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**Abstract**

**This paper reviews the effects of the presence of hard constraints in the stability of model predictive control (MPC). Assuming a fixed active set, we show that the optimal solution can be expressed in a general state-feedback closed form. This corresponds to a piecewise linear controller, for the linear model case. The changes introduced in the original unconstrained solution by the active constraints, as well as other eflFects related to the loss of degrees of freedom are clearly depicted in the current analysis. In addition to modifications in the unconstrained feedback gain, we show that the presence of active output constraints can introduce extra feedback terms in the predictive controller. This can lead to instability of the constrained closed-loop system with certain active sets, independently of the choice of tuning parameters used. To cope with these problems and extend the constraint handling capabilities of MPC, we introduce the possibility of considering soft constraints. Here we compare the use of the *I2* (quadratic), *l* (exact), and Zoo-norm penalty formulations. The analysis reveals a strong similarity between the constrained and**

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unconstrained control laws, which allows a direct extrapolation of the unconstrained tuning guidelines to the constrained case. In particular we show that the exact penalty treatment has identical stability characteristics to the correspondent unconstrained case, and therefore seems well suited for general soft constraint handling, even with nonlinear models. These extensions are included in the previously developed Newton control framework, allowing the use of the approach within a consistent framework for both linear and nonlinear process models, and increasing the scope of application of the method. Simple process examples are given to illustrate the capabilities of the proposed approaches.

# 1 Introduction

An important aspect in the application of model predictive control (MPC) is the effect of the presence of constraints in the stability of the resultant closed-loop system. Recent work by Zafiriou (1990-91) has shown this to be a significant issue, because of the frequency with which saturation can occur during routine operation. Considering the QDMC framework, he was able to identify several situations where the decrease in the degrees of freedom available to influence the process masked the effect of the tuning parameters for stabilization, rendering the constrained system unstable. A study of these effects becomes therefore an important issue in the design of constrained control systems.

Presently, the design of constrained predictive controllers is most often an iterative process. Typically, it starts with a candidate set of tuning parameters, representing an acceptable unconstrained design, obtained by taking into consideration e.g., the performance and robustness of the unconstrained system. The closed-loop performance of this controller then needs to be evaluated in the presence of the constraints which can potentially become active during the operation of the process. Depending on the results obtained, it might be necessary to adjust the initial parameters in order to achieve a good overall behavior with all possible active sets. These changes require in turn a re-evaluation of the modified design according to the criteria used in the first step. Hence it is essential in a good constrained design methodology to have the components to: i) assess the closed-loop performance of the process, especially its stability properties, in the presence of various types of constraints; ii) a systematic procedure for adjusting the tuning parameters, or if necessary adapt the constraint handling methodology used, in order to provide a good, global, closed-loop performance. Both of these issues are addressed in detail throughout this paper.

We start by presenting a comprehensive treatment of the effects of the presence of active constraints in the stability of the predictive control. This study is motivated by the pioneering work of Zafiriou (1990; 1991a, b) in this area. Similar to his analysis, the nonlinearities introduced by the presence of active constraints are handled through separate consideration of each different active set. A disadvantage of this approach is that a complete analysis requires checking all possible combinations of constraints, in order to guarantee global properties. This is essentially a combinatorial task, which introduces considerable difficulties in the analysis of predictive horizons of the size usually considered in practice. However, as described below, much of the same information can be derived from a significantly smaller subset of constraints. The present analysis also shows that the optimal solution can be expressed in a general state-feedback closed form, similar to the unconstrained case. Here the modifications introduced in the original solution by the presence of constraints are clearly displayed. Even so, a change in the adjustable parameters of the controller might not be sufficient to prevent, in the worst case, the instability of the closed-loop system with certain active sets, as illustrated in the examples included.

To deal with the stability problems caused by the presence of active hard constraints, we introduce in this work a constraint relaxation scheme based on the use of penalty functions. This feature is also created as an extension of the previously developed Newton control framework (Li and Biegler, 1989; Oliveira and Biegler, 1993), allowing a straightforward generalization of these results to the nonlinear case, because of the uniform treatment of both types of systems provided by the formalism.

This paper is organized as follows; a brief review of the Newton control formulation is given in Section 2. This is followed by the stability analysis of MPC in the presence of hard constraints, in Section 3. Assuming a fixed active set, we first show that the optimal input profile can be expressed in a general state-feedback form. In addition to changes in the unconstrained feedback structure, we demonstrate that the presence of active out-

**put constraints can introduce extra feedback terms in the predictive controller. Therefore together with a decrease in the degrees of freedom available, this can lead to stability problems of the constrained system with certain active sets, independently of the choice of tuning parameters used. A constraint relaxation treatment capable of preventing the occurrence of these problems is introduced in Section 4. Here we address first the common case of a quadratic penalty objective, for which simpler stability results are derived, compared to the treatment of Zafiriou (1991a). These relate directly the stability of the relaxed constrained problem to an equivalent modification in the tuning parameters of the original unconstrained problem. We show that in general only a maximum finite value of the penalty parameter can be tolerated for stability. Since the stability characteristics of the relaxed problem are still dependent on the current active set, this approach suffers from the same disadvantage as the hard constraint analysis, of being combinatorial with the length of the predictive horizon used.**

**To eliminate this last requirement we then introduce an alternative soft constraint treatment, that uses the *l*\ or Zoo-norm penalty formulations. In this case, we demonstrate that the resultant constrained system has identical stability properties to the corresponding unconstrained situation. Moreover if the original hard constrained controller is stable, the *l*\ strategy requires only a finite penalty parameter (larger than the norm of the Kuhn-Tucker multipliers) to match the solution of the original problem, in contrast with the quadratic penalty case which requires a parameter value of infinity. This characteristic allows much better control of the errors resulting from constraint relaxation, and simplifies considerably the use of the soft constraints, especially for nonlinear systems. We also discuss in this section the possibility of occurrence of steady-state oflfeets for large values of the penalty parameter and short horizons (due to the receding nature of the control law), together with sufficient conditions for their elimination. The use of alternative penalty formulations and their possible effects on the closed-loop stability properties are also considered here. Finally, these developments are illustrated with application to several process examples.**

## 2 Preliminary definitions

This section presents a short overview of the Newton control formulation used in the analysis of the stability properties of MPC in the next section; a more complete description can be found in Oliveira and Biegler (1993). The analysis presented here is based on the control law expressed in magnitude form, where the independent variable is *U*. This formulation is particularly convenient for studying the stability properties of the resultant controller.

We denote by $u$ G $R^{TM1}$ the vector of system inputs (manipulations), $x$ G $R^{n*}$ the vector of state variables, $y$ G $R^{n\circ}$ the vector of system outputs, $6$ € $R/^{1*}$ a vector of system parameters, and $d$ G $H^{nd}$ the vector of process disturbances. The lengths of the input and output predictive horizons are m and p, respectively, with $m \leq p$. The identity matrix of order $n$ is denoted here by /„ e $R^{n x n}$. We start by defining a discrete system operator as

$$Xk = \text{Sfc+i} = X(tk + T; t_{ky} Z^*, u_k, d \backslash 0),$$

where we assume that the transition function $x$ is continuous and differentiable with respect to all of its arguments. This operator can be obtained from a continuous plant model, provided that it satisfies the proper Lipschitz continuity conditions (Economou, 1985), or by direct application of discrete identification techniques to the system under consideration. In a similar form, we denote the operator induced by the control algorithm generically by Ufc+i $= ip(x_k, u_k, y_{rk})_y$ where $y_{rk}$ corresponds to the value of an external reference input (set-point), specified over a finite horizon (also in discrete time). Thus, together with the previous plant model, an augmented closed-loop system can be defined as

$$z_{k+1} = \begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix} = X(T \backslash z_{ky} y_{rk}) = \begin{bmatrix} \; \\ \psi(x_k, u_k, y_{rk}) \end{bmatrix}, \tag{2.1}$$

where we have omited the dependence of $z$ on T, $d$ and $0$ for clarity of notation.

More specifically, the selection of the control law in the Newton framework is based on a quadratic performance index in a moving horizon of length p, which corresponds to the solution of the following constrained quadratic programming (QP) problem

$$\min_{U} \quad J_2 = (V_{SP} - Y)^r Q_l (Y_{sp} - Y) + (U - U_r)^r Q_2 (U - U_r) \tag{2.2a}$$

$$\text{s.t.} \quad Y = Y' + S_m U \tag{2.2b}$$

$$U_t \leq U \leq U_U \tag{2.2c}$$

$$Y_t \leq Y <\_ \; Y_u. \tag{2.2d}$$

The solution of this QP corresponds to a Newton step towards the solution of the predictive control problem for a nonlinear process model, or the optimal profile in the linear case. Here $Q_x$ = diag{(?$_{yi}$} € R ^ * ^ * and $Q_2$ = diag{Q$_{ui}$} € *R^™)*^™)* axe adjustable weights in the objective. Also, capital letters *E,* A", *Y* and *U* are used throughout the paper to denote augmented vectors defined for the entire predictive horizon. Thus for example, *U* corresponds to the augmented input vector

$$U \in \mathbf{R}^{nm} \equiv \left[ u_k^T \quad u_{k+l}^T \quad \cdots \quad u_{k+m-1}^T \right]^T .$$

The vector $U_r$ defines a reference trajectory fot the inputs, similar to the role of *Y^* **for** the outputs. $S_m$ is the system *dynamic matrix.* This matrix can be formed directly from a linear process model or obtained from a sensitivity analysis of **a** nonlinear model **around a** nominal trajectory $\bar{U}$ (Oliveira and Biegler, 1993). It generates in the second case a linear time-varying (LTV) approximation of the original process model, as part of the Newton iteration. *Y** corresponds to the system response for a zero input. For linear models, *Y** can be expressed directly in terms of the initial conditions and sensitivity coefficients, as

$$Y^* = C^* x_k = \begin{bmatrix} C_{k+1}\Phi_k \\ C_{k+2}\Phi_{k+1}\Phi_k \\ \cdots \\ C_{k+p}\left(\prod_{j=1}^{p} \Phi_{k+p-j}\right) \end{bmatrix} x_k.$$

Replacing (2.2b) directly in the objective, leads to the following formulation

$$\min_{U} \quad J_2 = \{ET - <S_m £/)^T Qi(£^* - S_m U) + (U - U_r)^T Q_2 (U - U_r) \tag{2.3a}$$

$$\text{s.t.} \quad U_{\underline{i}} \le U \le U_U \tag{2.3b}$$

$$Y\hat{d} \le S_m U \le Y;_{d}, \tag{2.3c}$$

where we have defined $E^* = Y_{sp} - Y\backslash$ $Yf_d = Y_t - Y\backslash$ and $Y\& = Y_U - Y^*$.

For the unconstrained case, the analytical solution of the previous problem is

$$U = \{ftQiSm + (?_2)^{-1}(\wedge Q_1 \wedge + Q_2 U_r) \ \wedge Kxt + d\wedge + dr, \tag{2.4}$$

where

$$H = S_m^T Q_1 S_m + Q_2 \tag{2.5a}$$

$$K = -H^{-l} SlQ_x C^* \tag{2.5b}$$

$$d_{sp} = \mathbf{fr}^{1} \wedge ! \wedge \tag{2.5c}$$

$$d_r = H^{-l} SlQ_2 U_r. \tag{2.6d}$$

In the above solution, /f corresponds to a state feedback term, while dgp and $d_r$ can be seen as additional bias terms, denoting the fact that nonzero reference values are used for V^p and $U_r$. Clearly, only $K$ contributes to the stability of the closed-loop system, since the remaining terms are fixed and bounded. The receding nature of MPC also requires that only the first move in computed profile be implemented at a time, with the calculation repeated at the next sampling point, using any additional information available. This implies that the implemented manipulation at t* is $Uk = \hat{f/}_m \ 0 \ \cdots \ O] \ C/$, where $U$ is given by the solution of either (2.3) or (2.4).

# 3 Effect of active hard constraints in the closed-loop stability

**In the presence of active hard constraints, the optimal input profile needs to be found as the solution of (2.3). For a fixed (given) active set, we will denote the corresponding constraints as**

$$I^a U = f\!\!/_6^{\,a} \tag{3.1a}$$

$$S\wedge U = Y_b{}^a - Y*^a = Y_b{}^a - CTx_k. \tag{3.1b}$$

**The superscript $^a$ is introduced to denote active constraints. We represent the number of currently active input and output active constraints by $n_u$ and $n_y$, respectively. In this case, the matrices $I^a$ G $R^{nwX}\wedge{}^m$> and <S£ e $R^n v^x(^{n}\circ p)$ are obtained through selection of the rows of $/_{n,m}$ and <$S_m$ that correspond just to the currently active constraints. Here *Ug* and $Y_b^a$ represent the values of the active input and output bounds, respectively. These express either upper or lower bounds, or equality constraints which can also be specified for these variables. The above set of constraints can also be represented in a more compact form as**

$$AU = c_a, \tag{3.2}$$

**where**

$$A - \left\lfloor \; a \right\rceil, \qquad c_a = \begin{bmatrix} U_b^a \\ V^a - Y^{*a} \end{bmatrix}.$$

**The effects of the presence of constraints are better illustrated with a range and null-space decomposition, performed on the matrix of constraints *A*. Since this matrix can be considerably ill-conditioned, it is preferable to base the decomposition on a QR factorization of $A^T$ with column pivoting (Golub and Van Loan, 1989). The advantage of this algorithm is that the determination of the numerical rank of the matrix to be factorized can be done simultaneously in a numerically robust form, taking into consideration the**

precision of the data available. Assuming that the rank of $A$ is 7V $\leq$ n^ leads to

$$A^{\mathrm{T}}P = QR = \left[Q_y \quad \mathrm{a}\right] \begin{bmatrix} \mathcal{R} \\ 0 \end{bmatrix}, \tag{3.3}$$

where $Q$ G $K^{TliXni}$ is an orthogonal matrix satisfying $Q^TQ$ = $/_{n \cdot}$ and $Tl$ G R^1^ *(*«+»*> is an upper triangular matrix. $P$ G R^(n" +^n v)^x(^n" +^n v) is a permutation matrix, obtained by interchange of the columns of the identity matrix of the same order. $Q_y$ and $Q_z$ correspond to a partition of the columns of $Q$ with $Qy$ G $R^{niXTlr}$ and $Q_z$ G R^1* *(»<-"••), respectively.

Pre-multiplying both members of (3.3) by $Q^T$ gives $Q\%A^TP$ = $Tl$ and $QjA^TP$ = 0. This implies that

$$P^TAQ_y = Tl^T \qquad \text{and} \qquad AQ_Z = 0, \tag{3.4}$$

and therefore $Q_z$ corresponds to a basis for the null-space of $A$. Hence the full $U$ space can be partitioned into range and null-space components as

$$U = QyUy + Q_zU_{zy} \tag{3.5}$$

where $U_y$ G $R^{nr}$ and $U_z$ G R^n*""nr. The optimal solution for $U$ can then be found separately in terms of these two components, which can be combined in the end according to (3.5) to give the optimal profile. It should be noted that since the $U_y$ component is entirely determined by the current active constraints, the number of degrees of freedom in the input profile to be determined is just fi* — $n_r$.

