Sets of formulas valid in finite structures

Alan L. Selman
Carnegie Mellon University

Follow this and additional works at: http://repository.cmu.edu/math
SETS OF FORMULAS
VALID IN FINITE STRUCTURES

by

Alan L. Selman

Report 70-41

November, 1970
Sets of Formulas Valid in Finite Structures

Report 70-41

by

Alan Lo Seltaan

Errata «Sheefc

p* 2s line 15 Let \( K \subseteq \omega_0 \)

Po6* line 5 replace "5° chapter" by "paper"

Abstract, line ^4 d\{V_{\ldots}\} < d(V_{\ldots}) v d f V J,
SETS OF FORMULAS VALID IN FINITE STRUCTURES

Abstract

A function Ir is defined on the set of all subsets of \( u \) so that for each set \( K \), the value, \( Ir \), is the set of formulas valid in all structures of cardinality in \( K \). An analysis is made of the dependence of \( \sim \) on \( K \). It is easily seen that for all infinite sets \( K \), \( d(K) \neq d(K) \neq d(K)^1 \). On the other hand, we prove that \( d(U) = dIr = d(U) \), and use this to prove that for any two degrees \( a \) and \( b \), \( a \neq 1 \), \( a \leq b \leq a^1 \), and \( b \) r.e. \( a \), there exists a set \( K \) so that \( d(K) = a \) and \( d(V) = b \). Various similar results are also included.
B. A. Trachtenbrot [8] has shown that the set of formulas of first order logic valid in all finite structures is not recursively enumerable, although it is the complement of such a set. Let us define a function $I_K$ on the set of all subsets of $\mathcal{A}$ so that for each set $K$, the value, $I_K$, is the set of formulas valid in all structures of cardinality in $K$. A. Mostowski has asked (in conversation, 1966) what can be said of $I_K$ when $K$ is known. In particular, if the Kleene-Post degrees of the two sets $K$ and $J$ are identical, are the degrees of $I_K$ identical?

Let $\overline{K}$ denote the complement of the set $K$. (The universe of discourse is $<D$ throughout.) It is shown that for all infinite sets $K$, $d(K) \leq d(U)$, $< d(K)$. Nevertheless, in section 3 it is shown that there exist sets $K$ for which $d(V) < 3(V_{\overline{K}})$. This solves the above question in the negative. In section 4 we describe the extent to which $d(I_K)$ is independent from $d(K)$.

The principal result in this direction is Theorem 12. The techniques used to obtain our results involve both the writing of explicit algorithms and the application of standard theorems about the degrees of unsolvability.
It is assumed that we have at our disposal some first order language, £, with equality whose grammar contains an infinite list of k-ary predicate letters \( M, F, \ldots \), for each \( k \geq 1 \). \( (p, 0, \ldots) \) shall denote formulas of this language. \( (p(M, \ldots, M, F, \ldots), n) \) is a formula containing among its predicate letters one or more occurrences of the one-place predicates \( M, \ldots, M, F \).

Let \( 91 \) be an interpretation of the formula \( (p) \). \( 91 \) is a structure with domain \( A \) and k-ary relations \( M^{91} \) corresponding to predicate letters \( M \) occurring in \( (p) \). We write \( ||A|| \), for the cardinality of a set \( A \). By the cardinality of a structure \( 91 \) we mean the cardinality of its domain. A structure \( 91 \) is finite, if its domain is. We write \( f=\varnothing <p \), if \( <p \) is valid in \( 91 \).

Also, we will use the notation "\( A \prec X B \)" for "\( A \) recursive in \( B \)" and "\( A \prec m B \)" for "\( A \) is many-one reducible to \( B \)".

**Definition 1.** Let \( K \subseteq \varnothing \).

(i) \( ^K = C_{\varnothing:V} \),

(ii) \( \varnothing = \{ (P^\varnothing/\varnothing = \varnothing (D \& ||A||eK) \} \)

(iii) \( m_K = \{ \langle P, \prec 91 (\prec co \& ||A|| \prec to) - ||A||eK \rangle \} \).

**Lemma 1.** \( <p \in \prec ^K \prec p \in \prec ^K \prec p \in \prec ^K \prec p \in \prec ^K \). 

\( ^K \) and \( \varnothing \) defined above have conceptual interest, and, by Lemma 1, for each set \( K \), \( d(V \prec) = d(\varnothing \prec) = d(to \prec) \). In fact, we prefer to analyze the function \( to \), since as is easily seen, for all \( K \), \( to \) is r.e. in \( K \).
1. **Trachtenbrot's Theorem.**

This section is concerned with certain generalizations of Theorem 1 of [8].

Throughout this paper we equate computable with recursive.

As an instance, given a formula \( \varphi(M_1, \ldots, M_n, F, \ldots) \), \( n \geq 1 \), define \( p(m_1, \ldots, m_n, j, k) \) to be 0, if \( \varphi \) has a model \( M \) of cardinality \( k \) so that \( |M_{1,1}| = m_1, \) for \( i \leq n, \) and \( |F_{1,1}| = j, \) and 1 otherwise. \( p \) is recursive.

**Definition 2.** Let \( K \) be a non-empty subset of \( \mathbb{U} \). A formula \( \varphi(M_1, \ldots, M_n, F, \ldots) \) is a \( K \)-representation of an \( n \)-place function \( f \) if

\[
\begin{align*}
(i) & \quad \forall m_1, \ldots, m_n \exists j, k [p(m_1, \ldots, m_n, j, k) = 0 \land k \in K], \\
(ii) & \quad \forall k, m_1, \ldots, m_n \exists j, k [k \in K \land p(m_1, \ldots, m_n, j, k) = 0 \\
& \quad \land \exists m_{1 \leq i \leq n} f(m_1, \ldots, m_n) = j].
\end{align*}
\]

The proof of the following theorem is immediate.

