

1970

Prabir Roy's space Δ is not N-compact

Peter Nyikos
Carnegie Mellon University

Follow this and additional works at: <http://repository.cmu.edu/math>

Recommended Citation

.

This Technical Report is brought to you for free and open access by the Mellon College of Science at Research Showcase @ CMU. It has been accepted for inclusion in Department of Mathematical Sciences by an authorized administrator of Research Showcase @ CMU. For more information, please contact research-showcase@andrew.cmu.edu.

PRABIR ROY'S SPACE A IS NOT
N-COMPACT

by

Peter Nyikos

Report 70-40

October, 1970

**University Libraries
Carnegie Mellon University
Pittsburgh PA 15213-3890**

FEB 1 '71

HUNT LIBRARY
CARNEGIE-MELLON UNIVERSITY

Introduction* N-compact spaces were introduced by S. Mrowka in [M₁], where the general concept of an E-compact space was defined: given a Hausdorff space E, a space X is E-compact if it is homeomorphic to a closed subspace of E^M for some cardinal number M. Thus the I-compact spaces (where I is the closed unit interval) are the compact Hausdorff spaces, the R-compact spaces are the realcompact spaces, and the 2-compact spaces (where 2 denotes the discrete two-point space) are the 0-dimensional compact Hausdorff spaces. The N-compact spaces are those which can be embedded as closed subspaces in N^ℕ where ℕ is the set of natural numbers with the discrete topology.

The main properties of E-compact spaces were given in [EM], where it was asserted that the N-compact spaces are precisely the 0-dimensional realcompact spaces. (^f0-dimensional^f there, as here, means 'having a base of clopen sets'.¹) It is clear that every N-compact space is 0-dimensional and realcompact, but the proof of the converse in [EM] was incomplete. The purpose of this paper is to show that the converse is in fact false, that Prabir Roy's space A is a counterexample. It is not N-compact, but it is metrizable of cardinality 2^{ℵ₀} and hence realcompact [cf. GJ, p. 232] and it is 0-dimensional.

The space A was described by P. Roy in [R₁] and many of its properties were proven in detail in [R₀]* including its metrizability and its zero-dimensionality. The fact that A is not N-compact is the new result, first established by the author using the proof given below. It is based on the following characterization of N-compactness, first discovered by H. Herrlich

[H, Beispiele 6].

Theorem. A zero-dimensional Hausdorff space X is N -compact if, and only if, every clopen ultrafilter¹ on X with the countable intersection property² is fixed.

In what follows, we establish the existence of 2^{\aleph_0} distinct free clopen ultrafilters on A , any one of which is enough, by Herrlich's theorem, to establish that A is not N -compact.

Constructing the Ultrafilters. In order to facilitate comparison with [R₂], the numbers of the lemmas and theorems will begin with a 5, and if a lemma closely parallels a lemma or a definition in [R₂]* ^{at w i ^ 1 ^ e} given a similar numbering. (Thus, the first two lemmas follow right from Definitions 5.3.3 and 5.3.4 in [R₂1 and are numbered 5.3.5 and 5.3.6; the next three lemmas are like Lemma 5.4 in [R₂1* and are numbered respectively 5.4.0, 5.4.1, and 5.4.2. And a later lemma is a generalization of Lemma 5.7 in [R₂1 and is numbered 5.7^f.

All the notation used in this section, unless otherwise remarked, follows that of [R₂]. Not all the results in [R₂] will be used in proving the lemmas below; while familiarity with [R₂] up to the beginning of Section 5.3, p. 127 is certainly an asset, the reader will be able to get by with considerably less. Specifically, after the end of Section 2.1 only the following facts in [R₂J are used:

¹ an ultrafilter on the Boolean algebra of clopen subsets of x is called simply a clopen ultrafilter.

² Given a cardinal number \aleph_n , a filter \mathcal{F} is said to have the ~~countable intersection property~~ if every collection of \aleph_n or fewer sets in \mathcal{F} has nonempty intersection. The countable intersection property is the case $\aleph_n = \aleph_0$.