We will start by calculating {$7_y$, which can be obtained by replacing (3.5) in (3.2), giving $AQ_yUy = c_a$. This implies that $P^TAQ_yU_y = P^Tc_a$ or from (3.4),

$$\mathcal{R}^{\mathrm{T}}U_y = P^T \begin{bmatrix} U_b^a \\ Y_b^a - Y^{*a} \end{bmatrix} = P^{\mathrm{T}}c_a. \tag{3.6}$$

If TV = Tu + $ny$, i.e., $A$ is full row-rank, (3.6) constitutes a set of linear lower triangular equations which can be solved by forward substitution to obtain $U_y$. The solution can also be expressed analytically as

$$U_y \;= \mathcal{R}^{-\mathrm{T}} P^{\mathrm{T}} \begin{bmatrix} U_b^a \\ Y_h{}^a\text{-}Y^{*\,a} \end{bmatrix} = \begin{bmatrix} B_y & K_y \end{bmatrix} \begin{bmatrix} U_b^a \\ y_v^* \; - \; y^{*a} \end{bmatrix},$$

or

$$U_v = B_y U_b^a + K_y(Y_b{}^a \quad - Y^{*a}), \tag{3-7}$$

where we have defined $\lfloor B_V \quad K_y \rfloor$ as a compatible partition of the columns of $TZ^{\prime\prime T}P^T$. As with the unconstrained case, the contributions for $U_y$ in (3.7) can be grouped into a bias term (given by $B_yU\% + K_yY_v^*$), and a feedback term $—K_yY^{*a}$. It should be noted that this last term will only appear in the optimal profile if there exist output constraints which are active.

However if *rir* < *Uu* + riy, the matrix $TZ^T$ will have a lower trapezoidal structure, and the linear system (3.6) is overdetermined, corresponding in general to a disjoint active set. In this case it is possible to create the partitions

$$\mathcal{R}^{\mathrm{T}} = \begin{bmatrix} \mathcal{R}_u^{\mathrm{T}} \\ \mathcal{R}_l^{\mathrm{T}} \end{bmatrix} \qquad \text{and} \qquad c_a = \begin{bmatrix} c_{au} \\ c_{al} \end{bmatrix},$$

where ft£ G $R^{n\wedge xn\wedge}$ corresponds to the upper triangular part of ft, *Hf* G R($^n$<*+$n_y$-$n_r$)xnr $i_s$ the remaining rectangular matrix, and $c_{au}$ G R/^, *CQX* £ $R^{nu+n}$v~"$^{nr}$ constitute an equivalent partition of $c_a$. This allows us to solve the lower triangular system $V\wedge U_y = P^T c_{auy}$ to find $U_y$. The computed solution can then be plugged back in the remaining equations, to check the feasibility of (3.6). If this set of equalities is compatible, then $U_y$ can still be expressed in the form (3.7), with $\lfloor B_y \quad K_y \rfloor$ corresponding now to a column partition of 7£$_u^{\sim T}$$P^T$. It should be noted that in this case the choice of active constraints which are retained in *TZu* doesn't affect the stability properties of the resultant controller (although it apparently can induce a different feedback structure), since all choices of constraints produce the same value of $U_y$.

On the other hand, the term $U_z$ is determined by adjusting the remaining degrees of freedom in the profile to minimize the objective, if there axe any left. Prom (2.3), $J_2$ can be expressed as

$$J_2(U) = c + 2a^T U + U^T HU, \tag{3.8}$$

where $a = -S\wedge QiE* - Q_2 U_r$ and $H$ is given by (2.5a). Replacing (3.7) in (3.8) gives

$$J_2(U_Z) = \text{constant} + 2a^T Q_z U_z + {}'2U_y{}^r\$HQ_z U_z + UjQjHQ_z U_z.$$

Solving $VJ_2(U_Z) = 0$ leads to

$$U_z = -(Q^T{}_2 H Q_z Y^l Q^T{}_z\{a + HQyU_y). \tag{3.9}$$

Equations (3.7) and (3.9) can finally be combined to give the analytical solution of the predictive control law. Replacing these in (3.5) gives

$$U = K_h x_k + 4{,}sp + d_{hr} + d_{hu} + d_{hy}, \tag{3.10}$$

where we have defined

$$H_P = Q?(SlQiSm + Q_2)Q* \tag{3.11a}$$

$$B = Q_t H;^l Q^T{}_z \tag{3.11b}$$

$$K_h = -(/_{n<m} - B\wedge QyKyC** - BSlQ_x C* \tag{3.11c}$$

$$d_{hr8p} = BSTFQiFsp \tag{3.11d}$$

$$d_{hr} = BQ_2 U_r \tag{3.11e}$$

$$d_{hu} = (Intm - BH)Q_y B_y US \tag{3.11f}$$

$$d_{hy} = (I_{n_i m} - BH)Q_y K_y Y_b^a. \tag{3.11g}$$

Analogously to the unconstrained solution, the stability of the closed-loop law depends only on $Kh$, since the other terms are fixed within a given active set, and are bounded for all sets of constraints. Comparing (3.10) with the corresponding unconstrained solution (2.4) it is possible to observe that the general effect of the presence of constraints is to change the feedback structure of the system, replacing the unconstrained Hessian $H$ by JB, and introducing also additional bias terms $dh_U$ and $dh_y$ in it. In addition, the presence of active output constraints may yield extra feedback terms, as indicated by (3.11c). These feedback terms, which depend just on the current active set, can induce closed-loop instability in certain situations, as illustrated in the examples below. Also, as pointed out by Zafiriou (1990), we note that the stability properties of the predictive control law (3.10) with hard constraints are identical for each of the upper and lower bounds in the process variables, since the feedback term in it is not influenced by the magnitudes of the boimds $Ub$ and $Yb$ in (3.1).

The global stability properties of the constrained system can now be considered. As indicated by (3.10), the overall controller is piecewise linear in £*, with a structure which is only dependent on the current active set. This allows the use of the contraction mapping principle to show the following results.

**Theorem 1 (Economou, 1985)** *Consider the augmented closed-loop system (2.1)* defined *by the discrete LTV model* $x_{k+1} = \Phi_k x_k + \Gamma_k u_k$, *together with the state-feedback control law (3.10),* $u_{k+1} = $ tyfeZ**+$Y_k U_k + d(y_{sp}, U_r)$. *Define also an initial state* $z_0 = [x_j \quad v£]^T$ *and reference inputs* T/sp,fc+t = $y^*$> $u_{r,k+i} = $ tx*, $i = 1, \ldots, p$. *If*

$$\text{nnoii} = \left\| \frac{dX(T;x,u,y\%u')}{dz} \right\| = \left\| \begin{matrix} \frac{\partial \chi}{\partial x} & du \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial u} \end{matrix} \right\| \le 0 < 1, \qquad \forall z \in B(ZQ, r), \text{ where}$$

$$r \ge r_0 = \frac{\|X(T;z_o,y^*)-z_o\|}{(1-\theta)}, \qquad \text{and} \qquad 6(20, r) = \{z \in R^{n_x+n_u} : \|z - *>\| \le r\},$$

*then the system has a unique asymptotically stable equilibrium point* $z_e = [x\% \ uJ]^T$ *in* $B(zo,r)$. *Furthermore,* $B(ZQ,TO)$ *is a region of attraction for* $z_e$. *(Here* $\| \bullet \|$ *represents any consistent norm* definition).

**This theorem provides a sufficient condition for the stability of the closed-loop system, although its application with constrained systems is frequently limited in practice by the need of finding a consistent norm for different active sets. The next result is simpler to**

**verify in the general case, even though it represents only a necessary condition for stability.**

**Theorem 2 (Zafiriou, 1990)** *Consider the augmented closed-loop system (2.1) defined by the discrete LTV model Xk+\ = $kXk + FkUk, together with the state-feedback control law (3.10),* txfc+i = ^k^k + Yfctifc + *d(ysp, Ur), an initial state ZQ =* [xj *u$* $^T$ *and reference inputs* $y_{sPy}$*k+i* = *V*,* *V,fc+t-i = u*, t = 1,... ,p. *This system can only be asymptotically stable in B{ZQ^r) if*

$$Psr(F'(z)) = _{Psr} ( ** \; l^k ) \leq 6 < 1, \qquad V^* \in B\{z_0, r),$$

where $p_{sr}(^m)$ *represents the spectral radius of a matrix,* $p_{8r}(A)$ = max* |At(i4)|.

**As mentioned, this theorem provides just a minimal condition for global stability. Nevertheless, it is frequently used in practice, together with an implicit assumption about the rate of change of the system structure[1]. For example, when considering the closed-loop response of a LTI model with a fixed set-point, and sufficiently small disturbances such that no changes in the current active set occur, this last condition becomes also sufficient for stability, since a fixed active set induces a fixed control structure. Furthermore, since any induced norm constitutes an appropriate consistent norm, Theorem 1 becomes also easier to apply in this situation. Both of these results will be useful later in the context of the soft constraint handling methodology described below.**

## 3.1 Special cases

**We consider now some specific cases of (3.10), which introduce special characteristics in the closed-loop behavior, when some particular combinations of constraints are active.**

**Only input constraints active** **In this case, the only active constraints are of the form (3.1a). Because of the special structure of the matrix of constraints, the range and null**

[1]**Le., the so called *slowly varying* systems, (Vidyasagar, 1978).**

space decomposition is straightforward. For example, if

$$I^a = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$(I^a)^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$Qy \qquad Qz \qquad n/o$$

As can be observed, $Q_y$ may be taken in this situation as $(/^a)^T$, and $Q_z$ formed from the remaining columns of the correspondent identity matrix. Prom (3.5), this immediately implies that if there are no constraints active in the first interval of the horizon, the implemented control input *Uk* always comes from the $Q_z$ (or null) space, and corresponds to the first components of (3.9). Also in this case, $TZ = 2\wedge$, and there's no need for column pivoting during the QR factorization of *A*. Furthermore, the projected Hessian $H_p$ is simply formed by selection of the free columns and rows of if, according to (3.11a). The inverse of this matrix is later projected back into the full space during the computation of *B*, as indicated by (3.11b).

**First output constraint active, with single-input systems** Here the bases for the range and null spaces of the constraint matrix need to be found in general through the standard decomposition algorithm described above. However since $Q_z$ always corresponds to a basis for the null space of $A$, this implies that

$$AQ_Z = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,i_2} & 0 & \cdots & 0 \\ a_{1,i_2+1} & a_{2,i_2+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1,n_i-n_r} \\ z_{21} & z_{22} & \cdots & z_{2,n_i-n_r} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = 0,$$

and therefore the first row of $Q_z$ must be identically null, i.e., $[z_{11} \quad z_{12} \quad \cdots \quad z_{1,n_i-n_r}] = 0$. From (3.5), this implies that the null space contribution to $U$ is completely omitted in the implemented control $U_k$, and is determined by $U_y$ alone. As a consequence, the normal feedback structure of the controller is suspended, and the only possible source of feedback in the closed-loop system will be any active output constraints, as indicated by (3.11c). If there are only active input constraints, then the system will essentially behave as open-loop. This situation corresponds therefore to one of the most severe loss of degrees of freedom that can be induced by the presence of active constraints. It may easily induce closed-loop instability in several situations, such as with open-loop unstable plants and systems with non-minimum phase characteristics. This behavior is observed in some of the examples presented below.

**Maximum rate limits in the inputs** Maximum bounds in the rate of variation of the manipulated variables corresponds also to a frequently used direct control objective in MPC; most of the existing implementations provide some facility for the treatment of these constraints. These limits are especially useful to prevent aggressive control moves caused by strong changes during the process operation. The corresponding constraints can be expressed in the form

$$|\Delta u_{k+i}| = |u_{k+i} - u_{k+i-1}| \leq \Delta u_{\max} \qquad i = 0, \ldots, m-1,$$

with *Uk~i* given, and Attmax representing the maximum allowed input move during a unique sampling interval. These equations can also be written in matrix form as

$$-AC/_{max} \leq GU \leq AC/_{max},$$

where

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(n_i m)\times(n_i m)} \tag{3.12a}$$

$$\Delta U_{max,j} = \begin{cases} \pm t!fc\_l + \Delta u_{max} & j = 1 \\ \Delta u_{max} & j = 2, \ldots, m. \end{cases} \tag{3.12b}$$

When active, these constraints become equalities of the form $G^a U = AC/\wedge$, where $G^a$ and $AC/\wedge$ are formed from the active rows of $G$ and Al/max- The effects of their presence on the closed-loop stability, e.g. with a mixture of constraints of the other types can therefore be treated by the algorithm described previously. Applying a similar reasoning, it is possible to conclude that, analogous to the effect of active absolute bounds on the inputs, the stability properties with rate constraints depend just on the given active set, and not on the bounds themselves. However, the tuning parameters used can certainly influence whether or not a given active set of this type can become active (i.e., optimal). Also, we should notice that because of the special form of (3.12a), similar considerations to the constrained single-input system case can be made here. Having a rate constraint active during the first sampling interval will also cause the closed-loop system to behave essentially as open-loop during the same period of time. Hence possible stability problems with unstable plants can be anticipated in this situation as well.

## 3.2 Examples

We illustrate now the application of the previous method in the elucidation of the constrained stability properties of MPC, with a number of simple process examples. These particular models were selected in order to provide a broad view of the possible behaviors and stability effects that can result with active hard constraints. The results shown here were obtained through an implementation of the algorithm described above in the *Mathematica* language (Wolfram, 1991).

**Example 3.1** We will start with the first order plus time-delay SISO system given by the transfer function

$$G« = (7TT)^{-p^{-0.06a}}-$$

This corresponds to a frequently used class of models, that describes chemical processes with relatively simple and slow dynamics. The equivalent discrete pulse transfer function obtained with a sampling time $T = 0.1$ is

$$HG[Z) = \frac{0.03921(z+1.427)}{*(*-0.9048)}\ '$$

For this model, we consider only simple bounds in the input and output variables, of the form (3.1). The lengths of the predictive horizons used with it are m = $p$ = 5, with tuning parameters $Q_y$ = 1, $Q_u$ = 0.01. In Figure 1 we have represented the spectral radius of the resultant closed-loop system, for each possible active set that can occur with this type of bounds. In order to sweep all different combinations of constraints in a systematic form, the following procedure is used. We begin by appending the output horizon to the end of the input horizon, in order to form a unique extended horizon, as indicated in Figure 2. Then, starting at the left, we read the correspondent binary number obtained by assigning either the bit 0 or 1 to the position of each variable in the extended horizon, depending on whether the correspondent constraint is inactive or active. For example, the total number of constraints in this case is $2^{10}$ = 1024. This means that the active set 0 will correspond to the unconstrained system; the active set 1 denotes the situation where only the first input in the first interval of the horizon is active; the active set number 32 represents the first output constrained only during the first interval; finally the last active set (1024) represents the fact all input and output constraints are active simultaneously.

