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N-COMPACT

by

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<u>Introduction</u>. N-compact spaces were introduced by S. Mrowka in [M_1], where the general concept of an E-compact space was defined: given a Hausdorff space E, a space X is <u>E-compact</u> if it is homeomorphic to a closed subspace of E^{m} for some cardinal number m. Thus the I-compact spaces (where I is the closed unit interval) are the compact Hausdorff spaces, the R-compact spaces are the realcompact spaces, and the 2-compact spaces (where 2 denotes the discrete two-point space) are the 0-dimensional compact Hausdorff spaces. The N-compact spaces are those which can be embedded as closed subspaces in N^{m} where N is the set of natural numbers with the discrete topology.

The main properties of E-compact spaces were given in [EM], where it was asserted that the N-compact spaces are precisely the 0-dimensional realcompact spaces. ('0-dimensional' there, as here, means 'having a base of clopen sets.') It is clear that every N-compact space is 0-dimensional and realcompact, but the proof of the converse in [EM] was incomplete. The purpose of this paper is to show that the converse is in fact false, that Prabir Roy's space Δ is a counterexample. It is not N-compact, but it is metrizable of cardinality $2^{\aleph_0}$ and hence realcompact [cf. GJ, p. 232] and it is 0-dimensional.

The space Δ was described by P. Roy in [R_1] and many of its properties were proven in detail in [R_2], including its metrizability and its zero-dimensionality. The fact that Δ is not N-compact is the new result, first established by the author using the proof given below. It is based on the following characterization of N-compactness, first discovered by H. Herrlich

[H, Beispiele 6].

Theorem. A zero-dimensional Hausdorff space X is N-compact if, and only if, every clopen ultrafilter[1] on X with the countable intersection property[2] is fixed.

In what follows, we establish the existence of $2^{\aleph_0}$ distinct free clopen ultrafilters on A, any one of which is enough, by Herrlich's theorem, to establish that A is not N-compact.

Constructing the Ultrafilters. In order to facilitate comparison with [R_2], the numbers of the lemmas and theorems will begin with a 5, and if a lemma closely parallels a lemma or a definition in [R_2]* ^t w i ^ 1 ^e given a similar numbering. (Thus, the first two lemmas follow right from Definitions 5.3.3 and 5.3.4 in [$R_2$1 and are numbered 5.3.5 and 5.3.6; the next three lemmas are like Lemma 5.4 in [$R_2$1* and are numbered respectively 5.4.0, 5.4.1, and 5.4.2. And a later lemma is a generalization of Lemma 5.7 in [$R_2$1 and is numbered 5.7'.

All the notation used in this section, unless otherwise remarked, follows that of [R_2]. Not all the results in [R_2] will be used in proving the lemmas below; while familiarity with [R_2] up to the beginning of Section 5.3, p. 127 is certainly an asset, the reader will be able to get by with considerably less. Specifically, after the end of Section 2.1 only the following facts in [R_2J are used:

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[1] an ultrafilter on the Boolean algebra of clopen subsets of x is called simply a clopen ultrafilter.

[2] Given a cardinal number 4*\ , a filter 3 is said to have the ~~/*H-intersection property~~ if every collection of *n* or fewer sets in 3* has nonempty intersection. The countable intersection property is the case *AM* = N_o.

(1) Lemma 2.7, which shows that the regions R_x and $R_{\langle p,n\rangle}$ form a base for a topology, and which can be proven right after Lemma 2.1;

(2) Lemma 2.8, which shows that the regions $R_x, x \in T_1$ form a local base at a point $p \in P_1$ and the regions $R_{\langle p,n\rangle}, n \in I_2$ form a local base at a point $p \in P_2$, and which can be proven right after Lemma 2.1;

(3) A modest version of Property II [p.126]: given a point $p \in P_1$ let x_n be a member of X with $|x_n| = n$ and such that p extends x_n (this means $x_n(i) = p(i)$ for $i = 1,\ldots,|x_n|$); then $\bigcap_{n=1}^{\infty} R_{x_n} = \{p\}$; similarly, given a point $p \in P_2$, $\bigcap_{n=1}^{\infty} R_{\langle p,n\rangle} = \{p\}$. These facts can be proven right after the definitions of Section 1.