**Theorem 1.** If \( \varphi(M_1, \ldots, M_n, F, \ldots) \) is a \( K \)-representation of \( \varphi \), then

\[
f(m_1, \ldots, m_n) = j \iff 3k [k \in K \land p(m_1, \ldots, m_n, j, k) = 0].
\]

**Theorem 2.** If \( f \) has a \( K \)-representation, then \( f \) is recursive in \( K \). If \( f \) has a \( K \)-representation and \( K \) is r.e. in a set \( B \), then \( f \) is recursive in \( B \).
Proof. If \( f \) has a \( K \)-representation, then, using Theorem 1, the graph of \( f \) is r.e. in any set \( B \) which \( K \) is r.e. in. So \( f \) is recursive in \( K \), and if \( K \) is r.e. in \( B \), then \( f \) is recursive in \( B \).

Corollary 1. If \( f \) has an \( \langle D \rangle \)-representation, then \( f \) is recursive.

Corollary 1 is due to Trachtenbrot.

Theorem 3. If \( B \) is an infinite set and \( f \) is recursive, then \( f \) has a \( B \)-representation.

Proof. The proof is essentially a repetition of the proof of Theorem 1 in [8]. It is shown in [8] that for each recursive function \( f \) there is an \( \langle u \rangle \)-representation \( \phi \). To complete the proof, it suffices to observe for each \( \langle u \rangle \)-representation \( \phi \), that if \( 21 \) is a model of \( \phi \) with domain \( A \), and if \( 31 \) is extended to a structure \( 91' \) simply by enlarging the domain \( A \), then \( 21' \) is a model of \( \phi \). Since \( B \) is an infinite set, each \( \langle D \rangle \)-representation \( \phi \) has a model of cardinality in \( B \). Thus, \( cp \) is a \( B \)-representation.

Definition 3. The spectrum of a first-order formula \( \phi \), \( S(\phi) \), is the set of all natural numbers \( n \) for which \( \phi \) has a model of cardinality \( n \).

It is well-known [1] that each \( S(cp) \) is an elementary set.

Let \( \text{rng } f \) denote the range of a function \( f \).
Definition 4. The class of spectral functions of n-arguments,

\[ \text{Spr}_n = \{ f : f \in \wp(\omega^n) \land f \text{ has an } u^n\text{-representation } \wp(M_1^{1}, \ldots, M_n^{1}, F^{1}, \ldots) \text{ so that } \text{rng } f = S((p(M_1^{1}, \ldots, M_n^{1}, F^{1}, \ldots) \land VXF x) \} \].

Lemma 2. (1) The functions \( 2x, 2x^4-1, \) and \( x^2 \) belong to \( \text{Spr}_1 \).

The function \( u + x \) belongs to \( \text{Spr}_2 \).

(2) \( \text{Spr}_n \) is closed under substitution. More generally, if \( g \in \text{Spr}_n \) and \( f_1, \ldots, f_m \in \text{Spr}_n \), \( n, m > 0 \), then the function \( h \) defined by \( h(x_1, \ldots, x_1, x_2, \ldots, x_n, \ldots, x_m) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \) is contained in \( \text{Spr}_n \).

Proof. (1) We again cite [8]. By that paper, the functions listed in (1) all have \( \omega \)-representations. It is easy to see that these representations have the required property.

(2) Let \( f \) and \( g \) belong to \( \text{Spr}_1 \). Define \( h(x) = g(f(x)) \), \( f \) has \( u^n\)-representation \( \wp(M_1^{1}, F^{1}, \ldots) \) and \( g \) has \( u^n\)-representation \( 0(M_1^{1}, F^{1}, \ldots) \), both satisfying Definition 4. By [8], \( h \) has co-representation \( \wp(M_1^{1}, G^{1}, \ldots) \land 0(G^{1}, F^{1}, \ldots) \). Suppose \( y = h(x) \), for some \( x \). \( \wp(M_1^{1}, G^{1}, \ldots) \) has a model \( \mathfrak{S} \) of cardinality \( |\mathfrak{S}| = |\wp(M_1^{1}, G^{1}, \ldots) \land 0(G^{1}, F^{1}, \ldots) | = g(f(x)) = y \). As observed in the proof of Theorem 3, \( \mathfrak{S} \) can be extended to \( \mathfrak{W} \) so that \( \mathfrak{W} \) is a model of \( \wp(M_1^{1}, G^{1}, \ldots) \). Thus, ye \( \wp(M_1^{1}, G^{1}, \ldots) \land 0(G^{1}, F^{1}, \ldots) \land VXF x). \) It is immediate that if ye \( \wp(M_1^{1}, G^{1}, \ldots) \land 0(G^{1}, F^{1}, \ldots) \land VXF x), \) then \( y \in \text{rng } h \). Thus \( h \in \text{Spr}_1 \).
The proof of the second statement in (2) is identical.

2. **Elementary Properties of** $\text{to}$.  

Suppose a Gödel numbering is given for the set of formulas of $\mathcal{L}$ so that each number is used exactly once. Throughout this chapter let $R(x,k)$ be the number theoretic predicate $R(x,k) = \text{formula with Gödel number } x \text{ has a model of cardinality } k$. $R$ is a recursive predicate. Let $\langle \phi \rangle$ denote the Gödel number of $\phi$, and let $\langle \chi \rangle$ denote the formula with Gödel number $x$. For each set $K$, $\text{to}_K = \{ \langle \phi : \exists k (R(<\phi/\langle r \rangle, k) \land k \in K) \}$. But, in what follows we will instead denote $\{x | \exists k (R(x,k) \land k \in K) \}$ by $\text{to}_K$.

**Theorem 4.** (1) For each set $K$, $\text{to}_K \in \text{r.e. } K$. In fact, $\forall K \in \text{r.e. } K$.

(2) If $K$ is finite, then $\text{to}_K$ is recursive.

(3) $K \subseteq S_{n-1} \Rightarrow K \in \text{r.e. } E_n$.

(4) $K \subseteq \text{r.e. } E_{n+1}$.

(5) For each set $K$, $K \subset \text{to}_K$.

**Proof.** The proofs of the first four clauses are immediate.

Let $E$ be a first order formula asserting the existence of $x \in K$ exactly $n$ distinct elements. $x \in K \iff E \iff \text{to}_K$. Thus, $K \subset \text{to}_K$.