[Figure 1 about here.]

[Figure 2 about here.]

As can be observed, there exist certain active sets for this system which will make the resultant closed-loop system unstable. This is indicated by the existence of points above the unit spectral radius line in Figure 1. The problem occurs with the given tuning parameters, even though the spectral radius of the unconstrained system is about 0.33, well below the stability limit. A closer inspection of these situations reveals that the instability always occurs when the first output constraint in the horizon is active. This behavior was anticipated in the special cases considered before, since the discrete plant model has a zero outside the unit circle. Hence to avoid possible instability, this observation requires the use of a different approach for constraint handling, or even a system redesign such that these constraints will not become active during the normal plant operation. Furthermore, the example shows that all of the possible values of the closed-loop spectral radius are essentially grouped into various clusters, a fact that can be used to simplify significantly the elucidation of the stability properties of the constrained system. According to this observation, the most important information can usually be obtained through the investigation of only a much smaller subset of constraints than the total number of possible combinations. For instance in this case, it would be sufficient to evaluate the stability properties of the system with all input constraints active throughout the horizon (pg,. = 0.90), all output constraints active throughout the horizon (p^ = 1.43), and both sets simultaneously ($p_{sr}$ = 1.43), together with the unconstrained information (p^. = 0.33). This requires examining only 4 active sets, instead of the full 1024 cases. This guideline, although heuristic in nature, can potentially lead to tremendous reductions in the effort involved in a constrained stability analysis of a predictive controller with linear time-invariant (LTI) models. It also applies successfully to all of the remaining examples presented below.

**Example 3.2** We consider now the linear, SISO, open-loop unstable system described by the discrete transfer function

$$GH(z) = \frac{z+1}{z^2 - 2z + 0.5}.$$

Similarly to the previous example, we are also interested in assessing the possible effects that simple bounds in the input and output variables might have on stability. Figure 3 represents the spectral radius of the resultant closed-loop system, for each possible active set, when predictive horizons of length $m = p = 5$, and tuning parameters $Q_y = Q_u = 1$ are used. Because the model is open-loop unstable, stability problems during closed-loop operation are possible now when the input saturates. This is indicated by a value of $p_n = 1.71$ for all cases where the first input in the horizon is constrained. As in the previous example, the clustering of values of the spectral radius is clearly visible in Figure 3. This is further evidence about the applicability of the heuristic rule given before- As expected, the points located in the line $p\wedge = 1$ are generated by active output constraints, since the model has a zero at $z = 1$. Also with this system, additional stability problems can result if rate constraints specified for the inputs become active. For example, whenever a rate constraint is active during the first interval in the horizon, the correspondent spectral radius becomes $p_m = 1.71$, indicating that the system is behaving essentially as open-loop. This situation requires therefore also special attention to avoid instability of the closed-loop system.

[Figure 3 about here.]

**Example 3.3** We examine now the stability properties of MPC applied to the linearized model of a FCC (Fluid Catalytic Cracking) unit. This model was obtained by linearization, followed by normalization, of the Lee and Kugelman (1973) model around a nominal stable operating pont, as described in Oliveira and Biegler (1993). This example illustrates therefore the use of the previous methodology to study the local stability properties of a nonlinear model, around a fixed operating point. Using a sampling time $T = 0.05$ leads to the following discrete model

$$x_{k+1} = \begin{bmatrix} 0.07362 & 0.1148 & 0.01044 & 0.4390 \\ 0.03940 & 0.1492 & 0.008940 & 0.8111 \\ 0.2794 & -0.7511 & -0.003253 & -6.127 \\ 0.04545 & 0.1559 & 0.009798 & 0.8384 \end{bmatrix} x_k + \begin{bmatrix} -0.2495 & 0.6030 \\ 0.01680 & 0.2505 \\ -5.785 & 7.2139 \\ 0.01157 & 0.1217 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k,$$

where $x = \begin{bmatrix} C_{sc} & T_{rx} & C_{rg} & T_{rg} \end{bmatrix}^T$, and $u = \begin{bmatrix} F_a & F_c \end{bmatrix}^T$. Here $C_{sc}$ denotes the coke content in the spent catalyst, $T_{rx}$ the reactor bed temperature, $C_{rg}$ the coke content in the regenerated catalyst, and $T_{rg}$ is regenerator bed temperature. The manipulated variables are $F_a$ (air flow rate) and $F_c$ (catalyst recirculation rate). The tuning parameters selected are $Q_y$ = diag{1,0.1} and $Q_u$ = diag{0.01,0.01}. In order to limit the number of possible active sets, the lengths of the predictive horizons were fixed in this case as $m = p = 3$, with a total number of $2^{12}$ = 4096 distinct constraint sets. The spectral radius of the closed-loop system for each of these possible cases is plotted in Figure 4.

[Figure 4 about here.]

Several situations that may lead to closed-loop instability can also be observed in this figure. For instance, if the constraints in the first input $u_1$ become active throughout the predictive horizon, then the closed system has $\rho_{sr}$ = 1.12. Similar behavior occurs if the second input saturates, leading in this case to $\rho_{sr}$ = 1.01. The maximum value of spectral radius ($\rho_{sr}$ = 4.04) is obtained however when both $u_1$ and $y_1$ are at their bounds. This behavior with the input constraints could not be easily anticipated just by examining the unconstrained characteristics of the plant, since the model is open-loop stable and the unconstrained closed-loop system has $\rho_{sr}$ = 0.53. Looking now at the possible effects of the horizon lengths in the constrained stability shows that if m is kept equal to p, then the horizons need to be at least 7 intervals long to avoid problems associated with the saturation of $u_1$ (alone), and 16 intervals for case of $u_2$. These requirements are therefore much more restrictive than the needs for the simple stabilization of the unconstrained system. Nevertheless, the stability problem mentioned, that occurs when both $u_1$ and $y_1$ are constrained either during the first interval, or throughout the entire horizon, cannot be solved simply by increasing the length of the horizons used. If these constraints are active only during the first interval, it implies that the first two rows of $Q_z$ are null, according to the special cases considered before; if they are active throughout the entire horizon, then the null space has zero dimension, and the solution is entirely determined by the range space component. As a result, the closed-loop spectral radius will be well above the stability limit in both cases. These active sets require therefore special attention in the design of the control system, such that they do not occur during normal operation of the process, or even the use of a different approach for constraint handling, like the soft constraint formalism described below.

**Example 3.4** We center now our attention on the well-known Wood and Berry (1973) distillation model, represented by the transfer matrix

$$G_p(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{(16.7s+1)} & \dfrac{-18.9e^{-3s}}{(21.0s+1)} \\[2ex] \dfrac{6.6e^{-7s}}{(10.9s+1)} & \dfrac{-19.4e^{-3s}}{(14.4s+1)} \end{bmatrix}, \qquad G_d(s) = \begin{bmatrix} \dfrac{3.8e^{-8s}}{(14.9s+1)} \\[2ex] \dfrac{4.9e^{-3s}}{(13.2s+1)} \end{bmatrix}.$$

Using a sampling time $T = 1$, leads to the following discrete model

$$HG_p(z) = \begin{bmatrix} \dfrac{0.7440}{(z-0.9419)z} & \dfrac{-0.8789}{(z-0.9535)z^3} \\[2ex] \dfrac{0.5786}{(z-0.9123)z^7} & \dfrac{-1.302}{(z-0.9329)z^3} \end{bmatrix}, \qquad HG_d(z) = \begin{bmatrix} \dfrac{0.2467}{(z-0.9351)z^8} \\[2ex] \dfrac{0.3575}{(z-0.9270)z^3} \end{bmatrix}.$$

In order to use this model within the Newton framework, the discrete transfer matrix was first converted to a balanced state-space realization, using a standard technique based on a singular-value decomposition of the correspondent Hankel matrix (Chen, 1984). This resulted in a state-space model of order 22, One interesting characteristic of this model is the presence of large time-delays, which introduce a minimum limit in the length of the horizons that can be used for predictive control. Consequently, a large number of possible constraints need to be examined in a stability analysis of this system. For example, the present value of $T$ yields a minimum number of $2^{2+16} \simeq 262 \times 10^3$ active sets. This number can be further reduced taking into consideration the special structure of the model. For instance, the effect of the current control action will not affect the values of $y_2$ until at least 3 intervals in the future. Therefore the constraints where this variable is saturated during the first 3 intervals in the horizon do not need to be considered, since the correspondent rows of the dynamic matrix are identically zero. This brings down the minimum number of sets to $2^{15} \simeq 32 \times 10^3$. Even so, this number is significantly larger than the previous cases, making this example a good candidate for the use of the heuristic guideline given before.

Table 1 represents the spectral radius of the closed-loop system for the different active sets possible where a given variable is saturated throughout the entire horizon. According to the previous rule, this information provides a good indication of whether stability problems with hard constraints can be expected with this system. The tuning parameters used in this case are $Q_y = \mathrm{diag}\{1,1\}$, $Q_u = \mathrm{diag}\{0.1,0.1\}$, together with horizons of length $m = p = 10$. As can be observed, the closed-loop spectral radius is now below the stability limits in all cases. The discrete model is also open-loop stable, and has either no multivariable or individual zeros outside the unit circle. Therefore, under these circumstances, no significant constrained stability problems are anticipated with the present example.

[Table 1 about here.]

**Example 3.5** The last example consists of a laboratory system composed of two tanks and a connecting delay channel (Borrie, 1986). The continuous model for this system is described by the transfer function matrix

$$G(s) = \begin{bmatrix} \frac{1.5e^{-0.2s}}{(s+3)} & \frac{1.5s}{(s+1)(s+3)} \\ \frac{3.0e^{-0.2s}}{(s+3)} & \frac{-3}{(s+3)} \end{bmatrix}.$$

Here the output variables represent the signals from the flow and pressure transducers, while the inputs represent the control signals to the upper and lower valves, respectively. Using a sampling time of $T = 0.07$ results in the following discrete model

$$HG(z) = \begin{bmatrix} \frac{0.01478(z+5.409)}{(z-0.8106)z^3} & \frac{0.09136(z-1)}{(z-0.9324)(z-0.8106)} \\ \frac{0.02955(z+5.409)}{(z-0.8106)z^3} & \frac{-0.1894}{z-0.8106} \end{bmatrix}. \tag{3.13}$$

As in the last example, a balanced state-space realization was also formed from the discrete model, resulting in a state-space model of order 6. We plotted in Figure 5 the spectral radius of the resultant closed-loop system, for predictive horizons of length 3, with $Q_y$ = diag{l, 1}, $Q_u$ = diag{0.1,0.1}. A close look at these results reveals that there axe also several active sets that can induce closed-loop instability with the above tuning parameters. Depending on the amount by which the stability limit is violated, these situations fall essentially into one of two different categories.

In the first group, we have active sets with $p_a$ slightly above the stability limit (typically less than 1.05). Examples of this group are for instance the constraint sets OOOIOOOOIOOO2 (Psr = 1.004), OOOIOOOOIOO12 (Par = 1017), and $010000101001_2$ (A* = 1-041). Similar to the previous FCC example, the stability properties of these cases can be improved simply by increasing the lengths of the predictive horizons used* Furthermore, since they correspond to situations where a given variable is not constrained throughout the horizon, the likelihood of stability problems caused by their occurrence during normal operation is certainly low. In the second category, we have constraint sets with *p*^ well above the unit spectral radius line (around 5.41). This behavior is first observed with the constraint set OIOOOOIOIOIO2, and also occurs whenever both of the outputs are saturated simultaneously. Here possible stability problems need to be considered more seriously, since they are not affected by the tuning parameters used, especially the lengths of the horizons. This is a consequence of the discrete model having a multivariable zero at $z$ = 5.409, revealed by constructing the canonical Smith-McMillan form of (3.13). Hence special care is also needed to ensure that these hard constraints do not occur during normal operation, due to the incapability of MPC to handle them.

[Figure 5 about here.]

# 4 Strategies for soft constraint handling

In order to remove the dependence of the closed-loop stability of MPC on certain active sets, several approaches have been proposed. Ricker et al. (1988) and Zafiriou (1991a) first suggested the use of a quadratic relaxation of the output constraints in the origin of these problems. Based on this philosophy, the last author derived a soft-constrained stability analysis for the QDMC framework, to assess the existence of potential constrained stability problems. As with the hard constrained case, this analysis suffers from the disadvantage of being combinatorial in the length of the horizon used. More recently Rawlings and Muske (1991) proposed an alternative approach, based on the use of a fixed state-feedback law. This approach relies on the removal of the constraints that become infeasible in the beginning of the horizon. By keeping a finite input horizon, and extending the output horizon to infinity, it is shown that the resulting control law has guaranteed stability properties similar to the LQR framework. Moreover, since the problem has only a finite number of degrees of freedom, it can still be solved on-line as a quadratic program, provided that an upper bound on the time for constraint feasibility is used. However this bound is dependent on the controller itself, and is difficult to obtain without introducing considerable conservativeness in the measure itself. Additionally, the method does not provide a strict guarantee that the output constraints will eventually be enforced (especially in the presence of disturbances), due to the receding nature of MPC. In this group of approaches, one can also include the treatment of Mayne and Michalska (1990), which consists of the specification of a final equality constraint for the state vector at the end of the predictive horizon. This modification can be seen as a particular case of the Rawlings and Muske approach though, by noting that an infinite weight (equivalent to the specification of a final state constraint) is just a special case of the initial condition used for the recursive solution of the controller gain in the first method. As in the previous approach, guaranteed stability properties can be derived, provided that the constraint set remains always feasible.

