(t,T) = g_{Q}(t) + \frac{p}{1} \frac{2}{1-\theta g_{k}^{\prime}(t)} k_{r}(t) + \frac{p}{1} \frac{2}{1-\theta g_{k}^{\prime}(t)} k_{r}(t) + \frac{p}{1-\theta g_{k}(t)} k_{r}(t) + \frac{p}{1-\theta g_{k}(t$$

Thus, the function T(t) has a minimum at $t = t^*$, which implies that the point t^* satisfies the necessary optimality conditions

$$t_{j}\frac{\partial \Gamma}{\partial t_{j}} = t_{j}\frac{\partial g_{0}}{\partial t_{j}} + \sum_{l}^{p}\frac{b^{2}\theta}{(l-\theta g_{k})^{2}}t_{j}\frac{\partial g_{k}}{\partial t_{j}} = 0, j = 1, \dots, m.$$

Carrying out the differentiations, we see that at t*

Now, divide these equations by GO and use the tightness property $8g_k(t) + bT\mathbf{k} = 1$ to obtain the equations

$$2_1 \circ a_{ij} = 0, j = 1, \dots, m,$$

where we have let

$$\mathbf{6}_{j_} \mathbf{\underline{f}} \cdot \mathbf{a}_i / \mathbf{G}_0, \qquad \mathbf{i} = \mathbf{1}, \dots, \mathbf{n}_Q,$$

and

6.
$$\underline{\mathbf{A}} \stackrel{\mathrm{Su}_{i}}{-2} \stackrel{\mathrm{i}}{-2} \stackrel{\mathrm{i}}{-2} , \qquad \mathbf{i} = \mathbf{n} \mathbf{u}_{\mathbf{K}}^{3} \cdot \cdot \cdot \mathbf{3}^{\mathbf{n}} \mathbf{y}^{*} \quad \mathbf{k} \stackrel{\mathrm{i}}{-2} \mathbf{1}, \cdots, \mathbf{p} \ll \mathbf{k}$$

Also, let

then we see that

ⁿ ° P P b 1 £ 6. + £ A, = (g + L | -) - = 1. 1 ^x 1 ^k ° 1 ^Tk ^G0

Thus 6. and A, so defined satisfy the positivity, orthogonality, **1 k**

and normality constraints of program B .

Now, Theorem 1 is to be employed to show that $G_Q = V(6, A)$ with 6. and A_v defined as above. From those definitions we have $\frac{1}{k} = \frac{\mathbf{A}_k^K \Delta \mathbf{A}_k + \mathbf{A}_k \mathbf{A}_k$

Using this result, we may redefine 6. and A, by the following I \mathcal{JC} four relations:

$$\delta_{i} = \frac{u_{1}}{G_{0}} \qquad i = 1, \dots, n_{0},$$

$$\bullet_{i} = A_{k}0U_{i} \qquad i - m_{k}, \dots, n_{k}, \quad k = 1, \dots, p,$$

$$\Delta_{k} = \frac{b}{T_{k}G_{0}} \qquad k = 1, \dots, p,$$

$$\Delta_{k}^{'} = \Lambda_{k}bT_{k} \qquad k = 1, \dots, p.$$

min $g_{(t)} = \max 0^{CT} v(6)$ for 0 < 6 < 1.

<u>Proofo</u> The proof of Theorem 2 shows that there exist $6_1 > 0$ which satisfy the orthogonality conditions. Hence, the

satisfy both the orthogonality and the normality conditions. In other words program B is consistent.

For 0 < b, Theorem 2 shows that the augmented program A^+ has a minimum value

$$M(6,b) = V(6,A),$$

where 6 and A denote an optimal solution to program B^+ . Suppose that 9 > 9, and let V denote the corresponding dual function. By Theorem 1, we know that

M(9*,b) 2 V*(6,A),

so

$$\frac{\mathbf{M}(\mathbf{8^{\ast}},\mathbf{b})}{\mathbf{M}(\mathbf{9},\mathbf{b})} \stackrel{s}{\sim} \frac{\mathbf{V^{\ast}(6,A)}}{\mathbf{V}(6,A)} \stackrel{r}{\sim} r_{\mathrm{L}}^{\mathbf{81}} \stackrel{o}{\mathbf{9}_{\mathrm{J}}}^{\mathbf{0}}$$

It is obvious from the form of program A⁺ that M(8,b) decreases as b decreases. Thus, we infer the existence of $\lim M(9,b) \stackrel{f}{=} K$ and $\lim M(9,b) ^K$ as b -* 0⁻. Moreover, the canonicality of program A implies that K > 0 and K* > 0, so the preceding inequality on a shows that a is bounded as b -> 0⁺. Since a A f 61 is bounded and since f 6. + S A_y = 1[^] it $1^{^{m}} n_1$ follows that the 6. and A_y have limits 61 and A^{*} as b -> 0⁺ (through a suitable subsequence).