(4) Lemma 5.1, which shows that every R_x and every $R_{\langle p,n\rangle}$ is clopen. This can be proven after Section 2 and does require all the results in that section.

In addition to the notations used in [R] we adopt the following notation: if $x \in X$ and $x' \in X$ [resp. $p \in P_1$] are such that $|x'| \geq |x|$ and $x'(i) = x(i)$ for $i = 1,\ldots,|x|$ [resp. $p(i) = x(i)$ for $i = 1,\ldots,|x|$] then we write $x < x'$ [resp. $x < p$]. Similarly, if $\pi \in \Pi$ and $\pi' \in \Pi$ [resp. $x \in X$] [resp. $p \in P_1$] are such that $|\pi'| \geq |\pi|$ and $\pi'(i) = \pi(i)$ for $i = 1,\ldots,|\pi|$ [resp. $|x| \geq |\pi|$ and $x(i) = \pi(i)$ for $i = 1,\ldots,|\pi|$] [resp. $p(i) = \pi(i)$ for $i = 1,\ldots,|\pi|$] then we write $\pi < \pi'$ [resp. $\pi < x$] [resp. $\pi < p$]. Also, we adopt the convention that if $x \in X$ is the (unique) member of X with $|x| = 0$, then $R_x = A$.

For the convenience of the reader, the following definitions in [R~ z] are repeated, with the above notation used where possible.

5. 3.1. 7r = the set of all finite sequences of positive real numbers, defined on initial segments of the set of positive integers. If *ireH,* then |TT| = the greatest integer for which *IT* is defined.

5.3.2. K is an <u>indicator</u> means that K is a subset of *IT* with

(1) if 7r,7r^{f}fK, then |ir| = |TT^f| ^ and that integer is denoted by |K|

(2) {r:r = 7r(l) for some *veK)* is an infinite set, and

(3) if TreK and j is a positive integer with j < |TT|, then [r:r = TT^T(J+1) for some TT'GK with 7r^T(i) = 7r(i) for i = l,...,j} is an an infinite set.

5.3.3. If K is an indicator and xeX then

E(K,x±) = [R.,,:|x^T| = |x| + |K|,X < x^T, and for some TreK,

x^f (|x|+i) = ±7r(i) for i = 1,..., |K| }.

(Remark: this is a composite definition; the things actually defined are £(K,x+) and £(K,x-).)

5.3.4. If K is an indicator and R, $_N$€F$_O$ then (p^n) 2

E(K,(p,n)) = {R(q,$_m$^:m = n + |K|,q€R?p,n) , and for some

7T€K,q$_z$(n-l+i) =7r(i) for i=l, . . ., | K|) .

5.3.5. <u>Lemma</u>. Let K be an indicator.

(E(K,(p,n)))* c: R$_x$ « R$_{(p^n)}$e ^

<u>Proof</u>. ^: (E(K, (p,n)))* is a union of subregions of R, $_N$.

⇒:(Σ(K, (p,n)))* c R$_X$ =, R$_X$ n R?$_{(p*n)}$> ^ *0* =» R^$_X$ 3 R$_{(p,n)\#}$

5.3.6. Lemma. Let K be an indicator

(1) $(\Sigma(K,x+))^* \subset R_{(p,n)} - R_x \subset R_{(p,n)}$

(2) $(E(K,x-))^* \Leftarrow R_{(p,n)} - R_x \wedge R \wedge n$,

Proof. <*:(2(K,x+))* and (E(K,x-))* are both unions of subregions of R_x.

=*: Proof will be given of (1) in the case n > 1, the only case needed here.

Suppose $(\Sigma(K,x+))^* \subset R_{(p,n)}$, then for each $R_{x'} \in \Sigma(K,x+)$, we have $R_{x'} \subset R_{(p,n)}$ and so $R_{x'} \cap R_{y(p,n,\pm)j}$ for some j (2.1.3); furthermore, since $R_{x'} \subset R^{+}_{(p,n)}$ (2.1.3), we have $|x'| \geq |p_x|+n+1$ because $R_{x'}$ contains $q \in P_2$ of length $|x'|$ while all P_2-points in $R^{+}_{(p,n)}$ must have x-coordinates of length at least $|p_x|+n+1$. This implies $R_{y(p,n,\pm)j} \Rightarrow R_{x'}$ for exactly one j (2.1.1).