(1) Lemma 2.7, which shows that the regions R_x and $R, *$ form a base for a topology, and which can be proven right after Lemma 2.1;

(2) Lemma 2.8, which shows that the regions $R_{x \in T, i}$ form a local base at a point $p \in P^-$ and the regions $R_{(p, n) \in I}^2$ form a local base at a point $p \in P_p$, and which can be proven right after Lemma 2.1;

(3) A modest version of Property II [p.126]: given a point $p \in P-p$ let x_n be a member of X with $|x_n| = n$ and such that p extends x_n (this means $x_n(i) = p(i)$ for $i = 1, \dots, |x_n|$); then $\bigcap_{n=1}^{\infty} R_{x_n} = \{p\}$; similarly, given a point $p \in P_0$, $T_1, R, \dots, v = \{p\}$. These facts can be proven right after the definitions of Section 1.

(4) Lemma 5.1, which shows that every R and every $R_{V P^{*n}}$ is clopen. This can be proven after Section 2 and does require all the results in that section.

In addition to the notations used in $[R^{-1} * w^e]$ we adopt the following notation: if $X \in X$ and $x' \in X$ [resp. $p \in P^+$] are such that $I^{x'} \supset I^x$ and $x'(i) = x(i)$ for $i = 1, \dots, |x|$ [resp. $p(i) = x(i)$ for $i = 1, \dots, |x|$] then we write $x < x'$ [resp. $x < p$]. Similarly, if $T \in T$ and $T' \in T$ [resp. $x \in X$] [resp. $p \in P^+$] are such that $T' \supset T$ and $T'(i) = T(i)$ for $i = 1, \dots, |T|$ [resp. $|x| \wedge |T|$] and $x(i) = T(i)$ for $i = 1, \dots, |T|$ [resp. $p(i) = T(i)$ for $i = 1, \dots, |T|$] then we write $T < T'$ [resp. $T < x$] [resp. $T < p$]. Also, we adopt the convention that if $x \in X$ is the (unique) member of X with $|x| = 0$, then $R = A$.

x

For the convenience of the reader, the following definitions

in $[R_z]$ are repeated, with the above notation used where possible.

5.3.1. $7r$ = the set of all finite sequences of positive real numbers, defined on initial segments of the set of positive integers. If $ireH$, then $|TT|$ = the greatest integer for which IT is defined.

5.3.2. K is an indicator means that K is a subset of IT with

(1) if $7r, 7r^f \in K$, then $|ir| = |TT^f|$ and that integer is denoted by $|K|$

(2) $\{r : r = 7r(1) \text{ for some } veK\}$ is an infinite set, and

(3) if $TreK$ and j is a positive integer with $j < |TT|$, then $\{r : r = TT^T(j+1) \text{ for some } TT' \in K \text{ with } 7r^T(i) = 7r(i) \text{ for } i = 1, \dots, j\}$ is an infinite set.

5.3.3. If K is an indicator and $x \in X$ then $E(K, x_{\pm}) = [R_{\pm}, : |x^T| = |x| + |K|, X < x^T$, and for some $TreK$, $x^f (|x|+i) = \pm 7r(i) \text{ for } i = 1, \dots, |K| \}$.

(Remark: this is a composite definition; the things actually defined are $\varepsilon(K, x_+)$ and $\varepsilon(K, x_-)$.)

5.3.4. If K is an indicator and $R, N \in \mathbb{F}_0$ then $(p \wedge n)^2$
 $E(K, (p, n)) = \{R(q, m) : m = n + |K|, q \in R(p, n), \text{ and for some } 7T \in K, q_z(n-l+i) = 7r(i) \text{ for } i=1, \dots, |K|\}$.

5.3.5. Lemma. Let K be an indicator.

$$(E(K, (p, n)))^* \subset R_x \ll R_{(p \wedge n)} e^{\wedge}$$

Proof. $\wedge : (E(K, (p, n)))^*$ is a union of subregions of R, N .

$$\Rightarrow : (\Sigma(K, (p, n)))^* \subset R_x =, R_x \cap R_{(p \wedge n)} > \wedge 0 \Rightarrow R_x \supset R_{(p, n)} \#$$

5.3.6. Lemma. Let K be an indicator

$$(D \text{ CC}(K, X_+))^* \subset R_{(p, n)} - R_x \subset R_{(p, n)}$$

$$(2) (E(K, X_-))^* \subset R_{(p, n)} - R_x \subset R_{(p, n)}$$

Proof. $(E(K, X_+))^*$ and $(E(K, X_-))^*$ are both unions of subregions of R_x .