In order to generalize the constraint treatment, we need to consider in more detail the consequences of a potential constraint violation in the process. Depending on their importance, process constraints can usually be classified as hard (if no violations are allowed at any time), or *soft* (where violations might be tolerated to satisfy other objectives). Examples in the first category include actuator limits, or safety constraints. In the second category we have e.g., output bounds corresponding to product specifications. While the original MPC formulation allows the specification of hard constraints, it might be preferable to consider, in some situations, a reformulation of (some of) the original constraints

$j = 1, \ldots, n_o$, such that

$$r_{ij} = \begin{cases} 1 & j^{th} \text{ output constraint active at either the upper or lower bound, at } t_{k+i} \\ 0 & \text{otherwise.} \end{cases}$$

This definition allows us to eliminate the constraints (4.2b, c) in the above problem, by substituting them directly in the objective function. Doing so yields the following unconstrained problem

$$\min_U J\wedge \ = \ J_2 \ + \ p\{S_m U \ - \ Y\wedge)^T R\{S_m U \ - \ 1£), \tag{4.3}$$

where $Y\& = Y_b - Y^*$ is formed from the appropriate elements of $Y\backslash$ or $Y_u$, depending on which constraints axe active. This problem can be solved analytically, giving

$$\begin{aligned} U_{\min} &= \left(S_m^{\mathrm{T}}(Q_1 + \rho R)S_m + Q_2\right)^{T} \ (\$\&!\& \ + \ Q_2 U_r \ + \rho S_m^{\mathrm{T}} R Y_{bd}^*) \\ &= K_a X_k \ + d_{ar} \ + daw \ + {}^{d}ay> \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} H_a &= S_m^{\mathrm{T}}(Q_1 + \rho R)S_m + Q_2 \\ K_a &= -H_a^{-1} S_m^{\mathrm{T}}(Q_1 + \rho R)C^* \\ d_{ar} &= H;^l Q_2 U_r \\ d_{a,sp} &= H_a^{-1} S_m^{\mathrm{T}} Q_1 Y_{sp} \\ d_{ay} &= {}_P H;^l SlRY_h. \end{aligned}$$

A close look at the structure of the solution (4.4) reveals the following result:

**Theorem 3** *The stability characteristics of the relaxed constrained problem (4.2) are equivalent to the stability of the unconstrained problem (2.3a), with the tuning parameters* $Q_1^* \leftarrow Q_1 + \rho R$, $Q_2^* \leftarrow Q_2$.

The proof for this theorem is given in the Appendix. This result equates the effects of the presence of soft constraints to an equivalent change in the tuning parameters of the unconstrained system. This corresponds to a more straightforward result than the stability criterion given by Zafiriou (1991a). Since the term $pR$ depends on the current active set, different active sets have in general dissimilar effects on the closed-loop stability. Hence as in the hard constraint case, a complete stability analysis requires a combinatorial study of the effects of different constraints, which can be difficult to perform for large horizons. Because the approach affects only output constraints, the total number of combinations that need to be considered now is just $2^{n_o p}$, though. As in the hard constraint case, the stability characteristics of the constrained control law are identical for both boimds, since the feedback gain $K_a$ does not depend on Yj>.

In addition, this approach requires knowing how the tuning parameters affect the closed-loop stability of the unconstrained system, for a wide range of parameter values. This task can be readily accomplished for LTI systems, where a unique curve of the closed-loop spectral radius, as a function of the penalty parameter $p$ is needed. However this information is more difficult to obtain with other types of models (such as LTV or nonlinear systems), where it becomes dependent on the initial conditions or on the operating region considered. The following examples illustrate the possible application of the analysis with simple LTI models.

Example 4.1 Consider the model defined in Example 3.1. Using predictive horizons of length $m = p = 3$, and nominal tuning parameters $Q_y = 1$, $Q_u = 0.01$, we plotted in Figure 6 the closed-loop spectral radius as a function of the penalty parameter p, for all possible active sets. In addition to the unconstrained information, we have $2^P - 1 = 7$ curves to check.

[Figure 6 about here.]

From Figure 6, we observe that closed-loop stability is guaranteed for $Q\backslash \leq 41$ (approximately). This limit corresponds also to the case where the output constraint is just active in the first interval of the horizon. Therefore this implies that if the output constraints for this example are relaxed as soft constraints, the penalty parameter used must obey $p < 40$ in order to avoid closed-loop instability with some of the present active sets.

**Example 4.2 (Rawlings and Muske, 1991)** We consider now the discrete, open-loop stable realization

$$x_{k+1} = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \tag{4.5a}$$

$$vk = [-2/3 \quad 1]\, x_k, \tag{4.5b}$$

obtained from the continuous system

$$G\{S) = \frac{42 \sim 11s}{s^2 + 4s + 42},$$

with a sampling time $T = 0.1$. We assume a initial condition $x_0 = [3 \quad 3]^T$ and output constraints $\backslash yk\backslash \leq 0.5$, or equivalently for the state-space representation

$$\begin{bmatrix} -2/3 & 1 \\ 2/3 & -1 \end{bmatrix} x_k \leq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

This example exhibits also stability problems when we try to enforce the output bounds given, as hard constraints. For example, since the initial condition violates the output constraint, from (4.5) we need $VLQ\_ > 1.75$ to satisfy this constraint at $t\backslash$. However this condition causes also the state vector to increase in norm at $t\backslash$. This implies in turn that $u\backslash > w_0$, in order to keep inside the feasible region. The input and state sequences will consequently increase in magnitude, in an unbounded form. Furthermore, including maximum bounds in the input, in the form $\backslash uk\backslash \leq U_{max}$ will only make the constrained problem infeasible at some point in the future. Like most of the previous examples, this effect is independent of the tuning parameters used, since the control solution is always obtained from the range space of the constraints.

Using a quadratic penalty relaxation of the output constraints, with horizons of length 5, leads to $2^5 - 1 = 31$ possible active sets to consider. For $Q_y = Q_u = 1$, we plotted in Figure 7 the effects of the penalty parameter $p$ in the closed-loop spectral radius, for all possible constraint combinations. The stability limit is in this case $Q\backslash \leq 3.1$, corresponding again to the output constraint being active just during the first interval of the horizon.

[Figure 7 about here.]

Figure 8 illustrates the control profiles obtained with this constraint relaxation strategy. In all cases except with the highest value of *p* the system is stabilized, although this requires the output constraint to be violated during the first two intervals. To verify the conservativeness of the above stability limit, we tried to obtain numerically the minimum value of *p* that would make the closed-loop system unstable, from the given initial conditions. The value found in Figure 8 is $p_{C}$nticai £ [29.1,29.2], which is higher than the theoretical limit of *p* with all constraints active, respectively *pm^* = 21.2. The discrepancy between these values is due to the small value of $x_0$ used, since the spectral radius provides only a sufficient condition for stability. This difference can be decreased simply by increasing the norm of $x_0$, because the feedback and bias terms in the control law have roughly opposite effects in the magnitude of {/, in the present case.

[Figure 8 about here.]

As mentioned previously, one of the main disadvantages of the quadratic penalty formulation is that, even for LTI systems, the determination of the corresponding stability limits becomes a non-trivial task for large predictive horizons. Instead of explicitly enumerating all possible combinations of constraints, the evaluation of these limits would be faster if the problem was formulated as an equivalent optimization problem. However the resultant problem is difficult to solve, because the spectral radius is in general a nonlinear, nonconvex, and nondifferentiable function of the parameter p. Some methods under development for the constrained robustness analysis of LTI systems (Balakrishnan and Boyd, 1991; Young et al., 1992) show however some promise in this situation. Based on the use of a branch and bound algorithm, these methods are able to refine successively the estimates provided by approximate bounds, allowing the attention to be centered quickly in the regions of the parameter space that are of more importance. Their use for the determination of the stability limits of the quadratic penalty remains a topic of further research.

The above result can also be generalized to the case where a mixture of both hard and soft constraints are considered simultaneously. In order to do that, we define first the vectors $Y_h$ and $Y_3$ as the components of the augmented output vector $Y$ for which hard and soft constraints are specified, respectively. In a set sense, we assume that $Y_h \cup Y_3 = Y$, and $Y_h$ PI $Y_s = 0$. Using a similar line of reasoning to the previous analysis, it is possible to derive the following result:

**Theorem 4** *Consider the constrained quadratic penalty formulation, with a mixture of hard and soft output constraints*

$$\min \quad J_{2a,mix} = (E^* - S_m U)^T Q_h (E^* - S_m U) + (U - U_r)^T Q_2 (U - U_r) + \rho e^T e \tag{4.6a}$$

$$\text{s.t.} \quad U_l \leq U \leq U_u \tag{4.6b}$$

$$Y_{lh} \leq Y_h \leq Y_{uh} \tag{4.6c}$$

$$Y_{ls} - \epsilon < Y_a < Y_{Ha} + \epsilon \tag{4.6d}$$

$$\epsilon \geq 0. \tag{4.6e}$$

Tie *stability properties of the mixed constrained problem (4.6) are equivalent to the stability of the correspondent hard constrained only problem (4.6a-c) and e* = 0, *with the tuning parameters $Q_1 \longleftarrow Q_1 + \rho R$, $Q_1 \longleftarrow Q_2$-*

The proof for this theorem is given in the Appendix. It allows the stability properties of a predictive problem with mixed types of constraints to be related to the stability of the correspondent hard constrained only problem, for which the analysis presented in Section 3 can be applied. Using these results, it is therefore possible to perform a stability analysis of linear models in the presence of a large variety of constraints. The method provides also a systematic mechanism for the choice of appropriate values for the tuning parameters in order to avoid most of the problems described in the examples of Section 3. Also, although the previous formulation was targeted to the use of soft output constraints, the same approach can be used to treat input constraints, provided that the bounds that these represent can be relaxed. The analysis of this case using the methodology described previously is straightforward.

## 4.2 Exact penalty treatment

One of the properties of the quadratic penalty mentioned previously is that for finite values of p, the original output constraints may not be satisfied, which translates to non-zero values of *e*. This means that a violation of the original constraints is unavoidable with this formulation. Replacing the penalty term by the *l* (exact) penalty function, eliminates however the necessity of increasing the penalty parameter to infinity to recover the original constrained solution. A sufficient condition for this is to have p > |A|oo, where A is the vector of Lagrange multipliers of the constraints in the original problem (Fletcher, 1987). This formulation allows therefore a better control of the errors resultant from constraint softening.

Using the *l* penalty and starting by ignoring the presence of input constraints, allows us to express the predictive problem (2.2) as

$$\min_{U} J'_{2b} = h + r^T \max\{0, Y - Y_U\} + r^T \max\{0, -Y + F/\}, \tag{4.7}$$

with *J<i* given by (2.3a). Here r € $R^{n \circ p}$ = [p- • • p]$^T$ is a vector of penalty parameters. The penalty terms in (4.7) can be rearranged, leading to the equivalent constrained formulation

$$\min \quad J_2 b = h + r^T e \tag{4.8a}$$

$$\text{s.t.} \quad Y i - e \leq y \leq y_{tt} + e \tag{4.8b}$$

$$6 \geq 0, \tag{4.8c}$$

with € keeping its original definition from Section 4.1. The solution of (4.8) can also be obtained in analytical form, and expressed by

$$U_{\min} = (S_m^T Q_1 S_m$$
$$= K x_k + d_s p + dr + <ky, \tag{4.9}$$

where $K$, $d_{sp}$ and $dr$ are defined by (2.5b-d), and

$$d_{by} = -(S_m^{\mathrm{T}} Q_1 S_m + Q_2)^{-1} S_m^{\mathrm{T}} \lambda_1 / 2.$$

The derivation of (4.9) is presented in the Appendix. Here Ai is a vector of Lagrange multipliers for the active constraints. The main difference with respect to the quadratic penalty treatment is that the multiplier Ai replaces now the penalty parameter r in the feedback law. The presence of each active set introduces consequently a different bias term *dby* in the control law, dependent on the correspondent Lagrange multipliers. It is also noted in the Appendix that |Ai|oo $\leq$ p, which imposes an upper bound on the magnitude of the term *d^*. We therefore have the following result:

**Property 1 The** ***control law (4.9), for the exact penalty relaxation of the output constrained control problem with a*** **finite** ***p, has identical stability properties to the corresponding unconstrained control law (2.3a).***

This result is immediately established using the boundedness properties of $d_{sp}$, $d_r$ and *d\yy,* and noting that the feedback term is identical to the unconstrained case. As **a** consequence, the approach becomes considerably simpler to apply, especially with time-varying and nonlinear systems, when compared with the previous quadratic constraint relaxation strategy. Moreover, since it is possible to use arbitrary values for the penalty parameter *p* without changing the stability properties of the formulation, the original constrained solution can be approximated much closer, or even recovered for sufficiently large values of p, provided that the original hard constrained problem is stable. The following example illustrates this behavior.

**Example 4.3** Consider again the linearized model of a FCC unit described in Example 3.3. The previous analysis indicated that stability problems can occur in the presence of hard constraints, if both $u_2$ and *y* saturate simultaneously. This situation is illustrated in Figure 9, where we plotted the closed-loop response for $y_{8p}$ = [o.3 0.2]$^{\mathrm{T}}$ , with $m$ = $p = 20$, **using an initial condition** $x_0 =$ 0.2793 0.3000 0.0564 0.3200J , and remaining parameters identical to the previous example. The hard constraints are in this case $u<i$ $\leq$

—0.0397 and ţ/i $\leq$ 0.3. As can be observed, the controller starts by bringing $y_2$ closer to its set-point, which induces the appearance of oscillation, since both of the hard constraints become active. This causes the ouputs to move away from the set-points after $t$ = 0.25 (because of the finite accuracy used in the computation of the solution). The outputs converge then very slowly to their respective reference values, indicating that this problem might occur again in the future.

[Figure 9 about here.]

The correspondent results using a $l\backslash$ penalty relaxation of the output constraint are shown in Figure 4.3. In contrast with the hard constraint situation, the profiles do not show any visible oscillation, and the desired set-points are reached at the end of the first interval. This example illustrates therefore the ease of use of the $l\backslash$ penalty formulation. In contrast with the quadratic penalty formulation, no additional stability information is required now to choose the value of $p$ (except knowing that the input only constrained system is also stabilized by the present tuning parameters).

[Figure JO about here.]

A particular characteristic of this formulation is that in certain situations, as in the next example, large values of $p$ can produce undesirable steady-state offsets in the closed-loop response, due to the receding nature of the MPC law. These offsets correspond also to a violation of the original constraints, but they can be eliminated simply by increasing the length of predictive horizon used. More precisely, this property can be stated as follows:

**Property 2** ***The control law (4.9), correspondent to the exact penalty relaxation of the output constrained control problem (4.8), exhibits no steady-state offsets for a perfect model, and any finite value of the penalty parameter p, when the length of the output predictive horizon goes to in&nity.***

This property can be demonstrated by noting that with an infinite output horizon, the objective function (4.8a) can only be made finite if the last input in the control profile is able to satisfy the limit of the set-point trajectory ţ/*$_{sp}$, at some point in the output horizon. Therefore, as long as the set-point is feasible and reachable, the optimal solution will have no permanent constraint violation, since the existence of at least one feasible point corresponding to a finite objective is guaranteed in this case. Hence by choosing a sufficiently large output horizon, it becomes possible to use any value of p, in order to limit the error resultant from constraint relaxation. An additional way of eliminating these offsets is the use of integral action in the controller, e.g. as considered by Oliveira and Biegler (1993). The following example illustrates this behavior with different horizon lengths.