If $R_x \not\subset R_{(p,n)}$, l^*l £ $IP_x|+n$ (2-1-1)- pick x' such x"(i) = x'(i) for i= 0,...Jp_x|+n,x'(|p_x|+n+l) £ X'(|p_x|+n+l), and $R_{x''} \in \Sigma(K,x+)$. If n > 1 we have q(|p_v|+n+l) = p (|p_v|+n-l) for all $q \in P_j$ ∩ $R_{(p,n)}$. Now let $q' \in P_1$ ∩ $R_{x'}$ and $q'' \in P_1$ ∩ $R_{x''}$. q and q' cannot both be in $R_{y(p,n)}$ contradicting $\Sigma(K,x+) \subset R_{(p,n)}$.

5.4.0. Lemma. Let $\{r_\alpha\}_{\alpha\in Q}$ be an infinite set of positive real numbers. For each r_a let K_a be an indicator, with $|K_\alpha| = n$ for all a. Now let K = {π': π' | = n+1, π'(1) = r_α for some $a \in G$ and, for the same a, π'(1+i) = π(i) for some $\pi \in K_a$ }. K is an indicator.

Proof. (1) is clearly satisfied.

(2) {r:r = π(1) for some π∈K} is an infinite

set, namely $\{r_\alpha\}_{\alpha\in G}$.

(3) is also satisfied: for $j = 1$ it is satisfied because of property (2) of each K_α; for $j > 1$ it is satisfied because of property (3). of each K_α.

5.4.1. Lemma, If it is a collection of regions and R is such that, for uncountably many x' with $|x'| = |x|+1$, $x < x'$, $x'(|x|+1) > 0$ [resp. $x'(|x|+1) < 0$] there is an indicator $K_{x'}$ with $\Sigma(K_{x'},x'+) \subset M$ [resp. $\Sigma(K_{x'},x'-) \subset W$] then there is an indicator K such that $\Sigma(K,x+) \subset U$ [resp. $\Sigma(K,x-) \subset U$].

Proof. Since $\{M : M = |K_{x'}|$ for some $x'\}$ is a countable set, let N be a positive integer such that $|K_{x'}| = N$ for infinitely many x'. Let $\{r_\alpha\}_{\alpha\in G} = \{x'(|x|+1) : |K_{x'}| = N\}$ [resp. $\{-x'(|x|+1) : |K_{x'}| = N\}$] and let K be as defined in the previous lemma. Then $\Sigma(K,x+) \subset U$ [resp. $\Sigma(K,x-) \subset U$].

5.4.2. Lemma. If W is a collection of regions and $R^{(p,n)}$. is such that, for uncountably many $r\in R^+$ there exists $q^r \in R^{(p,n)}$ with $q^r(n) = r$ and an indicator K_r such that $\Sigma(K_r, (q^r,n+1)) \subset U$ then there is an indicator K such that $\Sigma(K,(p,n)) \subset U$.

Proof. Proof is as in 5.4.1.

5.6.1. Lemma. Let $(U_\beta)_{\beta\in B}$ be an open cover for R_x, with $|B| < C$ ($C = 2^{\aleph_0}$). Then there exists $\beta\in B$ and an indicator K such that $(\Sigma(K,x+))^* \subset U_\beta$, and there exists $\gamma\in B$ and an indicator K' such that $(\Sigma(K',x-))^* \subset U_\gamma$.

Proof. Given $\beta\in B$ let $\mathcal{U}_\beta = \{R_{x'} : R_{x'} \subset U_\beta\}$. We give the proof for the existence of a β and a K as defined above (the proof for γ and K' is analogous) by applying 5.4.1 to each