=*: Proof will be given of (1) in the case $n > 1$, the only case needed here.

Suppose $(E(K, X_+))^* \subset R_{(p, n)}$, then for each $R_x \supseteq E(K, X_+)_y$ we have $R_x \cap R_{(p, n)}$ and so $R_x \cap R_{y_j} \neq \emptyset$ for some j (2.1.3); furthermore, since $R_x \cap R_{y_j} \neq \emptyset$ (2.1.3), we have $|x^T| \geq |p_x| + n + 1$ (p*n)

because R_x contains $q \in P_2$ of length $|x^T|$ while all P_2 points in $R_{(p, n)}$ must have x-coordinates of length at least $|p_x| + n + 1$. This implies $R_{y_j} \cap R_{(p, n)} \neq \emptyset \Rightarrow R_x$ for exactly one j (2.1.1).

If $R_x \not\subset R_{(p, n)}$, pick x^T such $x''(i) = x'(i)$ for $i = 0, \dots, |p_x| + n$, $x^{lf}(|p_x| + n + 1) \notin X'(|p_x| + n + 1)$, and $R_{y_j} \in E(K, X_+)$. If $n > 1$ we have $q(|p_v| + n + 1) = p(|p_v| + n - 1)$ for all $q \in P_j \cap R_{(p, n)}$. Now let $q^{lf} \in P_1 \cap R_x$ and $q^T \in P_1 \cap R_x$. q and q^T cannot both be in $R_{(p, n)}$ contradicting $L(K, X_+) \subset R_{(p, n)}$.

5.4.0. Lemma. Let $\{r_a\}_{a \in \mathbb{Q}}$ be an infinite set of positive real numbers. For each r let K be an indicator, with $|K_a| = n$ for all a . Now let $K = \{r^f, 51r^T\} = n + 1$, $TT^l(1) = r^a$ for some $a \in \mathbb{Q}$ and, for the same a , $7r^f(1+i) = ir(i)$ for some $TTGK$. K is an indicator.

Proof. (1) is clearly satisfied.

(2) $\{r : r = TT(1) \text{ for some } TTGK\}$ is an infinite

set, namely $\{r_\alpha\}_{\alpha \in G}$.

(3) is also satisfied: for $j = 1$ it is satisfied because of property (2) of each K_α ; for $j > 1$ it is satisfied because of property (3) of each K_α .

5.4.1. Lemma. If \mathcal{R} is a collection of regions and R_x is such that, for uncountably many x^f with $|x^f| = |x|+1$, $x < x', x'(|x|+1) > 0$ [resp. $x'(|x|+1) < 0$] there is an indicator K_x , with $f(K_x, x^+) \subset M$ [resp. $f(K_x, x^+) \subset W$] then there is an indicator K such that $E(K, x^+) \supset \exists J$ [resp. $f(K, x^-) \subset W$].

Proof. Since $\{M: M = |K_x^f| \text{ for some } x^f\}$ is a countable set, let N be a positive integer such that $|K_x^f| = N$ for infinitely many x^f . Let $f^r_a\}_{a \in G} = \{x'(|x|+1): |K_x^f| = N\}$ [resp. $\{-x'(|x|+1): |K_x^f| = N\}$] and let K be as defined in the previous lemma. Then $f(K, x^+) \supset U$ [resp. $f(K, x^-) \subset W$].

5.4.2. Lemma. If \mathcal{W} is a collection of regions and $R_x^{(p,n)}$ is such that, for uncountably many $r \in R^+$ there exists $q^r \in R_x^{(p,n)}$, with $q^r(n) = r$ and an indicator K_r such that $E(K_r, (q^r, n+1)) \subset H^+$ then there is an indicator K such that $\Sigma(K, (p,n)) \subset U$.

Proof. Proof is as in 5.4.1.

5.6.1. Lemma. Let $\{R_x^i\}_{i \in J}$ be an open cover for R_x , with $|f_c| < C$ ($C = 2^{**0}$). Then there exists $j \in J$ and an indicator K such that $(f(K, x^+))^* \subset \bigcup_{j \in J} R_x^j$, and there exists $y \in f_0$ and an indicator K^1 such that $(E(K, x^-))^* \subset U_y$.