If $\langle \phi \rangle$ is a formula with one free variable, let $\exists x \langle \phi \rangle$ be the formula asserting that there are exactly $n$ distinct elements which satisfy $\langle \phi \rangle$. 
Theorem 5. If every function recursive in $K$ has a $K$-representation, then $\text{to}$ is a completion of $K$. Thus, $d(\text{to}) = d(K)^{\ell}$. 

Proof. By Theorem 4, to r.e. $K$. Suppose $P(x)$ r.e. $K$. $P(x) \equiv _K K K K 3k[f(k) = x]$, where $f$ is some function recursive in $K$. By assumption $f$ has a $K$-representation, say $cp(M,F,...)$. Let $g(n)$ be the number theoretic function defined by 

\[ g(n) = \varphi(M,F,...) \land \exists y F(y) \]

Then 

\[ P(n) = 3k[f^n(k) = n] = 3k[R(g(n),k) \land k \in K] \]

That is, $P(x) \not\leq \text{to}$. Thus, $\text{to}$ is a completion of $K$.

Corollary 2. If $f$ has a $K$-representation, then $\text{rng } f \subseteq \text{to}$.

If $K$ is an infinite set, then, by Theorem 3, every recursive function has a $K$-representation. Hence, the following Corollary 3 follows from Corollary 2.

Corollary 3. (1) $d(\text{to}) = 1$. In fact, $\text{to}$ is a complete $E_1$ set. (Theorem 2, [8]).

(2) If $K$ is infinite, then $d(\text{to}) \geq 1$.

Suppose $K \in T$. Then $k = \text{rng } f$, where $f$ is recursive in $\mathbb{N}$. That is $K$ is r.e. in a $H$-set. Thus, by Theorem 2, if $g$ is a function with a $K$-representation, then $g$ is recursive in a $E_1$-set. Thus, if $K \in S_1^n$ and $g$ has a $K$-representation, then $g \in A_7$. Hence, not every function recursive in $K$ has a $K$-representation.
This same conclusion follows from Theorem 5, since \( K \notin \mathfrak{F} \) implies \( \text{ID}_K \in 25_n \).

**Theorem 6.** \( 3B [A = B^j] \rightarrow \text{d} (U) \quad \approx \quad \text{d} (A) \).

**Proof.** \( A \subseteq B \). Thus, \( \text{to} \in S^B_x \) by Theorem 4(1). Hence, \( \text{to} \preceq A \), since \( A = B^j \). On the other hand, by Theorem 4(5), \( A \preceq \text{to} \).

The following corollary follows from Theorem 4 and Corollary 3.

**Corollary 4.** For all infinite sets \( K \), \( \text{d} (K) \& d (\text{to} ) \& d (K) \).

Corollary 3 and the following examples show that Corollary 4 gives the best possible upper and lower bounds to \( \text{d} (\text{to} ) \). By example 3, \( \text{d} (K) \) and \( \text{d} (K) \) are not the only possible values for \( \text{d} (\text{to} ) \).

**Examples.** 1. By Theorem 6 and Friedberg's characterization of the degrees greater than \( 0^\dagger \) [2],

\[
\text{Vd} > 0' \quad 3K \{ d (K) = d \& d (\text{to} ) \& d (K) \}.
\]

2. Also by the result in [2], given \( a > 0' \), choose \( b \) so that \( a = b' = b \& a \& b0^j \). Choose \( K \) so that \( \text{d} (K) = b \).

Then, \( \text{d} (K) \quad 0' = \text{S} (\wedge K) = \text{S} (^K) \).

3. By Theorem 4(3) and Corollary 3, if \( K \in \mathfrak{F} \) and \( K \) is infinite, then \( \text{d} (\text{to} ) = 1' \). By a theorem of Sack's [16, p. 107],

\[
3K \{ K \in \mathfrak{F} \& 0 < d (K) < 1 \& d (K) \} = 2'.
\]

Thus, \( aKTd (K) < d (u^j) < d (K)' \).
3. Relative Recursiveness

In the introduction to this paper we asked whether \( d(to) \) is a function of \( d(K) \). In this section we show that \( A \leq B \) does not imply \( to \leq to \), and, more strongly, show that \( d(to) \) is not a function of \( d(K) \). We then show (see Corollary 8 and Theorem 10) that for each degree \( d \) there exist sets \( A \) and \( B \) so that \( d(A) = d(B) \), \( d(to^A) = d(A) \) and \( d(to^B) = d(B) \lor 1 \).

Define \( p(x,y) = (x+y) + y \). Define \( t(x) = n \), where \( n \) is the largest square less than \( x \). Define \( s(x) = x - t(x) \) and \( u(x) = t(x) - s(x) \), \( x \not\geq 3 \). Then, \( u(p(x,y)) = x \) and \( s(p(x,y)) = y \). It follows from Definition 4 and Lemma 2 that \( p(x,y) \inSpr \).

Thus, \( p(x,y) \) has an \( a^-\)representation \( 1 \ll p(M,N,F,\ldots) \) so that \( \text{rng } p(x,y) = S(p(M,N,F,\ldots) \land VxF x) \).

Let \( CJ(X) \) be the number theoretic function defined by

\[
\phi(n) = \sum_{n=1}^{\infty} 3 \cdot \frac{1}{x} \cdot 4 \cdot x \cdot A \cdot cp(1,1,1,1,1) \cdot A \cdot VxF x^i.
\]

We have now the following lemma,

Lemma 3. \( R(a(n),k) = 3y[k = (n+y) + y] \).

Theorem 7. \( VA3B[B \leq A \land to = A^1] \).

Proof. Let a set \( A \) be given. Choose \( 3yS(x,y) \) to be a complete \( A^-\)generable predicate. By Lemma 3 and the definitions preceding Lemma 3,

\[
3yS^A(x,y) = 3y[R(a(x),y) \land S^A(u(y),s(y))].
\]
Define \( B = \{ y : S^A(u(y), s(y)) \} \). \( B \uparrow A \). \( 3y S^A(x, y) \equiv 3y[R(a(x), y) \land y \in B] \). Thus, \( 3y S^A(x, y) \leq_{to} \). \( to \) is r.e. \( A \) follows from Theorem 4(1), since \( B \notin A \). Thus, \( to \) is complete for \( A \).