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Suppose no such pair β,K exists. Then, for each j8efl there are at most countably many $x^f > x$ with $|x^!| = |x|+1, x^f(|x|+1) > 0$, for which there is an indicator K_{x}, with

$$(\Sigma(K_{x'}, x'+))^* \subset U_\beta .$$

Hence, altogether, there are fewer than C distinct $x^1 > x$ with $|x^!j = |x|+1, x^T(|x|+1) > 0$, for which there is some U_β and an indicator K_{x^f} with

$$(\pounds(K_{xf}, x<+))^* \subset U_\beta .$$

Now, pick any $x_1 > x$ with $|x_1| = |x|+1, x_1(|x|+1) > 0$ for which no such pair exists. The process repeats: there are fewer than C distinct x" with $|x"| = |x|+2, x^M > x_1, x^{11}(|x|+2) > o$ for which there is some U_p and an indicator $K_{x''}$ with

$$(E(K_{x''}, x''+))^* \subset U_\beta .$$

Pick any $x_2 > x_1$ with $|x_2| = |x|+2, x_2(|x|+2) > o$ for which no such indicator exists.

In this way we get a nested sequence

$$\bar{x} \quad \bar{x}_1 \quad \bar{x}_2$$

and the unique point $p \in P_1$ such that $x_n < p$ for all n is in their intersection. For no n is it true that

$$(\Sigma(K_{\bar{x}_n}, x_n+))^* \subset U_\beta \quad \text{for some } \beta \text{ and some } K_{\bar{x}_n} .$$

And, a fortiori, none of the R_{x_n} is contained in U_β for any β. But this violates the hypothesis that the U_β constitute as open cover of R_x, for $p \in R_x$ and the R_{x_n} form a local base at p (2.8).

5.6.2. Lemma. Let £u/?}/Jeft ^e an oPen cover for $R_{(p,n)}$ with $|H| < C$. Then there exists $\beta \in B$ and an indicator K such that $(E(K, (p,n)))^* \subset U_\beta$.

Proof. Proof is as in 5.6.1. What we obtain is, inductively, a sequence of $q^m \in R^\circ_{(q^{m-1}, n+m-1)}$ such that for no β and no indicator K_m is it true that

$$(\Sigma(K_m, (q^m, n+m)))^* \subset u^\wedge.$$

and we thus get a nested sequence

$$R_{(p,n)} \supseteq R_{(q^1,n+1)} \supseteq \cdots \supseteq R_{(q^m,n+m)} \supseteq \cdots$$

and now define $q \in P_2$ with $q_x = P_x, q_y = P_y, q_z(i) = p_z(i)$ for $i = 1,\ldots,n-1, q_z(n+m-1) = q^m_z(n+m-1)$. It is easy to see that $R_{(q,n+m)} = R_{(q^m,n+m)}$ and so the above nested sequence forms a local base at q, and its intersection is $\{q\}$, and $q \in R^\circ_{(p,n)}$. But for no m is it true that $R_{(q,n+m)} \subset u_\beta$ for some β, violating the assumption that the U_β form an open cover for $R_{(p,n)}$.

5.7. Lemma. If K is an indicator, and $x = y(p,n,\pm)j$ with $j \geq |K|+1$, then

$$(X(K,x))^* \subset (E(K,(p,n)))^*$$

while if $j < |K|+1$, $(L(K,x))^* \cap (S(K,(p,n)))^* = \emptyset$.

Proof. First, if $j \leq |K|$, it can be easily seen from the definitions that $R_x \cap R_{(q,m)}$ is empty, for all q in $S.S^\wedge j$ and hence a fortiori we get the result above. For the remainder of the proof, assume $j \geq |K|+1$.

For each $\pi \in K$, let $x_\pi \in X$ such that

$$|x_\pi| = |x|+|K|, x_\pi > x, x_\pi(|x|+i) = \pi(i) \quad \text{for } i = 1,\ldots,|\pi|,$$

and let $q_\pi \in P_2$ be such that $q_\pi > (p,n), q_{\pi z}(n+i-1) = \pi(i)$ for $i = 1,\ldots,|\pi|$.