Proof. Given $j \in J$ let $\#_p = \{R_x^i : R_x^i \subset U^+\}$. We give the proof for the existence of a $j \in J$ and a K as defined above (the proof for y and K^1 is analogous) by applying 5.4.1 to each

V

Suppose no such pair $j \in J, K$ exists. Then, for each

there are at most countably many $x^f > x$ with $|x^f| = |x|+1, x^f (|x|+1) > 0$, for which there is an indicator K_x , with

$$(\Sigma(K_x, x^f))^* \subset U_\beta.$$

Hence, altogether, there are fewer than C distinct $x^f > x$ with $|x^f| = |x|+1, x^f (|x|+1) > 0$, for which there is some U_β and an indicator K_x with

$$(\Sigma(K_x, x^f))^* \subset U_\beta.$$

Now, pick any $x^1 > x$ with $|x^1| = |x|+1, x^1 (|x|+1) > 0$ for which no such pair exists. The process repeats: there are fewer than C distinct x^2 with $|x^2| = |x|+2, x^2 (|x|+2) > 0$ for which there is some U_p and an indicator K_x , with

$$(E(K_x, x^2))^* \subset U_p.$$

Pick any $x_2 > x_1$ with $|x_2| = |x|+2, x_2 (|x|+2) > 0$ for which no such indicator exists.

In this way we get a nested sequence

$$x \subset x_1 \subset x_2$$

and the unique point $p \in \bigcap P_n$ such that $x_n < p$ for all n is in their intersection. For no n is it true that

$$(L(K_{x_n}, x_n))^* \subset X_{x_n} \text{ for some } \xi \text{ and some } K_{x_n}.$$

And, a fortiori, none of the R_{x_n} is contained in \mathbb{R} for any $f \in J$.

But this violates the hypothesis that the U^α constitute an open cover of R_x , for $p \in R_x$ and the R_{x_n} form a local base at p (2.8).

5.6.2. Lemma Let $\{U_\alpha\}$ be an open cover for R_{x_n} with $|H| < C$. Then there exists $J \in B$ and an indicator K such that $(E(K, (p, n)))^* \subset J$.

Proof. Proof is as in 5.6.1. What we obtain is, inductively, a sequence of $q^m \in R^0(q^{m-1}, n+m-1)$, such that for no β and no indicator K_m is it true that

$$(\Sigma(K_m, (q^m, n+m)))^* \subset u^\wedge.$$

and we thus get a nested sequence

$$R(p, n) \supset R(q^1, n+1) \supset R(q^2, n+2) \supset \dots$$

and now define $q \in P_2$ with $q_x = P_x, q_y = P_y > 3_2(i) = p^{\wedge * -}$ for $i = 1, \dots, n-1, q_z(n+m-1) = q_z^m(n+m-1)$. It is easy to see that

$$R(q, n+m) = R(q^m, n+m)$$

and so the above nested sequence forms a local base at q , and its intersection is $\{q\}$, and $q \in R^? \cdot$
 $vp^{\wedge n};$

But for no m is it true that $R_x \subset u_0$ for some fl ,
 $(q, n+m) \quad p$

violating the assumption that the U_p form an open cover for $R^?P, nr$

5.7^f. Lemma, if K is an indicator, and $x = y(p, n, \pm)j$ with $j \wedge |K|+1$, then

$$(X(K, xT))^* \subset (E(K, (p, n)))^*$$

while if $j < |K|+1$, $(L(K, xT))^* \cap (S(K, (p, n)))^* = fb$.

Proof. First, if $j \leq |K|$, it can be easily seen from the definitions that $R_x f) R(q, m)$ is empty, for all q in $S.S^{\wedge j}$ and hence a fortiori we get the result above. For the remainder of the proof, assume $j \geq |K|+1$.

For each $\tau \in TK$, let $x_{\tau} \in X$ such that

$$|x_{\tau}| = |x| + |K|, x_{\tau} > x, x_{\tau}(|x| + i) = \tau r(i) \quad \text{for } i = 1, \dots, |T|,$$

and let $\wedge P_2$ be such that $\forall i \in \tau, \tau^{\wedge z(n+i+1)} = \tau r(i)$ for $i = 1, \dots, |T|$.