**Corollary 5.** \( 3A, B[B \notin A \& to_A < \uparrow] \).

**Proof.** Choose \( A \) so that \( d(A) = d(to) \). Then apply Theorem 7.

**Corollary 6.** \( 3K[d(to) = d(K)'] \& d(to_\bot) = d(K) \).

**Proof.** Choose \( 3y Vz P(x, y, z) \) to be a complete \( S \) predicate.

\[
3y Vz P(x, y, z) = 3y[R(a(x), y) \& Vz P(u(y), s(y), z)] .
\]

Let \( K = \{ y : Vz P(u(y), s(y), z) \} \). Then, \( 3y Vz P(x, y, z) \leq_{to} \). But \( to \notin_{e f} \).

Thus, \( d(Lb) = 0 \). By Theorem 4(3) and Corollary 3(2), since \( K \) is the complement of an r.e. set, \( d(to_{\bot}) = 0 \).

Thus, \( to \) does not induce a function on degrees and \( to \) does not preserve relative recursiveness.

**Corollary 7.** (1) \( 3K[d(\neg K) = d(K) \& d(\neg K) = d(K)] \).

(2) \( 3K[d(U_{\bot}) = d(K) \& d(U_{\bot}) = d(K)] \).

**Proof.** Corollary 6 and Lemma 1.

Thus, the functions \( ft \setminus \) and \( \setminus s \) also do not induce functions on the degrees, and therefore do not preserve relative recursiveness.

**Definition 5.** Let \( \phi \) be a formula in prenex normal form and \( M \) a one place predicate letter, not occurring in \( \phi \). Define \( t^M \phi \), \( \phi \) relativized to \( M \), as follows:
(i) If \( \varphi \) is quantifier free and contains occurrences of the variables \( x_0, \ldots, x_n \) and no others, then \( \varphi_1 \) is:

\[
\varphi_1 = 1 \quad M \quad A \quad M \quad (x_0 A \ldots A M \quad (x^1 A) \).
\]

(ii) If \( \varphi \) is \( 3y0 \), \( \varphi_1 \) is \( 3y [M^1(y) A 0] \).

(iii) If \( \varphi \) is \( \forall y^1 \), \( \varphi_1 \) is \( \forall y [M^1(y) A 0] \).

An easy argument proves the following lemma.

**Lemma 4.** For every formula \( \varphi \), \( \varphi \) has a model of finite cardinality \( y \) if and only if \( \varphi_1 \) has a model \( M \) so that \( |M| = y \).

**Lemma 5.** For every function \( f \) belonging to \( \text{Spr}^1 \) there is a recursive function \( g \) so that

\[
VxVpVz3y[R(g(x),z) - (z = f(y) \& R(x,y))].
\]

**Proof.** Assume \( f \in \text{Spr}^1 \). By Definition 4, \( f \) has an \( \varphi \)-representation \( ^\varphi (M^1, \ldots) \) so that \( [^\varphi (M^1, \ldots) \& Vx{F}^1(x)] \) has a model of cardinality \( z \) if and only if \( 3y[z = f(y)] \).

Let \( g(x) = 0 \quad A \quad \varphi_1 \quad Vx{F}^1 x \), where \( 0 = \exists x y \) and suppose \( R(x,y) \). \( 0 \) has a model of cardinality \( y \). Thus, by Lemma 4, \( 0 \quad A \quad \varphi_1 \quad Vx{F}^1 x \) has a model \( M \) so that \( |M| = y \). Since \( f(y) > y \), \( 21 \) can be extended and expanded to a model of \( ^\varphi (M^1, F^1, \ldots) \) \( A \quad Vx{F}^1 x \) of cardinality \( f(y) \). Thus \( R(g(x), f(y)) \).

Suppose \( R(g(x), z) \). Then, \( 0 \quad A \quad \varphi_1 (M^1, F^1, \ldots) \) \( A \quad Vx{F}^1 x \) has a model \( 21 \) of cardinality \( z \). The restriction to \( M \) is a model of cardinality \( y \) of \( 0 \), where \( f(y) = z \). Thus \( R(x,y) \).
**Definition 6.** $A <^* B \iff A <^* B$ by a function $f \in \text{Spr}_n.$

$<$ is a reducibility. That is, $<$ is a reflexive and transitive subrelation of $\leq.$ In fact, if $A \leq B$ by $f \in \text{Spr}_n,$ and $B \leq C$ by $g \in \text{Spr}_n,$ then $x \in A \iff g(f(x)) \in C$. Therefore, by Lemma 2(2), $A \leq C$. Hence $<$ is transitive. Since the identity function belongs to $\text{Spr}_n,$ $\leq$ is reflexive.

**Theorem 8.** If $A <^* B,$ then $A <^* \text{ID}_B.$

**Proof.** Suppose $A <^* B$ by $f \in \text{Spr}_n.$ By Lemma 5, there exists a recursive function $g$ so that

$$R(x, y) \iff R(g(x), f(y))$$

and

$$\forall x, z \exists y[R(g(x), z) \iff (z = f(y) \land R(x, y)].$$

$$x \in A \iff \exists y[R(x, y) \land y \in A]$$

- $\exists y[R(g(x), f(y)) \land y \in A]$

- $\exists y[R(g(x), f(y)) \land f(y) \in B]$

- $\exists y[R(g(x), y) \land y \in B]$

- $g(x) \in A \

- g(x) \in \forall \text{ID}_B \

- \exists z[R(g(x), z) \land z \in B]$

- $\exists y[R(g(x), f(y)) \land f(y) \in B]$

- $\exists y[R(x, y) \land y \in A] \

- x \in \text{ID}_B.$
Thus to < to by g, concluding the proof of Theorem 8.