**Example 4.4** Consider again the system of Example 4.2. Using the same conditions as in the previous case, we plotted the control profiles correspondent to predictive horizons of length $m = p = 5$ in figure 11. The closed-loop system is stable for all values of the penalty parameter, although high values of $p$ produce a steady-state offset, as mentioned. However, when the length of the predictive horizon is increased to 10 intervals (in Figure 12), this problem disappears, and all curves reach now the desired set-point.

[Figure 11 about here.]

[Figure 12 about here.]

Similar to the quadratic penalty case, the stability properties with a mixture of both hard and soft constraints can be related back to the stability of the correspondent hard constrained only system. This is considered in the following theorem.

**Theorem 5** *Consider the constrained l\ penalty formulation, with a mixture of hard and soft output constraints*

$$\min_{U,\epsilon} \quad J_{2Mlix} = (E^* - SnUfQiiE^* - S_mU) + (U - U_r)^T Q_2 (U - U_r) + r^T e \tag{4.10a}$$

$$s.t. \quad Ui \leq U \leq U_u \tag{4.10b}$$

$$Yi_h \leq Y_h \leq Y_{uh} \tag{4.10c}$$

$$Yi_s - e \leq Y_s \leq Y_{us} + e \tag{4.10d}$$

$$e \geq 0. \tag{4.10e}$$

The *stability properties of the mixed constrained problem (4.10) are exactly identical to the stability of the correspondent hard constrained only problem (4.10a-c), with* $\epsilon = 0$.

The proof for this theorem can also be found in the Appendix. The identical stability properties of the $l_1$ penalty to the hard constraint only case make therefore the exact penalty approach considerably easier to use than the quadratic penalty formulation, since this approach doesn't require the knowledge of how certain changes in the tuning parameters affect the stability of the correspondent constrained system.

## 4.3 Constraint relaxation using other penalty formulations

The marked difference in the results derived in the previous sections for the two penalty formulations considered, can be seen essentially as a consequence of the use of quadratic versus linear penalty terms in the objective. The results obtained in this last case can therefore be expected to hold true for other penalty formulations which are linear in $\epsilon$, such as the case of the $l_\infty$ norm. In this case the soft constraint formulation can be expressed as

$$\begin{aligned} \min \quad & J_{2c} = J_2 + \rho\epsilon \\ \text{s.t.} \quad & \underline{Y}_l - \epsilon \underline{e} \leq \underline{Y} \leq \underline{Y}_u + \epsilon \underline{e} \\ & \epsilon \geq 0, \end{aligned}$$

with $\underline{e} = [1 \;\; 1 \;\; \cdots \;\; 1]^T_{1 \times p}$, and $\epsilon \in R^+$ now. A similar analysis to the one developed in the previous section shows that Properties 1 and 2, together with Theorem 5 also apply in the present case. The main difference now is the bound for the Lagrange multiplier $\lambda_i$, which can be shown to satisfy $|\lambda_i|_1 \leq \rho$ instead. Otherwise, the control law is also given by (4.9). The performance of this formulation with the model considered in the Example 4.4 is illustrated in Figure 13, with horizons of length $m = p = 10$, and identical conditions to the previous case. The behavior showed is similar to the one obtained previously with the $l_1$ penalty (including the appearance of steady-state offsets with small horizons). However since the infinity norm only weights the maximum constraint violation observed in the horizon, changes in *p* will induce now essentially the opposite effect in the characteristics of the closed-loop response. For instance making *p* larger will produce a lower peak, resulting in a slower output response, which stays outside the constraint bounds longer. This is a consequence of the inverse response nature of the system. We can also note that, contrary to the behavior of this system with the *h* penalty, all profiles in Figure 13, for p ^ O, have now a peak of smaller amplitude than the corresponding unconstrained response.

[Figure 13 about here.]

# 5 Conclusions

This paper presented a systematic analysis of the stability properties of MPC, in the presence of both hard and soft constraints. A perspective similar to treatment of Zafiriou (1990; 1991a, b) was used, which handles the nonlinearities introduced by the presence of active constraints through separate consideration of each different active set. Some important characteristics of the predictive control law were revealed here for the first time, These include the derivation of an explicit closed form expression for the optimal solution of the predictive problem in the presence of both hard and mixed types of constraints, as well as showing that active output constraints can introduce additional feedback terms in the constrained controller. As proposed above, the algorithm relies on a range and null space decomposition of the matrix of hard constraints, which is well suited for numerical (large scale) computation, and can be implemented in a numerically robust form, e.g. through a QR decomposition. The examples considered show also that, in most cases, the more important stability information relative to the presence of hard constraints can be derived by considering just a significantly smaller subset of constraints than the total number of possible combinations. This allows a considerable reduction in the effort required for a constrained stability analysis of a linear model.

In addition to providing tools for systematic diagnosis of possible stability problems with hard constraints, this paper presented some alternative constraint handling methodologies that enable these problems to be avoided. This is done through a relaxation of

**the problematic hard constraints, using a penalty formulation. This approach is most useful for output constraints, which represent frequently control objectives, rather than rigid limits in the process.**

**Starting with the case of a quadratic penalty, we showed that the stability properties with soft constraints can be related to the stability of the equivalent system with these constraints removed, by a simple change in the tuning parameters used. This corresponds in general to a finite maximum value of the penalty parameter that can be tolerated for stability. Also, this approach still suffers from the disadvantage of being combinatorial in the length of the predictive horizon used. The exact (or *h*) penalty eliminates this last requirement, leading to a constrained formulation that has the same stability properties as in the absence of the soft constraints. Because of the nature of the problem, this last result extends also to other penalty formulations which are linear in the constraint violation term. This is especially true with the case of the Zoo norm, for which a stronger bound for the Lagrange multipliers was shown to exist. This characteristic, together with the requirement of a finite value of the penalty parameter to match the solution of the original problem, simplifies considerably the use of soft constraints, especially for time-varying and nonlinear systems. We believe that the use of this exact penalty treatment has profound consequences on the design of constrained controllers, since it opens the possibility of using essentially all of the available tools for constrained control in this situation as well (e.g., robustness, etc.). This observation is particular pertinent to LTI models, for which a multitude of design methods is currently available.**

# Acknowledgments

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# Appendix

Proof of Theorem 3. The optimality conditions for (4.2) can be written as

$$£ = J_2 + pe^T e + X^T{}_u(S_m U - Y\wedge - e) + \backslash J(-S_m U + Y_t{}^*{}_d - e) - Xje$$

$$V_V C = 2\{SlQ_l S_m + Q_2)U - ttZQxET - 2Q_2 U_r + \langle S£(A_U - A_,) = 0 \qquad \text{(A.la)}$$

$$V,£ = 2pe - A_u - A, - A_2 = 0 \qquad \text{(A.lb)}$$

$$\lambda_{2i}\epsilon_i = 0 \qquad \text{(A.lc)}$$

$$\lambda_{ui}(S_m U - Y^*_{ud} - \epsilon)_i = 0 \qquad \text{(A.1d)}$$

$$\lambda_{li}(-S_m U + Y^*_{ld} - \epsilon)_i = 0 \qquad \text{(A.1e)}$$

$$A_{ui} \geq 0, \quad A« \geq 0, \quad A_{2i} \geq 0, \qquad \text{(A.1f)}$$

with $i = 1, \ldots, n_o p$. Assuming distinct upper and lower output bounds, the following combinations of values for the multipliers **are possible:**

1. $A_u = Aj = A_2 = 0$, $Vi = 1, \ldots, n_o p$. This corresponds to the unconstrained case **(6 = 0).**

2. $X_{ui} = \backslash u = 0$, $X_2 i > 0$. From (A.lc), this implies that $e^* = 0$. However from (A.lb), this implies that $X_{ui} + A/_i < 0$, which contradicts the initial assumption. Hence this combination of multipliers is not possible.

3. $X_{ui} > 0$, $Xu = 0$, $X_2 i = 0$. From (A.lb), this implies that $A\wedge = 2/9c_r$, and consequently $e_{\{} > 0$. In this case we have from (4.2b), $(S_m U)i$ — $Y\wedge + e_u$ and (4.3) can be used in this case with $Ri \wedge 0$, and $Y£_{di} = Y^*_{ud,i}$.

4. $X_{ui} = 0$, $A/* > 0$, $X_2 i = 0$. From (A.lb), this implies that $Xu = 2pe\wedge$ and consequently $e_{\{} > 0$. In this case we have from (4.2b), $(S_m U)i = Y\wedge - e_u$ and (4.3) can be used also in this case with $Ri \wedge 0$, and $Y\wedge = Y^*_{di}$.

5. $X_{ui} > 0$, $Xu = 0$, $A_2t > 0$. FVom (A.lc), $e_{\{} = 0$, and (A.lb) gives $X_{ui} + X_{2i} = 0$, which is inconsistent with the original assumption. This combination of multipliers is therefore not possible.

6. $X_{ui} = 0$, $Xu > 0$, $A_2t > 0$. Similarly to case 5, this leads to $A\wedge + X_2\{ = 0$, which is also inconsistent with the original assumption. Hence this combination of multipliers is not possible.

Hence (4.4) can be used with all consistent combinations of multipliers. This leads to

$$
\begin{aligned}
U_{\min} &= H_a^{-1}(S_m^{\mathrm{T}}Q_1E^{*} + QlUr + pS_m^{\mathrm{T}}RY_{bd}^{*}) \\
&= H_a^{-1}\left(S_m^{\mathrm{T}}Q_1(Y_{sp} - \mathrm{O} + Q_2U_r + {}_PSlR(Y_b - V)\right) \\
&= H_a^{-1}\left(-S_m^{\mathrm{T}}(Q_1 + {}_PR)Y^{*} + SfcQiY^{*} + Q_2U_r + pS\%RY_b\right),
\end{aligned}
$$

and therefore

$$
\wedge\mathbf{min} = (\,S\bar{m}\backslash Ql + pRj\wedge m + Q2) \quad {}^{\cdot}\backslash{}''\wedge\bar{m}\backslash Ql\ {}'+ \rho R)C^{*}x_k + S_m^{\mathrm{T}}Q_1Y_{sp} + Q_2U_r + \rho S_m^{\mathrm{T}}RY_b\Big). \tag{A.2}
$$

Comparing (A.2) with (2.4), we note that the feedback term becomes identical in **both** cases, if we replace $Q\backslash$ in (2.4) by $Qi + pR$. Therefore the stability properties of **the** relaxed controller are identical to the equivalent unconstrained controller with the new tuning parameters. •

**Proof of Theorem 4.** Similar to the hard constraint stability analysis of Section 3, we start by assuming that the present hard active set is known, and given by (3.1). Also, as in the soft-constrained only case, we define $R = \mathrm{diag}\{ri\}$, $i = 1,\ldots,p$, with r» = diag{ry}, $j = 1,\ldots,n_o$, such that[2]

$$
r_{ij} = \begin{cases} 1 & j^{th} \text{ output } \textit{soft} \text{ constraint active at either bound, at } tk+\% \\ 10 & \text{otherwise.} \end{cases}
$$

[2]**Clearly, values of *R* different of zero will only occur now with the elements of $Y_a$.**

This definition allows us to eliminate the soft constraints (4.6d) in the mixed formulation, by substituting them directly in the objective function. Doing so yields the following hard constrained problem

$$\min_{u} \quad J\wedge \wedge = J* + p(S_mU - Y£)^TR(S_mU - Y\&)$$

$$\text{s.t.} \quad ru = u\%$$

$$S\wedge U = Y_b^a - Y*^a.$$

This problem can now be treated using the hard-constrained approach described in Section 3. Since the hard constraints are identical in both cases, we will have the same bases for the range and null subspaces $Q_y$ and $Q_z$ in both situations, as well as the same range space solution $U_y$, given by (3.7). Prom (4.6a), the objective can be expressed as

$$J2a,mix\{U) = \text{constant} + 2a^T{}_8U + U^TH_SU, \qquad \text{(A.3)}$$

with $a. = -SlQ\wedge E* - Q_2U_r - pS\&RY\wedge$ $H_a = S\wedge(Qi + pR)S_m + Q_2$. The first order coefficient can also be expressed as

$$a_a = -SlQ_x (Y\wedge -Y)- Q_2U_r - pS_m^TR(Y_b - Y^*)$$

$$= S\wedge + {}_PR)V - SfcQiYv- Q_2U_r - pS\wedge RY*. \qquad (AA)$$

Replacing (3.5) in (A.3) gives

$$J2a,n\wedge(U_z) = \text{constant} + 2ajQ_zU_z + 2U_y{}^T\$H_sQ_zU_z + U_ZQ^T{}_ZH_SQ_ZU_Z.$$

Solving $VJ2a,mix(U_z) = 0$ leads to

$$U, = -\{Q^T{}_zH_sQ_z)\text{-}^lQ^T{}_z(a_s + H.QyUy). \qquad \text{(A.5)}$$

Equations (3.7) and (A.5) can finally be combined to give the analytical solution of the mixed constrained formulation. Replacing these equations in (3.5) gives

$$U = K_s x_k + 4,s_P + d_{hr} + d_{hu} + dsy, \tag{A.6}$$

where we have defined

$$H_{pa} = Q?H_a Q_t \tag{A.7}$$

$$B_a = Q_l H;_a{}^i Q^T{}_z \quad . \tag{A.8}$$

$$K. = -(/_{n<m} - B_a H_a) Q_y K_y C'^a - B_8 Sl\{Qi + pR) C* \tag{A.9}$$

$$d_{sy} = (I_{n_i m} - B_s H_a) Q_y K_y Y_b{}^a \quad + \rho B_s S\, lRY_b, \tag{A.10}$$

with $d/_{bS}p$, $dhr$ and $dh_U$ defined by (3.11d-f). Comparing now (A.6) with (3.10), it is possible to observe that the feedback term becomes identical in both cases if we replace $Qi$ in (3.10) by $Qi + pR$. Therefore the stability properties of the mixed formulation are equivalent to the stability of the correspondent hard constraint only formulation, with the new tuning parameters. D

**Derivation of (4.9).** We start with the optimality conditions of (4.8)

$$£ = J_2 + {}_r{}^T€ + Xl(Y* + S_m U - Y_u - e) + \backslash J\{-Y* - S_m U + Y_t - e) - Aje$$

$$\begin{aligned} \nabla_U \mathcal{L} &= \nabla J_2 + S_m^T(\lambda_u - \lambda_l) \\ &= 2\{SlQ_x S_m + Q_2) U - 2S£QiE* - 2Q_2 U_r + <S£(A_U - A,) = 0 \end{aligned} \tag{A. lla}$$

$$V_t £ = r - \backslash_u - \backslash i - A_2 = 0 \tag{A.11b}$$

$$\lambda_{2i}\epsilon_i = 0 \tag{A.11c}$$

$$\lambda_{ui}(Y - Y_u - \epsilon)_i = 0 \tag{A.11d}$$

$$Xni\text{-}Y + Yt\text{-}ejt\text{^}O \tag{A.lle}$$

$$A_{tt}i \geq 0, \quad A_K \geq 0, \quad A_{2i} \geq 0, \tag{A.11f}$$

with $i = 1, \ldots, n_o p$. From (A.llb,f) we conclude immediately that

$$|A_u|oo \leq P, \qquad |A/|oo \leq P-$$

Assuming distinct upper and lower output bounds, together with the non-degenerancy of the problem[3], the following combinations of values for the multipliers need to be considered:

1. $X_u = Xi = A_2 = 0$, Vz $= 1, \ldots, n_o p$. This is inconsistent with (A.llb,f), and therefore this combination of multipliers is not possible.