We have then by 5.3.3 and 5.3.4 that if $w \in K$ then

$$R_{x_{\pi T}} \in \Sigma(K, xT) \quad \text{and} \quad R_{\dots h} \in \Sigma(K, (p, n)).$$

Furthermore, in presence of the definitions of $x_{\pi T}^{\wedge}$ and q^{\wedge} , an inspection of 1.3.7-1.3.11 of [R₂] will show that

$$x_{\pi T}^{\wedge} = \gamma(q_{\pi, n+|K|, \pm})_{j-|K|}.$$

Again by 5.3.3 and 5.3.4 we have

$$E(K, xT) = \{R_{x_{\pi T}} : \pi \in K\}$$

and

$$E(K, (p, n)) = (R_{(\% j_{n+|K|})} : \pi \in K)$$

and so

$$(E(K, xT))^* \subset \{R^T(a_{\pi, n+|K|}) : \pi \in K\}^* \subset (E(K, (p, n)))^*.$$

5.13.1. Lemma. Let $\{V_r\}$ be a partition of R into clopen sets, $|G| < C$. Then there is an $a \in G$ for which there is an indicator K^{\wedge} such that $(L(K_1^{\wedge}, x+))^{\wedge} \subset v_a$ and there is a $\beta \in G$ for which there is an indicator K_2 such that $(E(K_2, x-))^* \subset v_{\beta}$. Furthermore $a = \beta$ and a is unique.

Proof. 5.6.1 guarantees the existence of V_a and V_{β} .

5.8 of [R₂] can now be used to show that V_a^{α} and V_{β}^{\wedge} must have a nonempty intersection: assume on the contrary $V_a^{\alpha} \cap V_{\beta}^{\wedge} = \emptyset$, then apply 5.8 to V_a^{α} and V_{β}^{\wedge} ; if $(E(K_2, x-))^* \subset v_{\beta}$ then a fortiori $(E(K_2, x-))^* \subset v_a^{\wedge}$ and now let $x_{jL} > x$ be such that

$|x_1| = |x_j| + 1$ and there exist indicators K_3 and K_4 such that $(\Sigma(K_3, x^{\wedge}))^* \subset v_{\beta}$ and $(S(K_4, x_{jL}^{\wedge}))^{\wedge} \subset V_a^{\wedge}$; by induction define $x_n > x_{n-1}$

with $|x_n| = |x_{n-1}| + 1$ so that there exist indicators K_{x_n} and

$(L(K_{x_n}, x_n+))^* \subset v_a, (S(K_{x_n}^{\wedge}, x_n-))^* \subset V_a^{\wedge}$ in this way we get a nested

sequence $\{R_{x_i}\}_{i=1}^{\infty}$, and each set in the sequence has a nontrivial intersection with V_{oc} and V_{oc}^c . Now $\bigcap_{i=1}^{\infty} R_{x_i} = \{p\}$ for some $p \in P$, and since both V_{oc} and V_{oc}^c are closed, p is in both of them, which is absurd. So $V_{oc} \cap V_{oc}^c = \emptyset$, and since these sets are members of a partition, $V_{oc} = V_{oc}$. Now suppose there exists V_{oc} , $y \in G$, and an indicator K such that $(\mathcal{E}(K, x+))^* \subseteq V_{oc}$. Then $V_{oc} \cap V_{oc} = \emptyset$ (same proof as above with y substituted for a) and so $V_{oc} = V_{oc}$. Similarly, if we have V_{oc} , $y \in G$ and an indicator K such that $(\mathcal{E}(K, x-))^* \subseteq V_{oc}$ then $V_{oc} \cap V_{oc} = \emptyset$, and $V_{oc} = V_{oc}$.

5.13.2. Lemma Let $\{V_{\alpha}\}_{\alpha \in Q}$ be a partition of $R_{(p,n)}$ into clopen sets, $|G| < C$. Then there is exactly one $a \in G$ for which there is an indicator K such that $[\mathcal{E}(K, (p,n))]^* \subseteq V_a$.

Proof Existence of a is shown by using 5.6.2. To show uniqueness, suppose $[\mathcal{E}(K_1, (p,n))]^* \subseteq V_a$, and $[\mathcal{E}(K_2, (p,n))]^* \subseteq V_b$. Let $x = y(p, n, i)j$ with $j \geq \max\{|K_1|, |K_2|\} + 1$. Then

$$(\mathcal{E}(K_1, xT))^* \subseteq [\mathcal{E}(K_1, (p,n))]^* \subseteq V_a$$

and

$$(\mathcal{E}(K_2, xT))^* \subseteq [\mathcal{E}(K_2, (p,n))]^* \subseteq V_b$$

Now the sets $\{V_{\alpha} \cap R_x\}_{\alpha \in Q}$ partition R_x into clopen sets and hence by 5.13.1, $V_{\alpha} = V_b$.