Define the recursive sup. of the two sets A and B by

\[ 2x \in A \lor B \iff x \in A \]

\[ 2x + 1 \in A \lor B \iff x \in B. \]

It is clear that \( A \subseteq A \lor B \), \( B \subseteq A \lor B \), and that \( d(A \lor B) \) is the least upper bound of \( d(A) \) and \( d(B) \).

**Lemma 6.** For any two sets A and B, \( A \subseteq A \lor B \) and \( B \subseteq A \lor B \).

**Proof.** By Lemma 2, the functions \( 2x \) and \( 2x+1 \) belong to \( \text{Spr}^\vee \).

The proof follows then from Theorem 8.

**Theorem 9.** \( \forall A \exists C [d(C) = d(A) \land \text{to is complete A-generable}] \).

**Proof.** By Theorem 1, \( \exists B [B \subseteq A \land \text{to is complete for A}] \).

Let \( C = A \lor B \). \( B \subseteq A \), thus \( d(C) = d(A) \). \( \text{to is r.e. in C} \)

\( \text{and C \subseteq A}, \), thus \( \text{to is r.e. in A} \). By Lemma 7, \( \text{to is complete for A. Thus, to is complete for A.} \)

**Corollary 8.** \( \forall d \exists A [d(A) = d(A) \land d(A') = d(A')] \).

The following theorem (obtained by Thomas Grilliot, in personal communication) gives a positive solution to a question raised in [6].

**Theorem 10.** \( \forall d \exists A [d(A) = d(A) \land d(A \setminus \{n\}) = d(A \setminus 1)] \).

**Proof.** By Corollary 3, we already have this result for the case \( d = \emptyset \). Therefore, assume that \( d > \emptyset \), and choose \( K \) so that \( d(K) = \emptyset \). Let \( \chi_K(n) \) denote the characteristic function of \( K \), and let \( \chi_K(n) \) (see [3, p. 231]) be the course-of-values
function for \( \text{Ch}_K(n) \). Then, define \( A \) to be the complement of \( \{ \text{Ch}_K(n) : \text{neu} \} \). \( K \notin A \), and \( A \subseteq \overline{K} \). Also, it is easy to see that \( 7? \) is recursive in every infinite subset of \( 75 \).

By Corollary 4, it suffices to show that \( \mathcal{L}(1^\mathcal{U}) \notin \mathcal{L}(A) \lor 1 \).

Let \( p \) be any formula of \( \mathcal{L} \). Since \( d(A) > 0 \), \( A \) is not recursive in \( S(p) \). Therefore \( S(p) \) cannot be an infinite subset of \( 75 \). Hence, either \( S(p) \) is finite, or \( \neg p \in \text{eto} \). That is, either \( 3yVz > y R(\neg p, y) \) or \( 3yfR(\neg \text{eto} \land \text{y} \land A) \). The function

\[
f(x) = \text{fly}[[R(x, y) \land y \in A] \lor Vz > y R(x, z)] \]

is recursive in \( A \) and \( 0^\mathcal{T} \), and

\[
V^c A \cup y \exists y f(C^\mathcal{U}) [RCo^\mathcal{U} \land ycA].
\]

Hence, \( d(to) \sim d(A) \lor 1 \).

4. Values of \( d(to) \) for \( K \) of a given degree

Are \( d(K) \lor 0^\mathcal{T} \) or \( d(K) \lor \) the only possible values for \( d(to) \) for any \( K \)? In this section we describe the extent to which \( d(to)^K \)
is independent from \( d(K) \), within the bounds given by Theorem 4 and Corollary 4.

Lemma 7. There is a recursive function \( f \) so that \( R(x, 2y) \leftrightarrow R(f(x), y) \).

Proof. Let \( x^0, x^1, \ldots \), be a complete list of the individual variables in \( \mathcal{L} \). Let \( S \) be a binary predicate letter and let \( a \) and \( b \) be individual constant letters. Given a formula \( p \) in \( \mathcal{L} \), let \( x^k \)
be the highest index variable which occurs in \( \varphi \). Let \( u^\dagger \) denote the variable \( x_{k+1+i} \). Then, none of the variables \( u_0, u^\dagger, \ldots \) occurs in \( \varphi \). Also, we may suppose without loss of generality that \( \varphi \) contains no occurrences of \( S, a \) and \( b \). (Otherwise, \( \varphi^\dagger \) can be found uniformly, where \( \varphi_1 \) contains no occurrences of \( S, a \) and \( b \), and \( R(\varphi, 2y) \prec R(\varphi_1, 2y) \).) We define a new formula \( \varphi \) as follows:

1. \((x^\dagger x^\dagger)^* \) is \( x_j = x_j \) and \( u_j = u_j \);
2. \( P_n(x_1, \ldots, x^\dagger)^* \) is \( P_n(x_1, u_1, \ldots, x_n, u_n) \);
3. \( (ip_1 \ A lb_1) \) is \( ib_1 \ A 0_2 \);
4. \((-0)^* \) is \( -r(\varphi) \);
5. \((3x_10)^f \) is \( 3x_1u_1[S(x_1, u_1) \ A 0^*] \).

Define

\[
T(\varphi) = \varphi^* A a \ ^* b \ A \ [VxS(x,a) \ A VxS(x,b) \ A VxVy(S(x,y) \ ^* y=a \ V y=b) \ A s(x_1, u_1) \ A \ldots A S(x_n, u_n)],
\]

where \( x_1, \ldots, x_n \) is a list of the free variables occurring in \( \varphi \).

Claim. \( \varphi \) is satisfiable in a structure of cardinality \( 2y \) if and only if \( T(\varphi) \) is satisfiable in a structure of cardinality \( y \).