From the proof of Theorem 2, we know that $Aj/k = {}^{A}i/{}^{G}Q^{*} {}^{so}$ $A_{r} \rightarrow o^{+}$ as b •* 0⁺. Moreover, this identity shows that the factor $A_{r} = {}^{2}A_{r}^{-}$ $A_{r} = {}^{A}i/{}^{G}Q^{*} {}^{so}$ $A_{r} = {}^{A}i/{}^{G}Q^{*}$ $A_{r} = {}^{A}i/{}^{G}Q^{*}$ $A_{r} = {}^{A}i/{}^{G}Q$

$$\log \ ^{2} - \frac{2}{T^{7}} 4^{k} \wedge K_{X}, A_{V}, + A_{k} \log G_{Q}.$$

From this identity it follows that \log $\sim * \circ$ ^{as b} "* \circ

As $b - \bullet \vec{0}$ we now see that

because the function V is continuous and because each $_{-}^{-} 1$. Also, the domain of the t variables is compact, so we can assume that t. ->> t as b ->> 0'. Then

$$c_{o}(t,T) -> g_{o}(t^{!})$$

because the proof of Theorem 2 shows that the extra terms in G_Q are of the form $b/T_{\mathbf{k}} = {}^{G}\mathbf{r}^{\mathbf{k}} k^{\text{whicll}}$ approach zero. Also,

$$8g_{k}(t) = 1 - bT_{k} \wedge 1$$
,

so $6g_k(t^!) \leq i$ lo We now see that $g_Q(t^T) = 8^- v(6^!)^{*}$ and this together with Theorem 1 completes the proof of Theorem 3.

4. The main theorems \circ We now have enough machinery to establish the main theorems of geometric programming.

Theorem 4. If program A is canonical and consistent, then program B is consistent, and

$$\min_{A} g_{u}(t) = \sup_{B} v(6).$$

Proof Theorem 3 asserts that program B is consistent. If t^{f} and 6^{f} denote optimal solutions whose existence is guaranteed by

Theorem. 3. ^T*^T have

 $g_o(t') = 9^a v(6') for 0 < 8 < 1.$

Letting 8 -» 1"_, we infer from compactness and continuity of $g_{\mathbf{k}}$ that t! has a limit point t" which satisfies the constraints of program A. Moreover, the preceding displayed relations imply that $g_n(t") \leq i \sup v(6)$. But Theorem 1 shows that $g \circ t^{ff} J \geq \sup v(6)$, B B

so the proof of Theorem 4 is complete.

Theorem 5, Let program A be canonical, and suppose that program B is consistent and has a finite supremum M^B . Then program A is consistent, and

 $\min_{A} g_n(t) = \sup_{B} v(6).$

<u>Proof</u>/ Since the dual objective function is

n c. ⁶ i p A, v(6) \pounds IK-jr[^]) II A^K, ~ 1⁵ i 0 *

and since

it is clear that v satisfies the identity

v(6) s [v(6/a)]^a

for any a > 0.

The consistency of program B clearly implies the consistency of program B⁺. Let 6• and A_v satisfy the constraints of 1 Rprogram B⁺; then the 6^ ^ ⁶i/^Ao ^{satisf}y ^{the} constraints of program B if program A_Q > 0. Choosing a = $\sim h_Q$ in our identity, we see that δ n c i p -A A p -A,

n c i p -A A p -A, n(-i) = v(6)n A, ^K = [v(6')] \circ n A, ^K 1 ⁵i 0 ^K 0 ^k so the augmented dual objective function