5.14.1. Lemma Let $U_x = \{U : U \text{ is clopen in } A \text{ and there exists an indicator } K \text{ with } (\mathcal{E}(K, x+))^* \subseteq U\}$. U_x is a clopen filter.

Proof; Clearly if $U \in U_x$ and V is clopen in A with $U \subseteq V$, then $V \in U_x$ and clearly $U \cap V \in U_x$. Now suppose U_1 and U_2 are both members of U_x , with associated indicators K_1 and K_2 ,

respectively. Consider the four clopen sets $U_1 \cup U_2, U_1 \cap U_2, U_1 \setminus U_2, U_2 \setminus U_1$ and $(U_1 \cup U_2)^c$. Their intersection with R_x gives a partition of R_x into clopen sets. Now $(\mathcal{E}(K, x))^* \subset R_x$ for all $x \in K$.
 $(U_1 \setminus U_2) \cap R_x \neq \emptyset$ iff $(U_1 \setminus U_2) \cap R_x \neq \emptyset$. But

$$(\mathcal{E}(K, x))^* \subset (U_2 \cap R_x) \cap U_x$$

and since $(U_1 \cap R_x) \cap (U_2 \cap R_x) = \emptyset$, we cannot have $(U_1 \cap U_2) \cap R_x \neq \emptyset$. In a similar way we eliminate $U_1 \setminus U_2$ and $U_2 \setminus U_1$.
 $U_1 \cap U_2 \in \mathcal{U}_x$ by 5.6.1.

5.14.2. Lemma Given $P \in P_2^A$ let $\mathcal{U}_{(p, n)} = \{u : U \text{ is clopen in } A \text{ and there exists an indicator } K \text{ with } [\mathcal{E}(K, (p, n))]^* \subset u\}$.
 $\mathcal{U}_{(p, n)}$ is a clopen filter.

Proof. Proof is as in 5.14.1.

5.15.1. Lemma \mathcal{U} is a free clopen ultrafilter with the ω -intersection property for all $M < C$, and $\mathcal{U}_x \cap \mathcal{U}_y$ are distinct

whenever $x \neq y$.

Proof That \mathcal{U} is an ultrafilter comes immediately from 5.6.1 and 5.14.1, and we can also use 5.6.1 to show that \mathcal{U} has the ω -i.p.: Suppose there is a family of $\{U_\alpha\}_{\alpha \in G}$ with $U_\alpha \in \mathcal{U}$ for all α and $|G| = \aleph_m < c$, then $\{U_\alpha^c\}_{\alpha \in G}$ forms a clopen cover for R_x and so one of them belongs to \mathcal{U}_x and so \mathcal{U}_x does its complement, a violation of 5.13.1. To show that \mathcal{U} is free, we show that for no R_x with $|x^f| > |x|$ is it true that $R_x \in \mathcal{U}_x$ and that for no R_x with $|x^f| > |x|$ and $R_x \subset R_x$ is it true that $R_x \in \mathcal{U}_x$: if u_x were fixed it would either be fixed on a point of P_1 , in which case it would have as members R_x for arbitrarily large $|x^f|$, or else it would be fixed on a

point $p \in \mathbb{R}_x^*$ which case it would have as members $R_{(p,n)}$ for arbitrarily high n , and since $p \in \mathbb{R}_x \subset U_x$ we must also have $R_{(p,n)} \subset R_x$ by 2.1.1.

First, if $|x^f| > |x|$, we have that $p(|x|+1)$ is the same real number for all $p \in \mathbb{R}_x \gg \mathbb{N} p_i$ while by (2) 5.3.2 and 5.3.3 it is possible to find infinitely many p in $(E(K,x))^* \cap \mathbb{R}_x^*$ with distinct values for $p(|x|+1)$. Hence $R_{(p,n)} \not\subset R_x$. Second, if $R_{(p,n)} \subset U_x$ then by 5.3.6 we have $R_{(p,n)} \subset R_x$ and by 2.1.2 and 2.1.3 we cannot have $R_{(p,n)} \subset R_x$. [Note: 5.3.6 was only proven in the case $n > 1$ but this is all we need.]