Proof. We first show that if \( \varphi \) is satisfiable in a structure of cardinality \( 2y \), then \( T(\varphi) \) is satisfiable in a structure of cardinality \( y \).
If a formula $\varphi$ holds in a structure of cardinality $2y$, then let

$$A = \{<1,1>,<2,1>,...,<y,1>,<1,2>,<2,2>,...,xy,2>\}$$

be the domain of such a structure, $\mathcal{A}$. Define a structure $\mathcal{B}$ with domain $B = \{1,2,...,y\}$ as follows:

1. If $R^A$ is a $k$-ary relation on $A$, then $R^B$ is a $2k$-ary relation on $B$ defined by

$$R^B(i_1,j_1,...,i_k,j_k) \iff R_A(<i_1,j_1>,...,<i_k,j_k>),$$

for $i_1,...,i_k \in \{1,2,...,y\}$, and $j^A = -j^A \in \{1,2\}$

2. $S_m = \{<i,j>: i = 1,...,y \& (j=1 \text{ or } j=2)\}$.

3. $a$ is 1, $b$ is 2.

It is clear that $[\forall x S(x,a) \land \forall x S(x,b) \land \forall x \forall y (S(x,y) \land y = a \lor y = b)]$ holds in $\mathcal{B}$.

We prove by induction that $\varphi$ is satisfiable in $\mathcal{B}$ if and only if $T(\varphi)$ is satisfiable in $\mathcal{A}$ (by an assignment $a$) if and only if $T(\varphi)$ is satisfiable in $\mathcal{B}$ (by an assignment $p$). Moreover, $a(x_i) = p(x_{\overline{i}})$.

**Case $\varphi$ is $x_i = x_{\overline{j}}$.** If $a$ satisfies $\varphi$ in $\mathcal{B}$, then for some $<s,t> \in h$, $a(x_i) = a(x_{\overline{j}}) = <s,t>$. Define $p$ by $p(x_{\overline{i}}) = p(x_{\overline{j}}) = s$ and $p(u_{\overline{j}}) = p(u_{\overline{j}}) = t$. Then $p$ satisfies $\varphi$ in $a$. Since $t = 1$ or $t = 2$, $p$ satisfies $T(\varphi)$ in $\mathcal{A}$. 
\( \varphi^* \) is \( x = x \). A \( u = u \). If \( \varphi \) satisfies \( T(\varphi) \) in \( \mathbb{B} \), then \( P(x_i) = P(x_j) = s \) and \( p(u_x) = p(u_j) = t \), for \( s, t \in \mathbb{B} \).

Also, \( s \land t \). Thus, \( <s, t> \in A \). Define \( a \) by \( a(x_i) = a(x_j) = <s, t> \), a satisfies \( \varphi \) in \( S_8 \).

**Case** \( \varphi \equiv P(x_n, \ldots, x_1) \). If there is an assignment \( a \) so that \( P(a(x_1), \ldots, a(x_n)) \), define \( p(x_i), p(u_i), i = 1, \ldots, n \), so that \( a(x_i) = <p(x_i), p(u_i)> \). Then, by definition of \( P \), \( V \in 1 \)

\( P_{\mathbb{B}}(x_1), P(u_1), \ldots, P(x_n), P(u_n) \). That is, \( p \) satisfies \( \varphi^* \) in \( S_8 \). Therefore, \( p \) satisfies \( T(\varphi) \) in \( S_3 \).

Suppose \( p \) satisfies \( T(\varphi) \) in \( S_3 \). \( \varphi \) is \( P(x_{11}, \ldots, x_{3n}, u_{11}, \ldots, u_{3n}) \).

\( P_{\mathbb{B}}(p(x_1), P(u_1), \ldots, P(x_n), P(u_n)) \), and \( S^*(p(x^*, p(u^*)), i = 1, \ldots, n \).

Thus \( <p(x_*), p(u_*)> \in A \), \( i = 1, \ldots, n \). Define \( a(x_i) = <p(x_i), p(u_i)> \).

\( a \) satisfies \( \varphi \) in \( S_8 \).

**Case** \( \varphi \equiv \Pi \). If \( \varphi \) satisfies both \( \Pi \) and \( Q \) in \( \mathbb{A} \), then by induction hypothesis \( p \) satisfies \( \Pi \) in \( S_3 \) and \( p \) satisfies \( \Pi \) in \( S_5 \), where \( p \) is defined so that \( a(x_1) = <p(x_1), p(u_1)>, i = 1, \ldots, n \). Thus \( p \) satisfies \( \Pi \) in \( S_8 \).

The other direction is identical. The case \( \varphi \equiv 0 \) is straightforward.

**Case** \( \varphi \equiv 3x.1I \). If \( f : 0 \) is satisfied in \( S_8 \), then some assignment \( a \) satisfies \( 0(x_3) \) in \( S_8 \). Thus \( p \) defined by

\( a(x_i) = <p(x_i), p(u_i)>, i = 1, \ldots, n \), for all \( j \), satisfies \( IB (x_4, u_2) \) in \( a \).
Also, \( S_{\chi}^{'0}(x.), p(u.) \). Hence \( p \) satisfies \( \exists x, u \left[ S(x, u) \land A \right] \) in \( \varphi \). Thus, \( g \) satisfies \( T(\varphi) \) in \( S \).

The other direction is similar.

We have shown that if \( tp \) is satisfiable in a structure of cardinality \( 2y \), then \( T(co) \) is satisfiable in a structure of cardinality \( y \). We show now that if \( T(\varphi) \) is satisfiable in a structure of cardinality \( y \), then \( \varphi \) is satisfiable in a structure of cardinality \( 2y \).

If \( T(\varphi) \) is satisfiable in a structure \( S \) of cardinality \( y \), we may assume that \( C = \{1, 2, \ldots, y\} \) is the domain, \( a \) is 1, \( b \) is 2, and \( S_{\xi}(i, j) = j = 1 \) or \( j = 2 \).

Define a structure \( S \) from \( \mathcal{F} \) as follows:

1. \( B = \{1, \ldots, y\} \), the domain of \( S \);
2. \( S^\uparrow = S_\varsigma \);
3. \( a \) is 1, \( b \) is 2;
4. \( P_{\xi}(i_1, j_1 \ldots, i_n, j_n) = P_\varsigma(i_1, j_1 \ldots, i_n, j_n) \land S^\uparrow(i_1^\downarrow j_1) \land \ldots \land S^\uparrow(i_n^\downarrow j_n) \).

(Note that only 2k-ary relations appear in \( T(\varphi) \).)