$$\text{vfft}, \mathbb{A}^{A} \stackrel{A}{=} 8^{\mathbf{\sigma}} \begin{array}{c} n \stackrel{c}{\mathbf{c}} \stackrel{b}{\mathbf{i}} p \stackrel{\mathbf{b}}{=} \frac{2}{\mathbf{v}} \mathbf{k} p \stackrel{A}{\mathbf{k}} \mathbf{k} \\ n \stackrel{c}{\mathbf{i}} \stackrel{i}{\mathbf{i}} \stackrel{n}{\mathbf{i}} \stackrel{\mathbf{b}}{\mathbf{k}} \stackrel{i}{\mathbf{i}} \frac{2}{\mathbf{k}} \mathbf{k} \\ \mathbf{1} \stackrel{\mathbf{b}}{\mathbf{i}} \stackrel{i}{\mathbf{i}} \frac{\mathbf{b}}{\mathbf{k}} \stackrel{i}{\mathbf{i}} \frac{\mathbf{b}}{\mathbf{k}} \stackrel{i}{\mathbf{i}} \mathbf{k} \\ \mathbf{1} \stackrel{i}{\mathbf{i}} \mathbf{k} \stackrel{i}{\mathbf{k}} \stackrel{i}{\mathbf{i}} \mathbf{k} \\ \mathbf{1} \stackrel{i}{\mathbf{i}} \mathbf{k} \stackrel{i}{\mathbf{i}} \frac{\mathbf{b}}{\mathbf{i}} \stackrel{i}{\mathbf{i}} \mathbf{k} \\ \mathbf{1} \stackrel{i}{\mathbf{i}} \mathbf{k} \stackrel{i}{\mathbf{i}} \mathbf{k} \\ \mathbf{1} \stackrel{i}{\mathbf{i}} \mathbf{k} \\ \mathbf{i} \stackrel{i}{\mathbf{i}} \mathbf{k} \stackrel{i}{\mathbf{i}} \mathbf{k} \\ \mathbf{i} \stackrel{i}{\mathbf{i}} \mathbf{i} \\ \mathbf{i} \stackrel{i}{\mathbf{i}} \mathbf{k} \\ \mathbf{i} \\ \mathbf{i$$

can be rewritten as

$$V(6, A) = [v(6^{!})^{\bullet}] U_{A_{0}}^{\bullet}] \prod_{1}^{p} (7_{A_{k}}^{\bullet})^{2A_{k}}] [(1 + T_{A_{k}}^{\bullet})^{A_{k}}] [(A_{V} + \Delta_{k})^{A_{k}}]^{\bullet} (\theta^{\lambda})].$$

Since $A_0 \wedge 1$ and $A_{\cdot \mathbf{k}} \wedge 1_3$ the factors in square brackets have bounds independent of $\mathbf{6_i}$ and $A_{\mathbf{k}}$ when 0 < 0 < 1; in particular, $[v(\mathbf{6}^f)] \stackrel{\mathbf{0}}{} \wedge \mathbf{M_R} \stackrel{\mathbf{-}}{} \max\{1 j \mathbf{M_B}\} \cdot Also$ $\mathbf{if} \stackrel{2A}{=} \leq \max\{1, \mathbf{b}^{\circ}\} (\frac{i}{\Delta})^{2A}$ $(1 + \frac{\Delta}{\lambda})^{\lambda} \leq (1 + \frac{\mathbf{f}}{\lambda} \stackrel{\mathbf{A}}{},$ $(\mathbf{x} + \mathbf{A})^{\lambda} \mathbf{e}^{\mathbf{A}} (\mathbf{A} + \mathbf{i})^{\lambda} \mathbf{e}^{\mathbf{A}} (\mathbf{A} + \mathbf{i}) \mathbf{e}^{\mathbf{A}}$

Clearly the functions on the right sides of these inequalities are continuous on the positive real axis and have finite limits at 0 and oo. Thus these functions are uniformly bounded. Likewise $[A_0^{"\lambda}0]$ is seen to be uniformly bounded. Consequently, if 6^ and A_{k} satisfy the constraints of program B_{3}^{+} and if $A_{0} > 0_{5}$ then

where the function K is defined for 0 < b and 0 < 0 < 1. However, V is a continuous function of 6, so the preceding bound remains true for $\langle \mathbf{o} = 0$.

For fixed b > 0, program A^+ is clearly consistent if 9 is chosen very small. If program A^+ is not consistent for all 6 < 1 then there is a $8_1 < 1$ such that the constraints $8g_k + bT$, $\kappa < 1$

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can be satisfied for $6 < 6_{11}$ but not for $6 > e^{-1}$ If $0 \to 0_x$ from below it is obvious that some $T_k \to 0$ and hence min $T(t) \to +\infty$. However, using the proof of Theorem 2 and the bound K, we have

min T(t) = max V(6,A)
$$\pounds$$
 K(b,8).
B⁺

But K(b,0) does not approach +oo as 0 -> 8^{*}, so this is a contradiction.