To conclude the proof of the Lemma, we note that if $x \neq x^f$, then in the case $|x| < |x^f|$ we have either $R_{(p,n)} \subset U_x$ or $R_{(p,n)} \not\subset U_x$ depending on which $|x|, |x^f|$ is bigger, while we do have $R_{(p,n)} \subset U_x$ for all $x \in X$. If $|x| = |x^f|$ but $x \neq x^f$, $R_{(p,n)} \cap R_{(q,m)} = \emptyset$ and so $R_{(p,n)} \not\subset U_x$.

5.15.2. Lemma. $U_{(p,n)}$ is a free clopen ultrafilter with the w -intersection property for all $M < C$, and $U_{(p,n)}, U_{(q,m)}$ are always distinct, while $U_{(p,n)} = U_{(q,m)}$ iff $R_{(p,n)} = R_{(q,m)}$.

Proof. To show $U_{(p,n)}$ is free, we show that no $R_{(p,n)}$ with $|x| > |p|$ is in $U_{(p,n)}$, and that no $R_{(q,m)}$ with $|q|+m > |p|+n$ is in $U_{(p,n)}$. freeness follows from the same considerations as above.

If $R_{(p,n)} \in U_{(p,n)}$ then $\exists K$ such that $(\mathcal{E}(K, (p,n)))^* \subset R_{(p,n)}$ which implies that $R_{(p,n)} \subset R_x$. By 2.1.1 and 2.1.2 this implies $|x| \leq |p|$.

If $R_{(q,m)} \in U_{(p,n)}$ then for some $K, (\mathcal{E}(K, (p,n)))^* \subset R_{(q,m)}$

which implies that $R_{(p,n)} \text{ fl } R_{(q,m)}$. Suppose first $R_{(p,n)} \text{ fl } R_{(q,m)}^0$ then we have $|q_x| = |p_x|$. Now all $q^T \in P_2 \text{ fl } R_{(q,m)}^0$, have the same (k)th coordinate for q_x if $k < m$ while by 5.3.2 and 5.3.4 it is possible to find an infinite set of $q^f \in P_9 (1 \text{ fl } (K, (p,n)))^*$ with $|q_x| = |p_x|$ and with distinct values for $a_x(n)$. Since the only $q^f \in P_9 \text{ fl } R_{(q,m)}$ with $|q_x| = |p_x|$ are those in $R_{(q,m)}$, it follows that if $(\text{fl}(K, (p,n)))^* \subset R_{(q,m)}$ then $m \leq n$, so that in this case $|q_x| + m \leq |p_x| + n$.

Second, suppose $R_{(p,n)} \text{ fl } R_{(q,m)} = \emptyset$, then $R_{(p,n)} \wedge R_{(q,m)} = \emptyset$ for some j which implies by 2.1.1

that $|q_x| + m + 1 \leq |p_x|$ and so here too $|q_x| + m \leq |p_x| + n$.

The /H-intersection property follows from 5.6.2 and an argument like that in the previous lemma. Similarly, $U_{(p,n)}$ is an ultrafilter,

To show that U_x and $U_{(p,n)}$ are always distinct, note that $R_x \in U_x$ and $R_{(p,n)} \in U_{(p,n)}$ so that if $R_x \text{ fl } R_{(p,n)} = \emptyset$ we are done. If $R_x \wedge R_{(p,n)} \neq \emptyset$ then $R_x \in U_{(p,n)}$, which rules out

$R_x \in U_{(p,n)}$. Hence by 5.3.6 $R_x \in U_x$ and so the two clopen-set ultrafilters are distinct in this case. If $R_x \text{ fl } R_{(p,n)} = \emptyset$ but $R_x \wedge R_{(p,n)} \neq \emptyset$ then $|q_x| + m > |p_x| + n$ and as shown above $R_x \wedge R_{(p,n)} \neq \emptyset$

Finally, suppose $U_{(p,n)} = U_{(q,m)}$. Then, as shown above, this implies $|q_x| + m \leq |p_x| + n$, and also $|p_x| + n \leq |q_x| + m$, so $|p_x| + n = |q_x| + m$. Also, $R_x \wedge R_{(q,m)} \neq \emptyset$, otherwise $|q_x| + m + 1 \leq |p_x| + n$ as shown above. This implies $R_{(p,n)} \wedge R_{(q,m)} \neq \emptyset$ by 2.1.4.