2. $X_U i = Xu = 0$, $X_2 i > 0$. Prom (A.llc), this implies that e, = 0, and from (A.llb) we have $A_{2i} = p$. This corresponds therefore to the unconstrained case.

3. $A_U t > 0$, $Xu = 0$, $A_2 t = 0$. Prom (A.llc), this implies that e; > 0, and we will have some constraint violation. Prom (A.llb), we obtain $A_u$» = p. In this case (A.lla) becomes

$$V_V C = V J_2 + \text{<S£Ax} = 0, \tag{A.12}$$

with Ait = $A_u$i.

4. $X_U i = 0$, A/t > 0, $A_2 j = 0$. Prom (A.llc), this implies that e* > 0, and we will have some constraint violation. Prom (A.llb), we obtain $Xu = p$. In this case (A.12) applies with AH = —*XU*.

5. $X_{ui} > 0$, $X_H = 0$, $A_{2i} > 0$. FVom (A.llc), $e_{\{} = 0$. Also from (A.lld), Y- = $Y_{ui}$, which corresponds to the exact solution of the original constrained problem. In this case (A.12) is also valid, with A^ = $A_u$* < $p$.

6. $X_U i = 0$, $Xu > 0$, $A_{2i} > 0$. Similarly to case 5, this corresponds to the exact solution of the original constrained problem, now at the lower bound. Here (A.12) is also valid with $Xu$ = —A* < p.

---

[3]**Le., there axe no redundant constraints (either linearly dependent, or with the same solution as the unconstrained formulation).**

Hence (A.12) is also valid for all consistent combinations of constraints. Ai G $R^{nop}$ can now be defined as

$$\lambda_{1i} = \begin{cases} 0 & \text{if } A_{ui} = X_u = 0 \\ Ki & \text{if } A_{ui} > 0 \\ -A,_t & \text{if } A\text{»} > 0 \end{cases}, \quad i = 1, \dots, n_o p.$$

Solving (A.12) for $U$ gives finally

$$U_{\min} = (S_m^T Q_1 S_m + Q_2)^{-1}(S_m^T Q_1 E^* + Q_2 U_r - S_m^T \lambda_1 / 2).$$

D

**Proof of Theorem** 5. As previously, we start by assuming that the present hard active set is known, and given by (3.1). Defining $Y_s = Y_s^* + S_{ma}U$ (i.e., the outputs for which soft constraints are specified), the optimality conditions of (4.10) can be written as

$$C = J_2 + r^T e + \lambda_{su}^T(Y_s^* + S_{ms}U - Y_{us} - e) + \lambda_{sl}^T(-Y_s^* - S_{ms}U + Y_{ls} - e)$$
$$+ \lambda_{hu}^T(I^a U - U_b^a) + \lambda_{hy}^T(S_m^a U - Y_b^a + Y^{*a}) - \lambda_2^T \epsilon$$

$$\nabla_U \mathcal{L} = \nabla J_2 + (I^a)^T \lambda_{hu} + (S_m^a)^T \lambda_{hy} + S_{ms}^T(\lambda_{su} - X_{sl}) = 0 \tag{A.13a}$$

$$\nabla_\epsilon \mathcal{L} = r - A_{su} - X_s i - A_2 = 0 \tag{A.13b}$$

$$\lambda_{2i}\epsilon_i = 0 \tag{A.13c}$$

$$\lambda_{su,i}(Y_s - Y_{us} - \epsilon)_i = 0 \tag{A.13d}$$

$$\lambda_{sl,i}(-Y_s + Y_{ls} - \epsilon)_i = 0 \tag{A.13e}$$

$$\lambda_{su,}i \geq 0, \quad X_a,i \geq 0, \quad A_{2i} \geq 0, \tag{A.13f}$$

with $i = 1,\dots, n_{os}$, where $n_{os}$ represents the total number of output variables for which soft constraints are specified. Prom (A.13b,f) it is also possible to conclude that

$$\|\lambda_{s}\|_\infty \leq \rho, \qquad \|\lambda_{sl}\|_\infty \leq \rho. \tag{A.14}$$

Considering now the possible values for the multipliers A^, A_sj and A2, in a similar fashion to the derivation of (4.9), shows that (A.13a) can also be expressed in the form

$$\nabla_U \mathcal{L} = \nabla h + (I^a)^T \lambda_{hu} + (\wedge)^T \lambda_{hy} + S_{ms}^T \lambda_1 = 0, \tag{A.15}$$

by defining in this case

$$\lambda_{1i} = \begin{cases} 0 & \text{if } \lambda_{su,i} = \lambda_{sl,i} = 0 \\ \lambda_{su,i} & \text{if } \lambda_{su,i} > 0 \\ -\lambda_{sl,i} & \text{if } \lambda_{sl,i} > 0 \end{cases}, \quad i = 1,\dots, n_{os}.$$

Comparing (A.15) with the optimality conditions of the hard-constraint only case, it is possible to observe that the the only difference between these two cases is the presence of the additional bias term $S_{ms}^T\lambda_1$ in (A.15). Since this term doesn't affect the feedback gain and is bounded in magnitude according to (A.14), the mixed constraint formulation possesses therefore identical stability properties to the correspondent hard constraint only formulation. •

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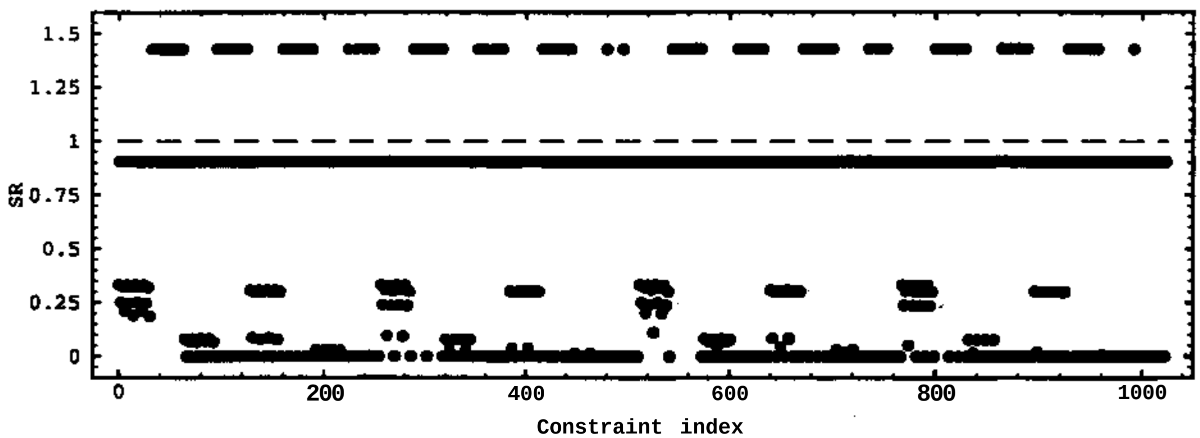



**Figure 1: Closed-loop spectral radius (SR) of Example 3.1 for all possible combinations of constraints.**

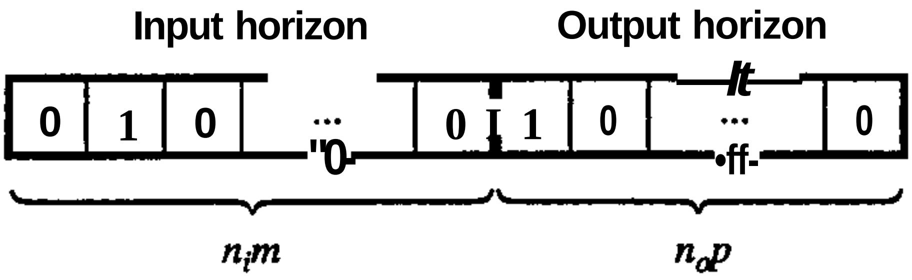



Figure 2: Building the constraint index.

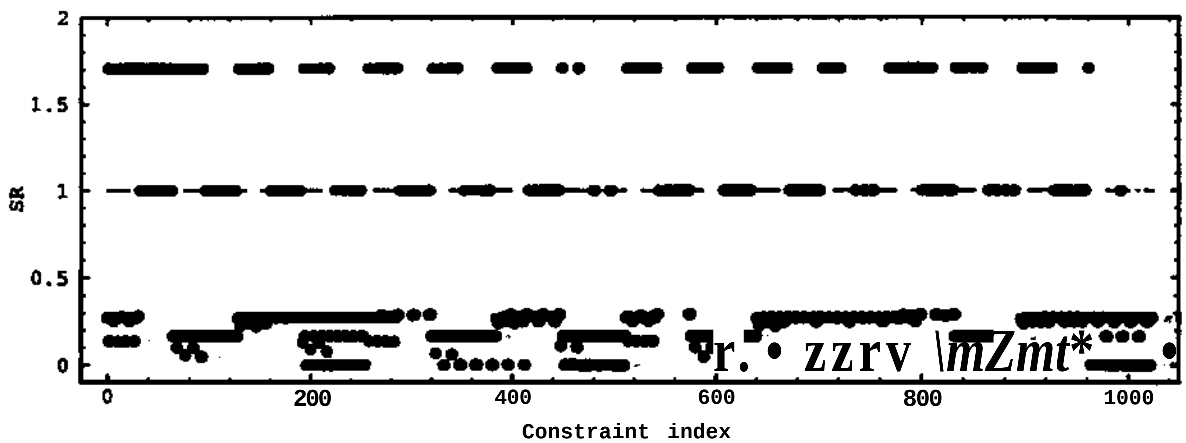



**Figure 3: Closed-loop spectral radius (SR) of Example 3.2 for all possible combinations of constraints.**

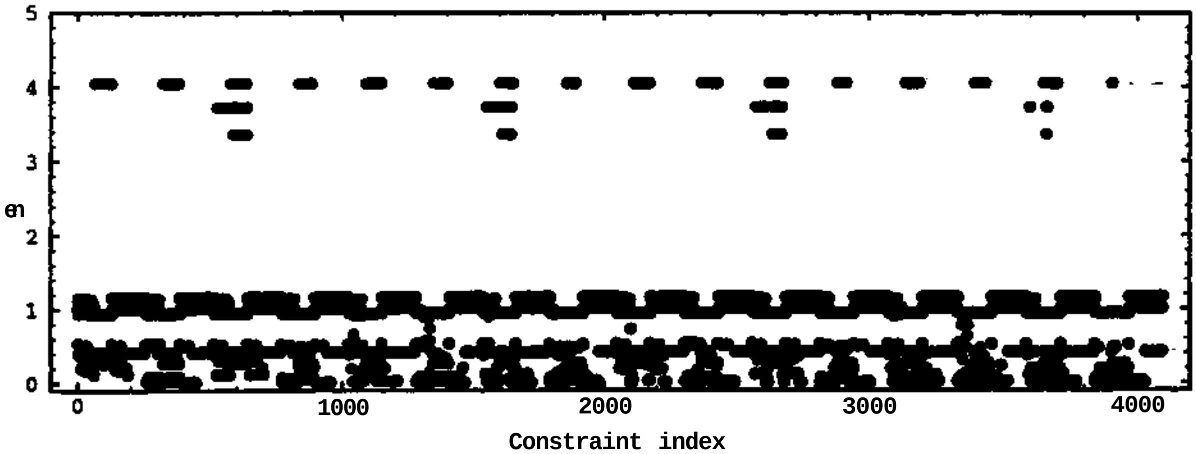



**Figure 4: Closed-loop spectral radius (SR) of Example 3.3 for all possible combinations of constraints.**

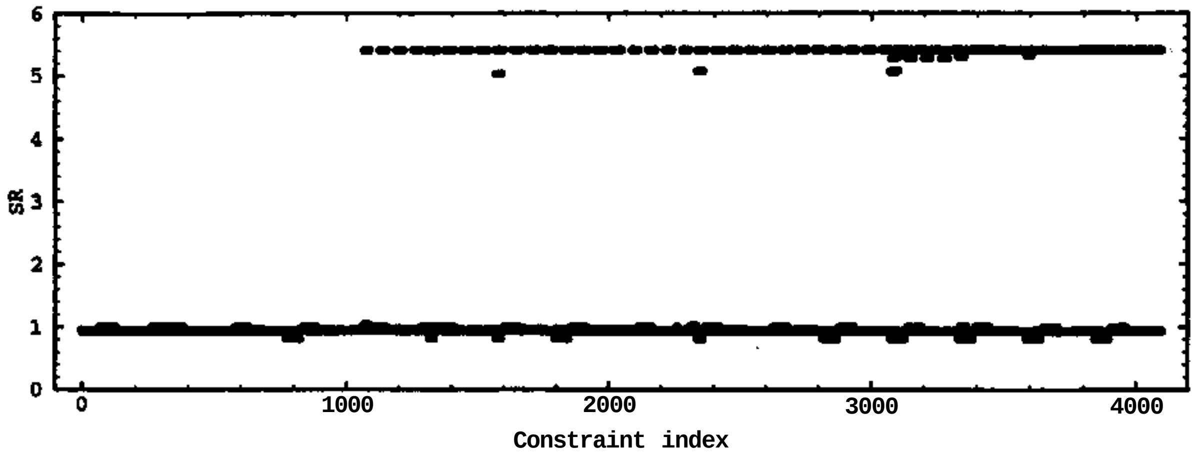



**Figure 5: Closed-loop spectral radius (SR) of Example 3.5 for all possible combinations of constraints.**

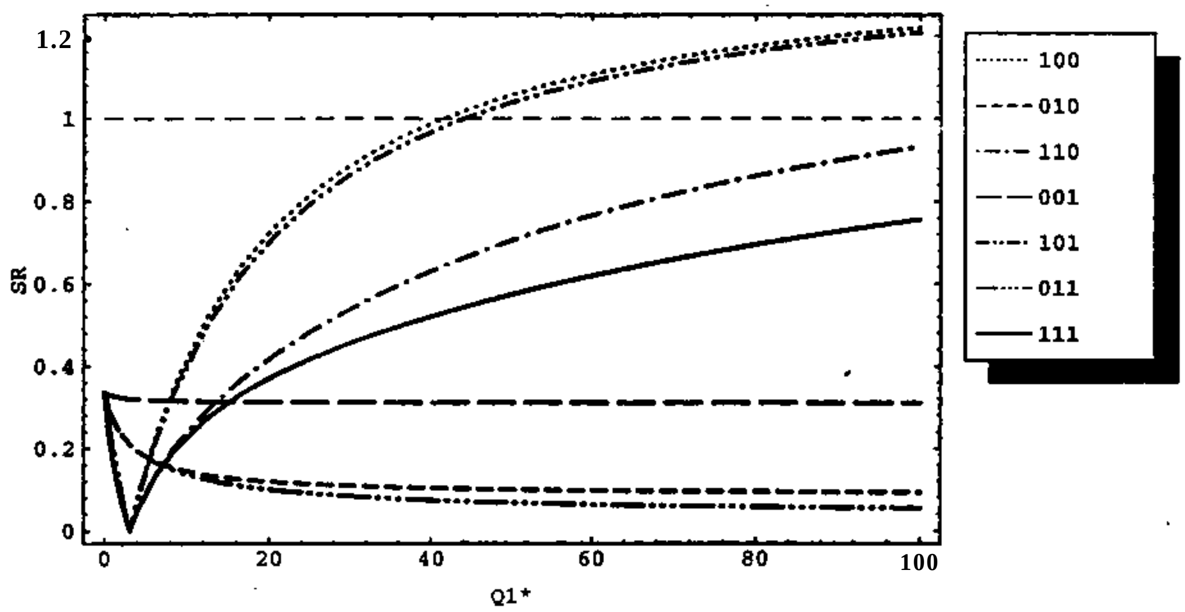



**Figure 6: Effect of the tuning parameter $Q\backslash = Q\backslash + pR$ on the closed-loop spectral radius of Example 3.1, for different active constraint sets.**