We show by induction that every assignment \( y \) which satisfies \( T(\varphi) \) in \( \mathcal{F} \) also satisfies \( T(\varphi) \) in \( S \), and every assignment \( p \) which satisfies \( T(p) \) in \( S_{\varsigma} \) also satisfies \( T(\varphi) \) in \( S \).

Our result follows easily from this, because \( S \) is obtainable from a structure \( 31 \) of cardinality \( 2y \) as in the previous part of
the proof, and we know that $T(<p)$ is satisfiable in $33$ iff and only if $<p$ is satisfiable in $91$.

If $<p$ is $x_i = x_j$, there is nothing to show, since $33$ and $E$ have the same domain.

Case ($p$ is $P(x_n, ..., x_m)$. Assume $T(<p)$ is satisfiable in $E$.

Then there is a $2n$-ary relation $P_r$ on $\{1, ..., n\}$ and an assignment $y$ to $E$ so that $p_r(y^1, y^2, ..., y^{m, n}, y^{n, n})$ and so that $S_r(y(x_i), y(u_i)), i = 1, ..., n$. Thus

$$S_r(y(x_i), y(u_i), ..., y(x_n), y(u_n)).$$

That is, $y$ satisfies $T(<p)$ in $33$.

It is obvious that an assignment satisfying $T(<p)$ in $33$ also satisfies $T(<p)$ in $E$. This direction is clear in the following cases too.

Case $<p_i < 1$ or $2$. Suppose an assignment $Y$ satisfies $T(<p)$ in $S$. $S_r(y(x_i), y(u_i)), i = 1, ..., n$. Thus $Y$ satisfies $ib$ and $0_2$ in $33$, and $Y$ satisfies $T(<p)$ in $33$.

Case $<p_j < 0$. Suppose $T(\neg p)$ is satisfied by an assignment $Y$ in $E$. $S_r(y(x_i), y(u_i)), i = 1, ..., n$. Thus, as above, $Y$ is an assignment to $S$. $Y$ satisfies $**(0)$ in $E$. Thus, $Y$ does not satisfy $0$ in $E$. By induction hypothesis, $Y$ does not satisfy $0$ in $33$. Thus $y$ satisfies $\neg(0) = ^n0$ in $33$. That is, $T(<p)$ is satisfied by $y$ in $E^*$. 

Case $\varphi$ is $3x.0$. If $T(\varphi)$ is satisfied in $E$, then $0$ is satisfied by some $y$ in $\preceq$. $S(x_i, u_i)$. Thus $y$ is an assignment to $S_3$ and $^\wedge (x_i, u_i)$ is satisfied by $y$ in $S_3$. Thus $T(\varphi)$ is satisfied by $y$ in $S_3$.

The proof of the claim is complete. Let $d(\varphi)$ denote the universal closure of $\varphi$. $\varphi$ is valid in a structure $\mathfrak{A}$ of cardinality $2^\gamma$ if and only if $Cl(\varphi)$ is satisfiable in $\mathfrak{A}$ if and only if $T(d(\varphi))$ is satisfiable in a structure $S_3$ of cardinality $\gamma$ if and only if $T(d(\varphi))$ is valid in $S_3$ (since $T(Cl(\varphi))$ is closed).

Define $f(x) = T(Ot(\varphi)r, \text{for } x = \overset{\sim}{\varphi}\p$ . Then, $R(x, 2^\gamma) = R(f(x), \gamma)$.

**Lemma 8.** There is a recursive function $g$ so that $R(x, 2^\gamma + 1) \Rightarrow R(g(x), \gamma)$.

**Proof.** As in the proof of Lemma 7, given $\varphi$ in $SL$, let $x^\p$ be the highest index variable which occurs in $\varphi$ and let $u_i$ denote the variable $X_{k+1+i}^\p$ all $i$. Again as in the proof of Lemma 7, we can suppose without loss of generality that $\varphi$ contains no occurrences of the binary predicate letter $S$ and $\varphi$ contains no occurrences of the individual constant letters $a$, $b$, and $c$. The formula $\varphi$ is defined for $\varphi$ as in the previous proof. Define

$$T(\varphi) = \varphi \wedge a \neq b \wedge a \neq c \wedge b \neq c$$

$$A \ [VxS(x, a) \ A VxS(x, b) \ A S(c, c) \ A VxVy(S(x, y) \Rightarrow (y = a \ Vy = b V (x = c \ A y = c) \ ) A S(x, u_i) \ A, \ A S(x, u_i) \ ],$$
where \( x, \ldots, x \) is a list of the distinct free variables occurring in \( \varphi \).

**Claim.** \( \varphi \) is satisfiable in a structure of cardinality \( 2y+1 \) if and only if \( T(\varphi) \) is satisfiable in a structure of cardinality \( y \).

If \( \varphi \) holds in a structure of cardinality \( 2y+1 \), then let

\[ A = \{<\text{i},1>, \ldots, <\text{y},1>, <\text{i},2>, \ldots, <\text{y},2>, <3,3> \} \]

be the domain of such a structure, \( \mathcal{A} \). Define a structure \( \mathcal{B} \) with domain \( B = \{1, \ldots, y\} \) as follows:

1. If \( R_A \) is a \( k \)-ary relation on \( A \), then \( R_B \) is a \( 2k \)-ary relation on \( B \) defined by

   \[ R_B(i_1, j_1, \ldots, i_k, j_k) \iff R_A(<i_1, j_1>, \ldots, <i_k, j_k>), \]

   for \( i > \to \{i \to j \to y\} \) and \( 3 \to k \to 1 \to 2 \to 3 \).

2. \( S_m \) is a binary relation defined by \( S_m(i,1) \) and \( S_m(i,2) \) for all \( i = 1, 2, \ldots, y \) and \( 8^{(3,3)} \).

3. \( a \) is 1, \( b \) is 2, and \( c \) is 3.