We have then by 5.3.3 and 5.3.4 that if $w \in K$ then

$$R_{x_{\pi\mp}} \in \Sigma(K, x\mp) \quad \text{and} \quad R_{\ldots h} \in \Sigma(K, (p,n)).$$

Furthermore, in presence of the definitions of $x_{\pi\mp}$ and q_π, an inspection of 1.3.7-1.3.11 of [R_2] will show that

$$x_{\pi\mp} = \gamma(q_\pi, n+|K|, \pm)_{j-|K|}.$$

Again by 5.3.3 and 5.3.4 we have

$$E(K,x\mp) = \{R_{x_{\pi\mp}} : \pi\in K\}$$

and

$$E(K,(p,n)) = \{R_{(q_\pi, n+|K|)} : \pi\in K\}$$

and so

$$(E(K,x\mp))^* \subset \{R^\mp(q_\pi, n+|K|) : \pi\in K\}^* \subset (E(K,(p,n)))^*.$$

5.13.1. Lemma. Let $\{V_\gamma\}_{\gamma\in G}$ be a partition of R into clopen sets, $|G| < C$. Then there is an $\alpha\in G$ for which there is an indicator K_1 such that $(L(K_1,x+))^* \subset V_\alpha$ and there is a $\beta\in G$ for which there is an indicator K_2 such that $(\Sigma(K_2,x-))^* \subset V_\beta$. Furthermore $\alpha = \beta$ and α is unique.

<u>Proof</u>. 5.6.1 guarantees the existence of V_α and V_β. 5.8 of [R_2] can now be used to show that V^α and V_β must have a nonempty intersection: assume on the contrary $V^\alpha \cap V_\beta = \emptyset$, then apply 5.8 to V^- and V^-; if $(E(K_2,x-))^* \subset V_m$ then <u>a fortiori</u> $(E(K_2,x-))^* \subset V_\beta$ and now let $x_1 > x$ be such that $|x_1| = |x| + 1$ and there exist indicators K_3 and K_4 such that $(\Sigma(K_3,x_1+))^* \subset V_\alpha$, $(E(K_4,x_1-))^* \subset V_\beta$; by induction define $x_n > x_{n-1}$ with $|x_n| = |x_{n-1}| + 1$ so that there exist indicators K_{x_n} and $(L(K_{x_n},x_n+))^* \subset V_\alpha, (\Sigma(K_{x_n},x_n-))^* \subset V_\beta$; in this way we get a nested

sequence $\{R_{x_i}\}_{i=1}^{\infty}$, and each set in the sequence has a nontrivial intersection with V_α and V_α^c. Now $\bigcap_{i=1}^{\infty} R_{x_i} = \{p\}$ for some $p \in P_x$, and since both V_α and V_α^c are closed, p is in both of them, which is absurd. So $V_\alpha \cap V_\beta \neq \emptyset$, and since these sets are members of a partition, $V_\alpha = V_\beta$. Now suppose there exists V_γ, $\gamma \in G$, and an indicator K such that $(\Sigma(K,x+))^* \subseteq V_\gamma$. Then $V_\gamma \cap V_\beta \neq \emptyset$ (same proof as above with γ substituted for α) and so $V_\gamma = V_\alpha$. Similarly, if we have V_γ, $\gamma \in G$ and an indicator K such that $(\Sigma(K,x-))^* \subseteq V_\gamma$ then $V_\gamma \cap V_\alpha \neq \emptyset$ and $V_\gamma = V_\alpha$.

5.13.2. ~~Lemma~~. Let $\{V_\alpha\}_{\alpha \in Q}$ be a partition of $R_{(p,n)}$ into clopen sets, $|G| < C$. Then there is exactly one $\alpha \in G$ for which there is an indicator K such that $[\Sigma(K,(p,n))]^* \subseteq V_\alpha$.

~~Proof~~. Existence of α is shown by using 5.6.2. To show uniqueness, suppose $[\Sigma(K_1,(p,n))]^* \subseteq V_\alpha$, and $[\Sigma(K_2,(p,n))]^* \subseteq V_\beta$. Let $x = y(p,n,i)j$ with $j \geq \max\{|K_1|, |K_2|\}+1$. Then

$$(\Sigma(K_1,x+))^* \subseteq [\Sigma(K_1,(p,n))]^* \subseteq V_\alpha$$

and

$$(\Sigma(K_2,x+))^* \subseteq [\Sigma(K_2(p,n))]^* \subseteq V_\beta.$$

Now the sets $\{V_\alpha \cap R_x\}_{\alpha \in Q}$ partition R_x into clopen sets and hence by 5.13.1, $V_\alpha = V_\beta$.