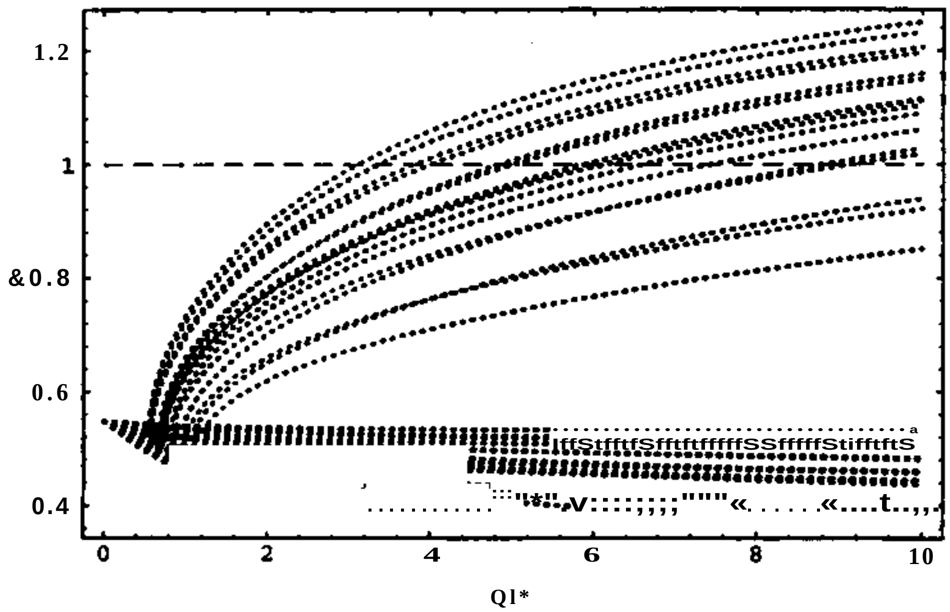



**Figure 7: EflFect of the tuning parameter *Q* = <Ji + p/2 on the closed-loop spectral radius of Example 4.2, for different active constraint sets.**

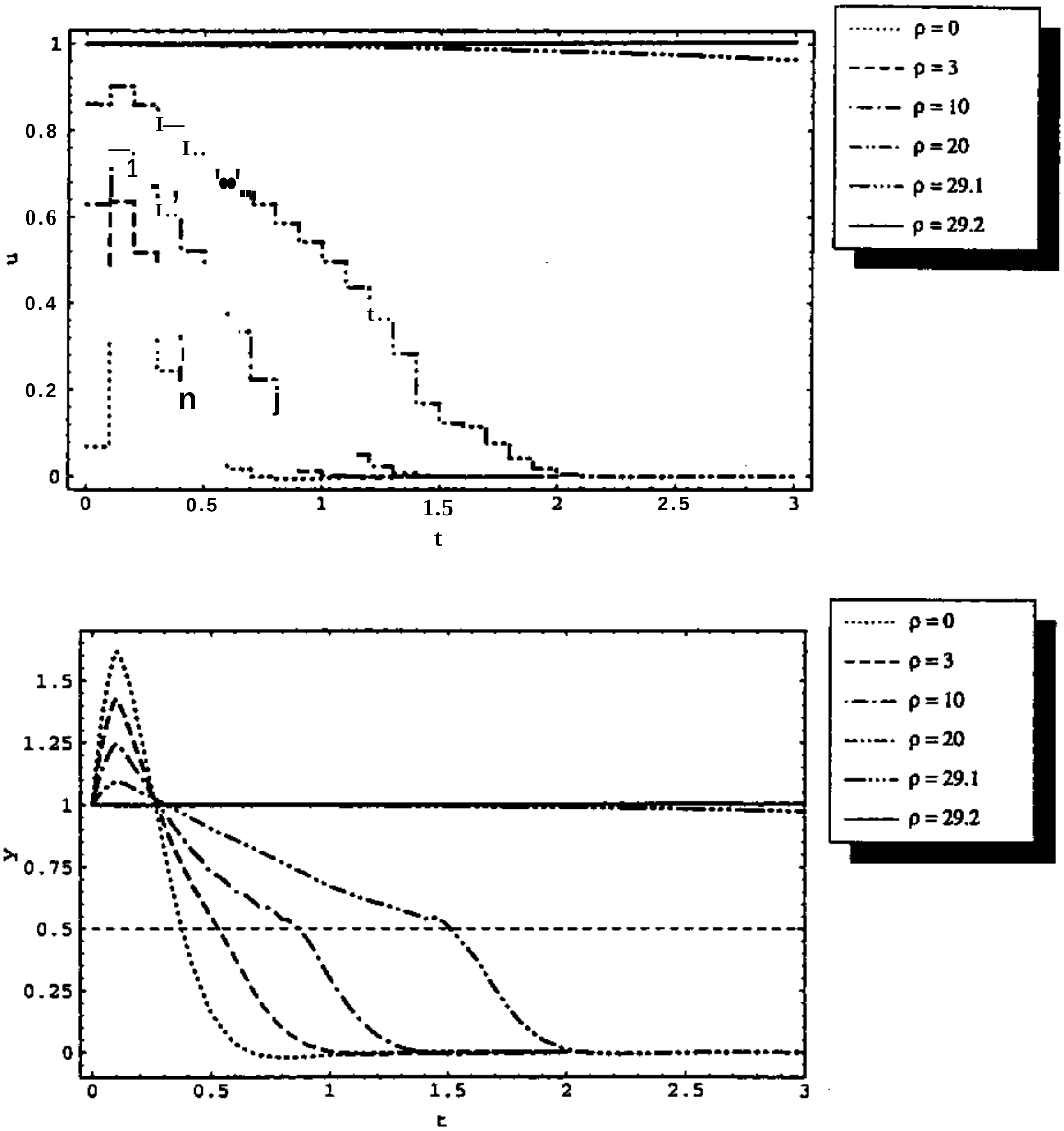
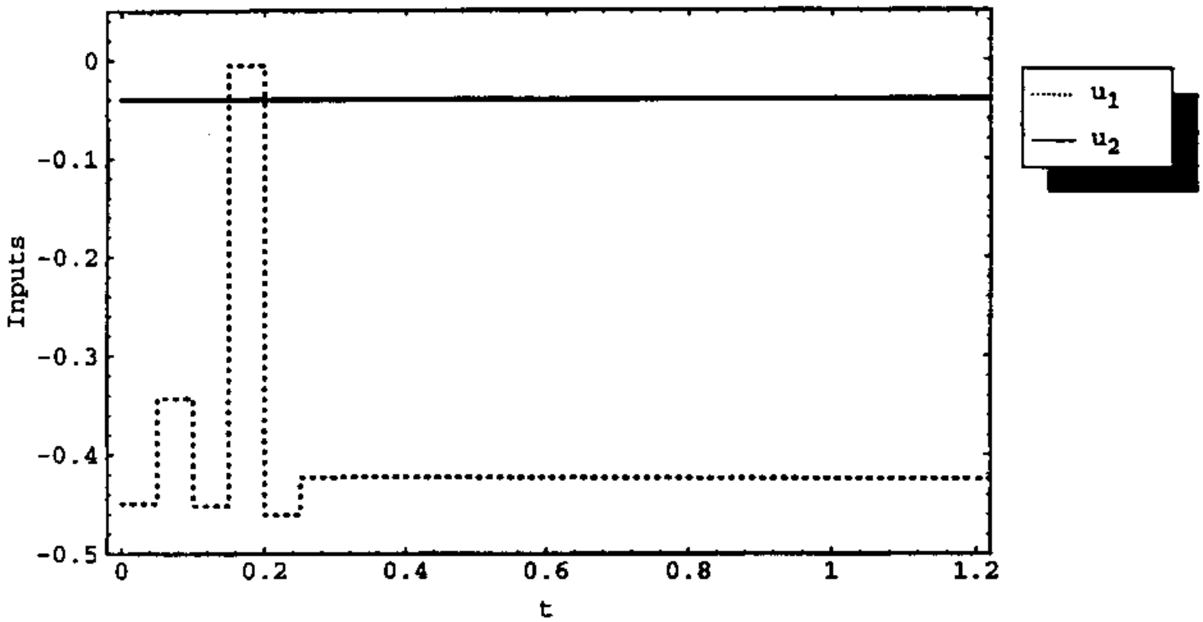



**Figure 8: Input and output profiles for Example 4.2.**



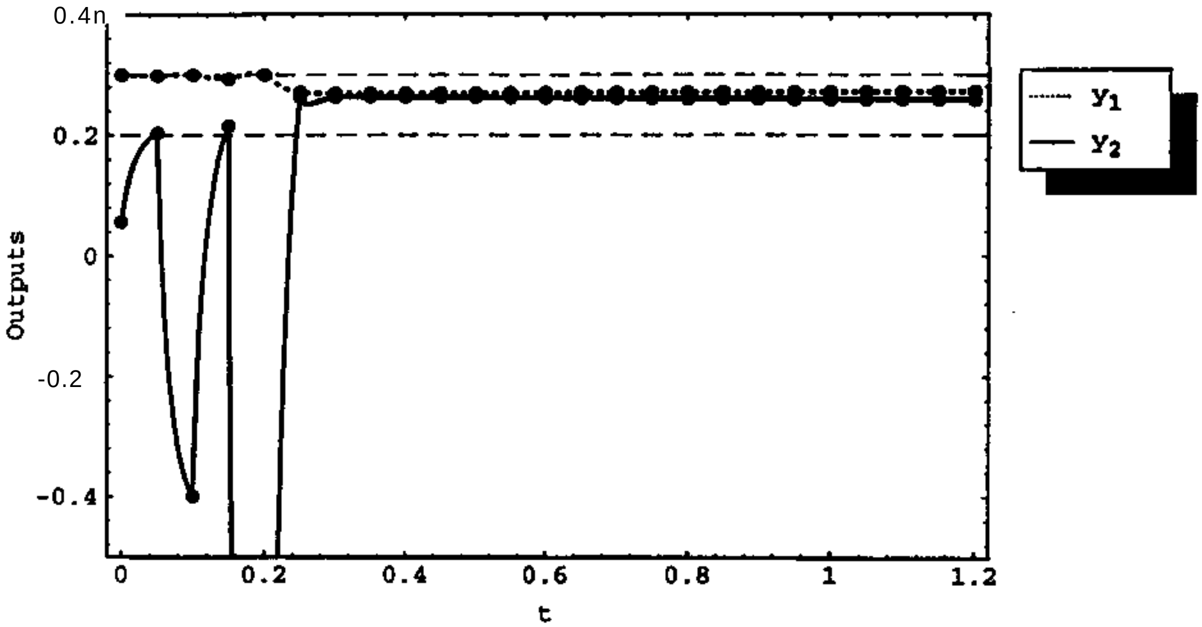




**Figure 9: Input and output profiles for Example 4.3 with hard constraints. The dots indicate the discrete sampling points considered.**