The preceding contradiction shows that the constraints $0g^{(t)} f \mathbf{l}$ can be satisfied for all 0 < 1. Then, the compactness property of canonical programs and the continuity of the $g_{\mathbf{k}}$ imply that the constraints can also be satisfied for 0 = 1. This shows that program A is consistent. But then Theorem 5 is a consequence of Theorem 4.

Theorem 6. Program A is canonical if, and only if, there is a vector 6^f with strictly positive components that satisfy the constraints of program B.

<u>Proof</u>o If program A is canonical it follows that it is consistent for 0 small. Then the proof of Theorem 2 shows that there are strictly positive 6, which satisfy the constraints of program B⁺. Hence the $6\frac{1}{1} \stackrel{n}{=} 6 - /Tn^{0}$ 6. are strictly positive and satisfy the constraints of program B.

Conversely, suppose that $6^!$ is a positive vector satisfying the constraints of program B_\circ By orthogonality

Thus, no term u, can vanish without causing other terms u. to \mathbf{x} 1 approach plus infinity, so program A is canonical and hence the proof of Theorem 6 is complete.

The preceding theorems constitute the major part of the duality theory of geometric programming. The theory may be completed by employing the concept of a subconsistent program. Program A is said to be <u>subconsistent</u> if program A^8 is consistent for all 6 < 1. Using this terminology the preceding theorems may be restated so as to apply to degenerate programs. The details may be fourid in references [1],[2], and [3].

Convergence of the numerical methodo Although Kochenberger and 5. Woolsey have been obtaining approximate numerical solutions to geometric programs by solving the appropriate augmented programs, they have not shown that the approximations can be made arbitrarily accurate by choosing b sufficiently close to zero. That such convergence is in fact the case can be readily established by examining the proof of Theorem 3. Such an examination should convince the reader that the following theorem is valid. Suppose that program A is canonical and consistent, and Theorem 7. 0 be fixed so that 0 < 8 < L Then the augmented program A⁺ let and its geometric dual program B^+ have optimal solutions t(b)and (6(b), A(b)) respectively when 0 < b. Moreover, when $b - 0^+$ ^through any sequence₃ the corresponding sequences $(t(b)\}$ and $\{6(b)\}$ each have at least one limit point t[!] and 6' respectively. Furthermore, each pair of limit points t^1 and $6^!$ generated in this manner are optimal solutions to programs A⁶ and B⁸

pectively.

g consistency" of program A for 0 < 8 < 1 (that is, there is a feasible solution t to program A such that $6g_k(t) < 1$, k = 1, ..., p). However, it is clear that Kochenberger^Ts method can be applied only to superconsistent programs, because the augmented program A for a consistent program A that is not superconsistent is obviously not consistent when 0 < b and 8 = 1. Moreover, it is obvious that every superconsistent geometric program A^n can be formulated Qas a program A corresponding to a consistent geometric program A by choosing each coefficient c. = c./8 for some 8 < 1 that is

Note that the consistency of program A implies the "super-

sufficiently close to 1. Needless to say, this restriction of the applicability of Kochenberger^Ts method to superconsistent programs is a rather insignificant limitation*.

Finally, we should mention that Kochenberger and Woolsey no longer use the augmented programs A^+ and B^+ for numerical minimization. Experimental investigations [7] indicate that it is numerically better to introduce an additional positive parameter r r -r -1

and add $bT^{+} bT_{k}$ (instead of just bT_{k}) to the objective function g_{n} . In particular, they have obtained sufficiently accurate approximate optimal solutions to a number of programs of practical significance by choosing b = r = .01.

It is clear that there are many other posynomials in T-. $^{\mathbf{k}}$ that produce tight constraints when added to the objective function g_0 ; the only requirement on such posynomials is that they are not themselves minimized by any $T^{*} < 1$. There is little doubt that such methods converge in the sense of Theorem 7, but the proofs are probably more complicated than the proof given here and hence probably do not provide an even simpler proof of the refined duality theory of geometric programming. Of course, each such numerical method corresponds to the use of a different penalty function¹. Moreover, each penalty function is known to produce a numerical method [4] for solving the primal program directly. Perhaps, a hybrid of the purely penalty function approach and Kochenberger^Ts approach would be most effective_o Such a hybrid method could conceivably exploit the fact that the primal constraint is slack when its corresponding dual positivity constraints are tight*

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