5.14.1. Lemma. Let $U_x = \{U : U$ is clopen in A and there exists an indicator K with $(\Sigma(K,x+))^* \subseteq U\}$. U_x is a clopen filter.

~~Proof~~; Clearly if $U \in U_x$ and V is clopen in A with $U \subseteq V$, then $V \in U_x$ and clearly $\emptyset \notin U_x$. Now suppose U_1 and U_2 are both members of U_x, with associated indicators K_1 and K_2, respectively. Consider the four clopen sets U.⊥ 0 U2*U-i⊥\U2'U2^U1' and (U1 U U2) „ Their intersection with R_x gives a partition of R_x into clopen sets. Now (£(K,x+))* c R_x for all • k. (U1\u2)eUx iff ((l^V^) n Rx)eux- But

$$(E(K_2, x+))^* \subset (u_2 \cap R_x) \in U_x$$

and since ((U^t^) 0 Rx) n (U2 0 R^) = f>, we cannot have (U^U^el$_x$. In a similar way we eliminate U9\ui an<* (ui ^ ^2^°' sofc^at U, 0 U9€U by 5.6.1.

5.14#2. Lemmao Given P€P2^ let $U_{(p,n)}$ = {u : U is clopen in A and there exists an indicator K with [£(K,(p,n))]* c u}. U . is a clopen filter.

Proof. Proof is as in 5.14.1.

5.15.1. Lemmao U_x is a free clopen ultrafilter with the yffi- intersection property for all M < C, and U_x ^U_{x^f} are distinct whenever x ^ x^f.

Proofo That U is an ultrafilter comes immediately from 5.6.1 and 5.14.1, and we can also use 5.6.1 to show that U has the /w-i.p. : Suppose there is a family of fU } ~ with U eli for all a and |G| £ m < c, then {U^c}a€G forms a clopen cover for R_x and so one of them belongs to U_x and so does its complement, a violation of 5.13.1. To show that U- is free, we show that for no R with |x^f| > |x| is it true that RV»€Uv* and that for no Rf (p,n; with n > 1 and R_{p_x} c R_x is it true that R(p,n)€Ux: if ux w©re fixed it would either be fixed on a point of P_1, in which case it would have as members Rx. for arbitrarily large |x^f|, or else it would be fixed on a point $p \in \overline{P}^*$ in which case it would have as members $R_{(p,n)}$ for arbitrarily high n, and since $p \in R_x \subset U_{x}$ we must also have $R_{p_x} \subset R_x$ by 2.1.1.

First, if $|x'| > |x|$, we have that $p(|x|+1)$ is the same real number for all $p \in R_x$, while by (2) 5.3.2 and 5.3.3 it is possible to find infinitely many p in $(E(K,x+))^* \cap P$ with distinct values for $p(|x|+1)$. Hence $R_{x'} \not\subset R_x$. Second, if $R_{x'} \notin U_{(p,n)}$ then by 5.3.6 we have $R_x \subset R_{(p,n)}$ and by 2.1.2 and 2.1.3 we cannot have $R_{p_x} \subset R_x$. [Note: 5.3.6 was only proven in the case $n > 1$ but this is all we need.]

To conclude the proof of the Lemma, we note that if $x \neq x'$, then in the case $|x| \neq |x'|$ we have either $R_{xx} \notin U_{x'}$ or $R_{x'x} \notin U$ depending on which $|x|, |x'|$ is bigger, while we do have $R_x \in U_x$ for all $x \in X$. If $|x| = |x'|$ but $x \neq x'$, $R_x \cap R_{x'} = 0$ and so $R_{x'} \notin U_x$.

5.15.2. Lemma. $U_{(p,n)}$ is a free clopen ultrafilter with the ω-intersection property for all $M < C$, and $U_{(p,n)}$ are always distinct, while $U_{(p,n)} = U_{(q,n)}$ iff $R_{(p,n)x} = R_{(q,m)x}$.

Proof. To show $U_{(p,n)}$ is free, we show that no R_x with $|x| > |p_x|$ is in $U_{(p,n)}$, and that no $R_{(q,m)}$ with $|q_x|+m > |p_x|+n$ is in $U_{(p,n)}$; freeness follows from the same considerations as above.