An induction argument shows that \( \varphi \) is satisfied in \( \mathcal{A} \) (by an assignment \( a \)) if and only if \( T(\varphi) \) is satisfied in \( \mathcal{B} \) (by an assignment \( p \)). Moreover, for all \( i \), \( a(x_i) = \langle p(x_i), p(u_i) \rangle \). It follows that if \( \omega \) is satisfiable in a structure of cardinality \( 2y+1 \), then \( T(\varphi) \) is satisfiable in a structure of cardinality \( y \).

Conversely, if \( T(\varphi) \) is satisfiable in a structure \( \mathcal{B} \) of cardinality \( y \), we may assume that \( C = \{1, \ldots, y\} \) is the domain, \( a \) is 1, \( b \) is 2, \( c \) is 3, and \( S_C(i,j) \) (\( j = 1 \) or \( j = 2 \) or \( i = 3 \) and \( j = 3 \)). Define a structure \( \mathcal{F} \) from \( \mathcal{E} \) as follows:
(1) \( B = \{1, \ldots, y\} \);

(2)

(3) \( a \) is 1, \( b \) is 2, \( c \) is 3;

(4) \[ \forall \forall \wedge i \rightarrow \forall \forall \]

\[ \text{As in the proof of Lemma 7, an induction argument shows that if } T(\varphi) \text{ is satisfied in } E \text{ by an assignment } y, \text{ then } T(<\varphi) \text{ is satisfied in } SS \text{ by } Y \text{ and conversely.} \]

\( S \) is obtainable from a structure \( \mathcal{S}_1 \) of cardinality \( 2y+1 \) as in the previous part of the proof; and we know that \( T(\varphi) \) is satisfiable in \( \mathcal{S}_1 \) if and only if \( \varphi \) is satisfiable in \( M \). Thus, if \( T(\varphi) \) is satisfiable in a structure of cardinality \( y \), then \( T(\varphi) \) is satisfiable in a structure of cardinality \( 2y+1 \). This completes the proof of the claim.

Define \( g(x) = \forall \left( T(C^\varphi) \right)^n, \text{ for } x = \forall (p^n). \text{ Then} \]

\[ R(x, 2y+1) \preceq R(g(x), y). \]

Theorem 11. \( d(to) \sim d(te) \lor d(to). \)

Proof. By Lemma 6, \( d(to) \lor d(\mathcal{B}) \lor d(to). \)

\[ \sim A \sim B \sim AVB \]

\[ 3y[R(x,y) \land y \in AVB] \leftrightarrow 3y[R(x,2y) \land 2y \in AVB] \]

\[ \lor 3y[R(x,2y+1) \land 2y+1 \in AVB] \]

\[ \leftrightarrow 3y[R(x,2y) \land yeA] \lor 3y[R(x,2y+1) \land yeB]. \]
By Lemmas 7 and 8, let \( f \) satisfy \( R(x, 2y) \Leftrightarrow R(f(x), y) \) and let \( g \) satisfy \( R(x, 2y + 1) \Leftrightarrow R(g(x), y) \). Then,

\[
3y[R(x, 2y) \& yeA] \Rightarrow 3y[R(f(x), y) \& yeA];
\]

and

\[
3y[R(x, 2y + 1) \& yeB] \Rightarrow 3y[R(g(x), y) \& yeB].
\]

Thus, \( x \in to \Leftrightarrow f(x) \in ID \lor g(x) \in to \). Thus, \( to \subseteq to \lor to \).

We are now ready to prove our main results.

**Theorem 12.** \( \forall a, b \exists 0^* \forall b3K[(a \land b \land a' \& b \land e. a) \Rightarrow (d(K) = a \land d(to^a) = b)] \).

**Proof.** (see Figure) Let \( a \) and \( b \) satisfy \( a \land 0', a \land b \land a' \), and \( b \land e. a \). By Friedberg's characterization [2], \( 3c a = c^1 \).

By Theorem 6, choose \( A \) so that \( d(A) = a(l_{\bowtie A}) = a \). By Corollary 8, choose \( B \) so that \( d(B) = d \) and \( ^{\bowtie B J = ^a \land B} = ^a \) \( B \land A \).

Let \( K = A \lor B \). \( d(K) = d(A) = a \). By Theorem 11, \( d(l_{\bowtie K}) = d(l_{\bowtie A \lor B}) = d(l_{\bowtie A}) \lor d(l_{\bowtie B}) = a \lor b = b \).

\[
\begin{align*}
a' \\
\d(l_{\bowtie K}) = b = d(K) = d(l_{\bowtie A}) \\
\d(l_{\bowtie B}) = b = d(l_{\bowtie B}) \\
K = A \lor B \\
\end{align*}
\]

**Figure**
Theorem 13. \( \text{Va} \vee \text{Vb}3K \{ a \leq b \land a' - (d(K) = b \land d(U) = a') \} \).

\[ \text{Proof, By Corollary 8 and Theorem 10, choose sets A and B so that } d(A) = a, d(B) = b, d(U) = a', \text{ and } d(U) = b \lor 1. \text{ Let } K = A \lor B. \text{ Then, } d(K) = d(A \lor B) = d(B) = b. \text{ By Theorem 11, } d(U) = d(U) = d(U) = a' \lor b \lor 1 = a'. \]

Theorem 14. \( \text{Va} \lor \text{Vb}3K \{ (a \leq b \land a' \land b \land a) - (d(K) = a \land d(U) = a') \} \).

\[ \text{Proof, Using [2] and [7], as in the first paragraph of the proof of Theorem 12, } 3c, d[c \leq d \leq d' = a \leq d' = b \land a' \]. \text{ By Corollary 8 and Theorem 10, choose A and B so that } d(A) = d, d(B) = c, d(U) = d \lor 1, \text{ and } d(U) = c \lor a. \text{ Take } K = A \lor B. \]
Bibliography


AMS Subject Classifications:

Primary 0270;
Secondary 2204

Key Phrases: first order formulas, finite structures, K-representation, spectrum, spectral functions

1. This paper is part of the author's doctoral dissertation directed by Professor Paul Axt and partially supported by NSF Grant GP7077. The results in this paper were first announced in [5].

2. The author is presently a postdoctoral research fellow at Carnegie-Mellon University.