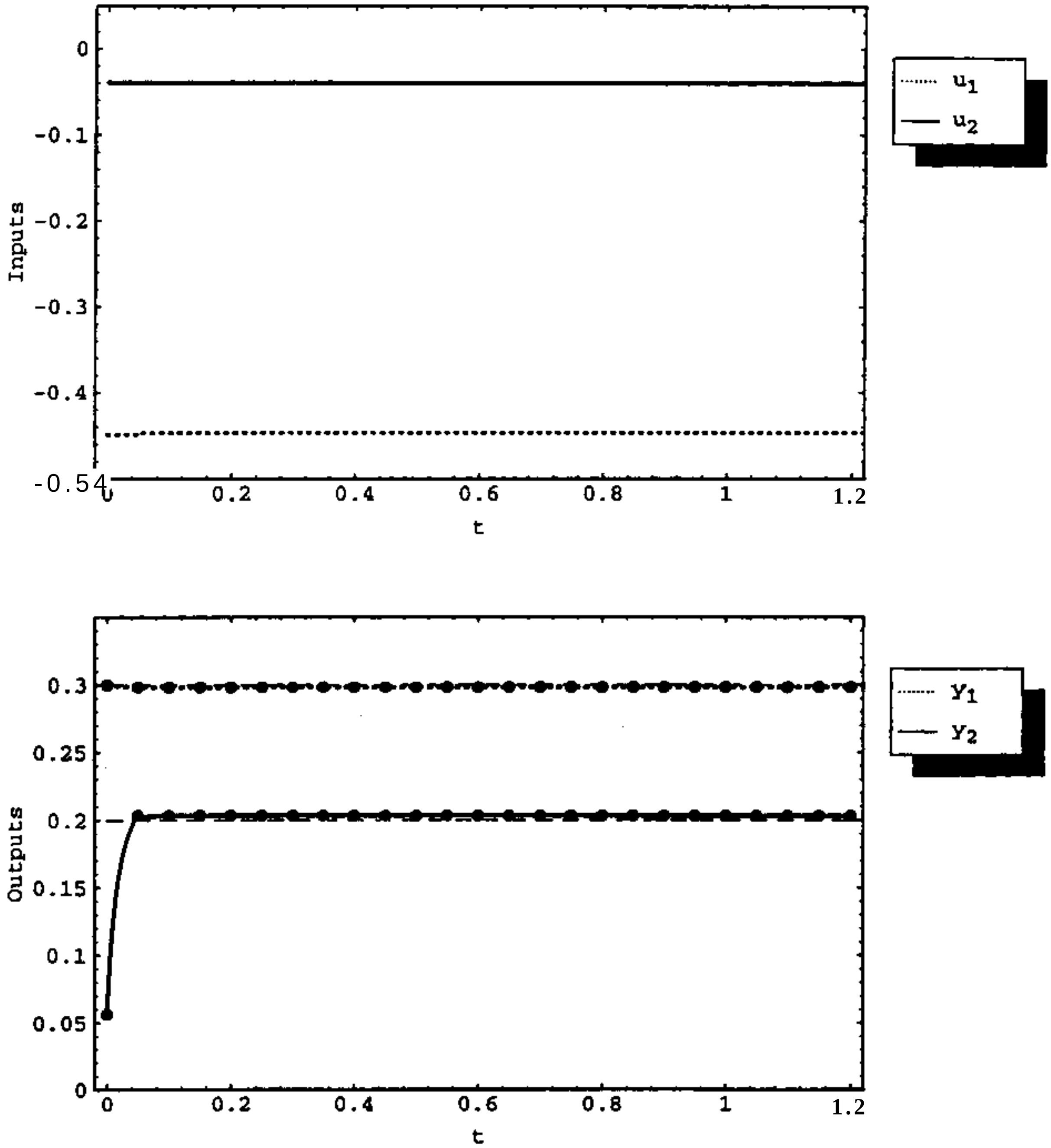
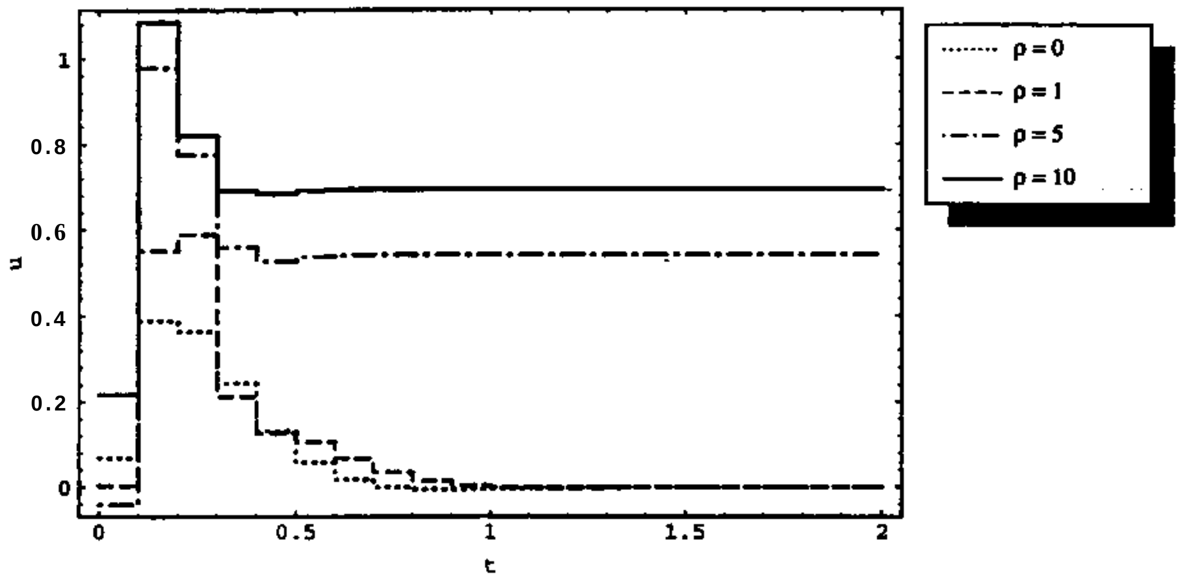



**Figure 10: Input and output profiles for Example 4.3 using the $l_1$ penalty relaxation of the output constraints.**



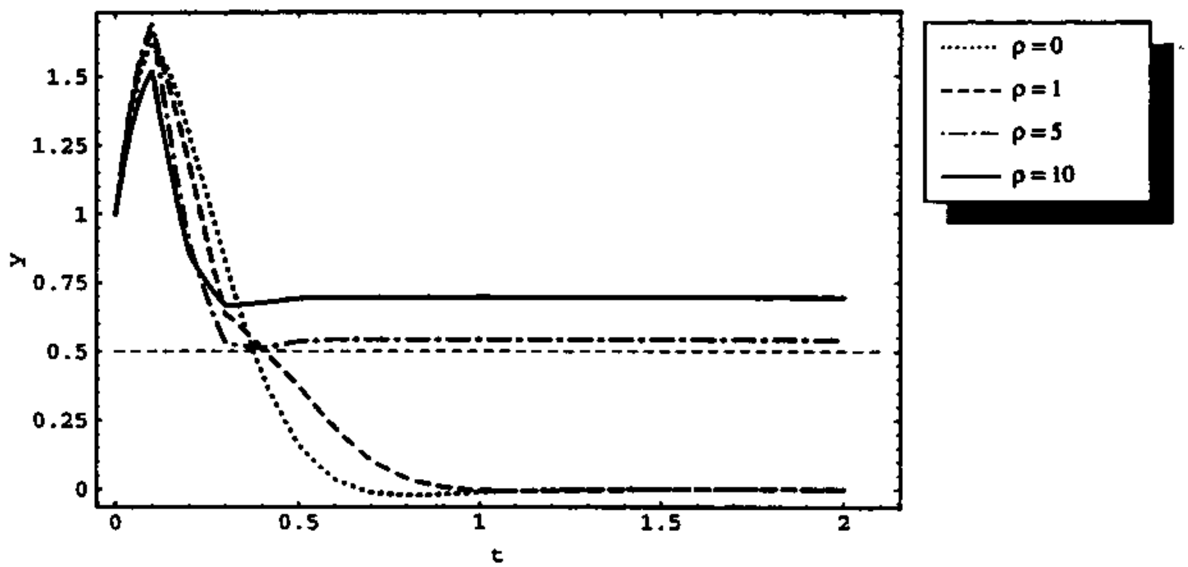




**Figure 11: Input and output profiles for Example 4.4.**

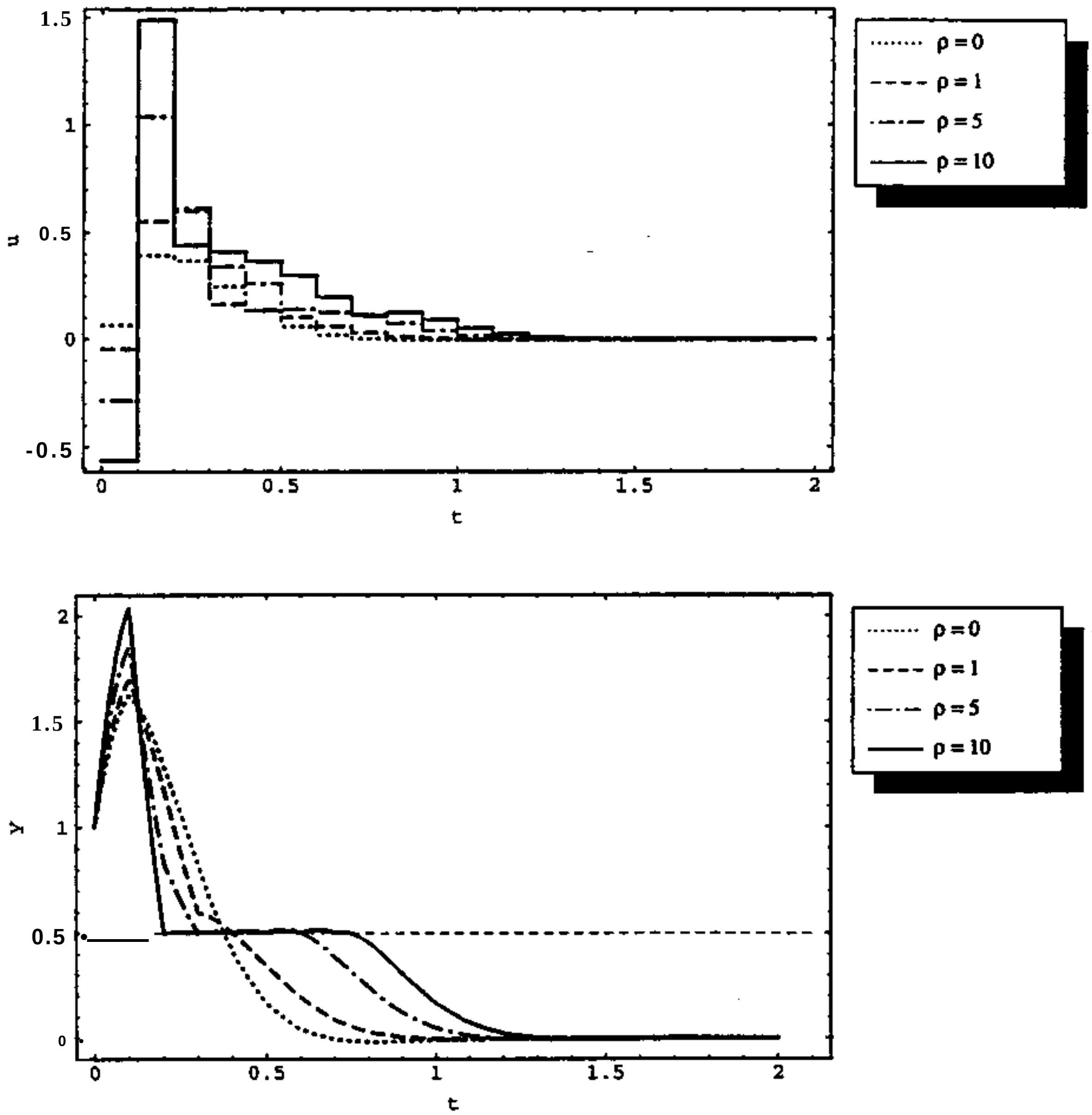



**Figure 12: Input and output profiles for Example 4.4.**

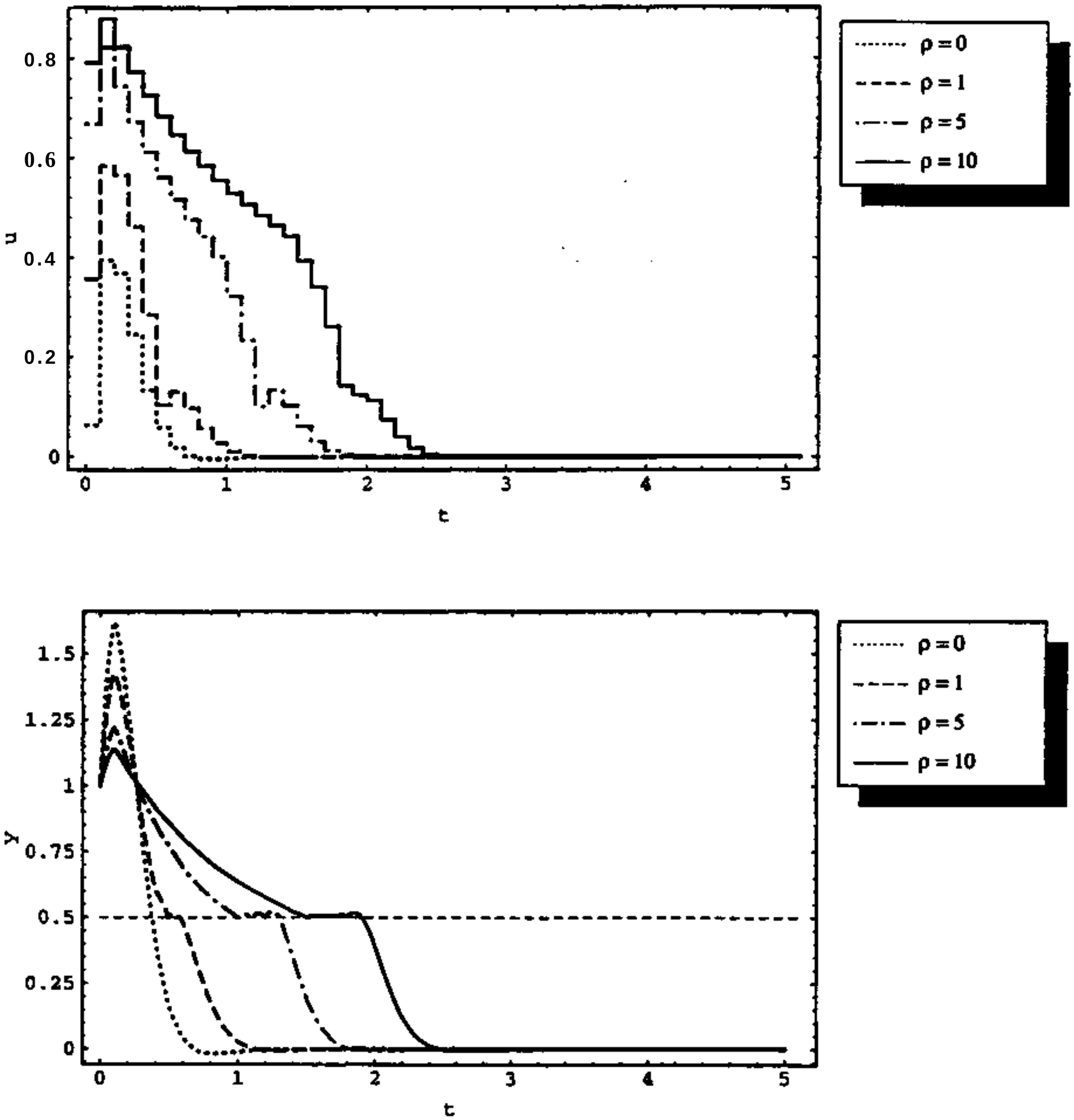



**Figure 13: Input and output profiles for Example 4.4 with the /«> penalty.**

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| Constrained variables | | | | |
| --- | --- | --- | --- | --- |
| *V2* | *y* | $u_2$ | *Ui* | *Psr* |
| no | no | no | no | 0.961 |
| no | no | no | yes | 0.947 |
| no | no | yes | no | 0.953 |
| no | no | yes | yes | 0.953 |
| no | yes | no | no | 0.961 |
| no | yes | no | yes | 0.942 |
| no | yes | yes | no | 0.953 |
| no | yes | yes | yes | 0.942 |
| yes | no | no | no | 0.961 |
| yes | no | no | yes | 0.953 |
| yes | no | yes | no | 0.953 |
| yes | no | yes | yes | 0.953 |
| yes | yes | no | no | 0.961 |
| yes | yes | no | yes | 0.953 |
| yes | yes | yes | no | 0.961 |
| yes | yes | yes | yes | 0.953 |

**Table 1: Closed-loop spectral radius of Example 3.4 for the cases where a given variable is constrained throughout the horizon.**