If $R_x \in U_{(p,n)}$ then $\exists K$ such that $(\Sigma(K,(p,n)))^* \subset R_x$ which implies **that** $R_{(p,n)} \subset R_x$. By **2.1.1** and 2.1.2 this implies $|x| \leq |p_x|$.

If $R_{(q,m)} \in U_{(p,n)}$ then for some $K, (\Sigma(K,(p,n)))^* \subset R_{(q,m)}$

which implies that $R^?_{(p,n)} \cap R_{(q,m)} \neq \emptyset$. Suppose first $R^?_{(p,n)} \cap R^\circ_{(q,m)} \neq \emptyset$ then we have $|q_x| = |p_x|$. Now all $q' \in P_2 \cap R^\circ_{(q,m)}$, have the same (k)th coordinate for q if $k<m$ while by 5.3.2 and 5.3.4 it is possible to find an infinite set of $q' \in P_9$ (1 (£(K, (p,n)))* with $|q'_x| = |p_x|$ and with distinct values for $a_{c^*}^{f}(n)$. Since the only $q' \in P_9 \cap R_{(q,m)}$ with $|q'_x| = |q_x| = |p_x|$ are those in $R^{(q,m)}$, it follows that if $(£(K(p,n)))^* \subset R^{(q,m)}$ then $m \leq n$, so that in this case $|q_x| + m \leq |p_x| + n$.

Second, suppose $R^{?(p,n)} \cap R^{(q,m)} = \emptyset$, then $R^?_{(p,n)} \cap R_{j(q,m)} \neq \emptyset$ for some j which implies by 2.1.1 that $|q_x| + m + 1 \leq |p_x|$ and so here too $|q_x| + m \leq |p_x| + n$.

The /H-intersection property follows from 5.6.2 and an argument like that in the previous lemma. Similarly, $U_{(p,n)}$ is an ultrafilter.

To show that U_x and $U_{(p,n)}$ are always distinct, note that $R_x \in U_x$ and $R_{(p,n)} \in U_{(p,n)}$ so that if $R_x \cap R_{(p,n)} = \emptyset$ we are done. If $R^\wedge \cap R_{(p,n)} \neq \emptyset$ then $R_x \ni R_{p_x}$, which rules out $R_x \subset R_{(p,n)}$. Hence by 5.3.6 $R_{(p,n)} \notin U_x$ and so the two clopen-set ultrafilters are distinct in this case. If $R_x \cap R^?_{(p,n)} = \emptyset$ but $R_x \cap R_{(p,n)} \neq \emptyset$ then $|x| > |p_x|$ and as shown $U_x \neq U_{(p,n)}$.

Finally, suppose $U_{(p,n)} = U_{(q,m)}$. Then, as shown above, this implies $|q_x| + m \leq |p_x| + n$, and also $|p_x| + n \leq |q_x| + m$, so $|p_x| + n = |q_x| + n$. Also, $R^\wedge_{(p,n)} \cap R^\wedge_{(q,m)} \neq \emptyset$, otherwise $|q_x| + m + 1 \leq |p_x|$ as shown above. This implies $R_{(p,n)} = R_{(q,m)}$ by 2.1.4.

5.16. ~~Property VI~~; A is not N-compact.

The proof consists of either 5.15.1 or 5.15.2 together with H. Herrlich's theorem quoted in the Introduction.

Concluding Remarks> With the problem quoted in the Introduction thus solved in the negative, two other problems, also having their roots in [EM], become better defined.

First, as pointed out in $[M_2]$, what [EM] really showed about N-compact spaces is that if X is realcompact and /?X is zero-dimensional (a strictly stronger condition the zero-dimensionality of X) then X is N-compact. (An explicit statement and proof of this may be found in [H, Beispiele 5,6].) The first unsolved problem before us is the converse of this statement. In other words, given a closed subset X of N ^{m}, /m any cardinal number, is it true that j8X is O-dimensional? (The other condition, realcompactness of X, does hold [cf« CJ, pp. 119-120, and p. 72]o) A counter-example would still be zero-dimensional and realcompact, of course, and the author is unaware of any spaces other than A itself and spaces trivially obtainable from A which are zero-dimensional and realcompact and whose Stone-Čech compactification is not zero-dimensional. There are, however, spaces which may be of this sort and which are moreover known to be N-compact. One example is the Sorgenfrey plane: it is the product of two copies of the real line with intervals of the form [a,b) as a base for the topology. Each factor is Lindelof and thus is both realcompact [cf. GJ, p.115] and has zero-dimensional Stone-Čech compactification [cfo GJ, pp.245-7] and is thus N-compact. Hence the Sorgenfrey plane is N-compact. But is its Stone-Cech compactification O-dimensional? Another possible counterexample is the N-compactification of A. (For a definition and construction of the E-compactification of a space, cfo [EM] or [H, Kapitel I, §3,§9].)

For further discussion of spaces with zero-dimensional Stone-Čech compactification, see [GJ, ch.16] (where these spaces are called simply fzero-dimensionalf9 while the spaces which this paper calls zero-dimensional are simply designated as 1having a base of open-and-closed setsr)9 [E, ch,6^§2] (where these spaces are called 'strongly O-dimensionalf), [H, Beispiele 5,6]3 and **[N,§2].**

The second problem is this: is there a single space E such that the class of O-dimensional realcompact spaces is the class of E-compact spaces? In this paper we have shown that if such a space exists it cannot be N, or any other N-compact space. Might it be A ?

This problem is admittedly less attractive than the first one. If the answer is affirmative, a proof of this result might have to depend on the construction of a non-N-compact, O-dimensional realcompact space that is substantially easier to work with than A *I*

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CORRIGENDUM

The statement of the proof of 5^3₀6 is not quite precise* It. is fcrme that n > 1 is the only case needed to prove that £ is not N-eompactj, but I do eventually make *use* of the case n ~ 1 in order to **show** that an **ultrafilter of the form U** $_x$ **i&** alv;< y distinct f^fom one of the form li, .» **There** are also a few typographical errors,, Here is a corrected proof-.

£?% Suppose (S(K_9.xv)> c R-(p, n). . Than for each Rx,*:E{X,:-r>;- we 'have R , c *R,* . and so R $_f$ fl R -, - 4 ,0 for serase j (2..1,3); furthermore^ since R c R-?^ n)' (2.1,3), we **have** '•**(J: »'R I ^ n 4- 1 because sx contains CrP^ tfifch 'c^j «« j^i: while all P^ points in R7 ^ $_{K pp n}$ must ha-^e x-coordxnates of Ie rç:h at least [p f + n •+ l> This implies R » • * 3 R , for exactly one j (2olol;

If R ..<f: R, -t*ix| < !pvi + n (2..1.1) , Pick x^J' such that x^{rf}(i) « x''(i) for i » 0^o,o,-p^: 4 n, $x^{9)}$ {j p j * n ^ 1; x^l(!p_xl ^ n -^ 1) and Py?r{K;x+), If n > i we have q(;p_xl + n + 1) ~ P^^n - I) for all q-^ n *i<* (p, n) , New let q^fp^ H R_x« and **q^r-$Pj** H R - *$q^n$* and q cannot bcth *he ii<.* K/^ ^^ (Pi- n) « contradicting £{K,X4-) c R(p9 n')

If n « 1 we distinguish three cases; . *x'* » jp j i.r ix[- |p ic |xj < jp i* If ; « |p j 4^ 1 then we take q^{*1} and q in Cs(K^x4-))* n P_1 With q^{r9}C;pj - 2/ ^ q'dP i ^ 2/ Since the ClP_x^ ^ D-coordinates of $q^{c?}$ and q are the sam-c-. and since any P_1^points in R,(p,3',) with identical CiP_x! ^ li-coordinates also have icientxcal (ip^f-^- 2)-coordinates (1,3,.9-10)

q^{fl} and q^* cannot both be in $R^*_{1>0}$ If jxf =* {p_vj the same argument works since I K| j£ 2 and so we can find q^M and q- that differ in the (|p_v(+ 2)-coordinate. The last case is disposed of by finding q€(£(K*xf)) * n Pj^ such that q({p_x() >= P_K(|p_xf).

Now the coitmient in brackets on page 12, lines 9-10 becomes superfluous* as does "with n > l^n on page 11, four lines fron the bottom*

CS
12/15/70