5.16. ~~Property VI~~; A is not N-compact.

The proof consists of either 5.15.1 or 5.15.2 together with H. Herrlich's theorem quoted in the Introduction.

Concluding Remarks> With the problem quoted in the Introduction thus solved in the negative, two other problems, also having their roots in [EM], become better defined.

First, as pointed out in [M₂], what [EM] really showed about N-compact spaces is that if X is realcompact and $\dim X$ is zero-dimensional (a strictly stronger condition than the zero-dimensionality of X) then X is N-compact. (An explicit statement and proof of this may be found in [H, Beispiele 5,6].) The first unsolved problem before us is the converse of this statement. In other words, given a closed subset X of \mathbb{N}^m , m any cardinal number, is it true that $\dim X$ is 0-dimensional? (The other condition, realcompactness of X , does hold [cf. CJ, pp. 119-120, and p. 72].) A counter-example would still be zero-dimensional and realcompact, of course, and the author is unaware of any spaces other than A itself and spaces trivially obtainable from A which are zero-dimensional and realcompact and whose Stone-Čech compactification is not zero-dimensional. There are, however, spaces which may be of this sort and which are moreover known to be N-compact. One example is the Sorgenfrey plane: it is the product of two copies of the real line with intervals of the form $[a,b)$ as a base for the topology. Each factor is Lindelof and thus is both realcompact [cf. GJ, p.115] and has zero-dimensional Stone-Čech compactification [cf. GJ, pp.245-7] and is thus N-compact. Hence the Sorgenfrey plane is N-compact. But is its Stone-Čech compactification 0-dimensional? Another possible counterexample is the N-compactification of A . (For a definition and construction of the E-compactification of a space, cf. [EM] or [H, Kapitel I, §3,§9].)

For further discussion of spaces with zero-dimensional Stone-
 Čech compactification, see [GJ, ch.16] (where these spaces are
 called simply ^fzero-dimensional^f, while the spaces which this
 paper calls zero-dimensional are simply designated as ¹having
 a base of open-and-closed sets^x), [E, ch,6^§2] (where these spaces
 are called 'strongly 0-dimensional^f'), [H, Beispiele 5,6], and
 [N, §2].

The second problem is this: is there a single space E
 such that the class of 0-dimensional realcompact spaces is the
 class of E -compact spaces? In this paper we have shown that if
 such a space exists it cannot be N , or any other N -compact space.
 Might it be A ?

This problem is admittedly less attractive than the first
 one. If the answer is affirmative, a proof of this result might
 have to depend on the construction of a non- N -compact, 0-dimensional
 realcompact space that is substantially easier to work with than A I

Bibliography

- [E] R. Engelking, Outline of General Topology, Amsterdam, North Holland Publishing Co., 1968.
- [EM] Engelking and Mrówka, On E-compact spaces, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys. J5 (1958), 429-436.
- [GJ] Gillman and Jerison, Rings of Continuous Functions, Princeton, N.J., Van Nostrand Co., 1960.
- [H] H. Herrlich, (g-kompakte Räume II Mathematisches Institut der Freien Universityt Berlin, 1965. Reprinted in Math. Zeitschr. 96 (1967), 228-255.
- [M₁] S. Mrówka, On universal spaces, Bull. Acad. Polon. Sci. ± (1956) 479-481.
- [M₂] _____, On E-compact spaces. II, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys. 1± (1966), 597-605.
- [N] P. Nyikos, Not every 0-dimensional realcompact space is N-compact, Report 70-30, Department of Mathematics, Carnegie-Mellon University.
- [R₁] P. Roy, Failure of equivalence of dimension concepts for metric spaces, Bull. Amer. Math. Soc. j58 (1962), 609-613.
- [R₂] _____i Nonequality of dimensions for metric spaces, Trans. Amer. Math. Soc. jL3f (1968), 117-132₀

CORRIGENDUM

The statement of the proof of 5³06 is not quite precise* It is forme that n > 1 is the only case needed to prove that f is not N-compactj, but I do eventually make use of the case n = 1 in order to **show** that an **ultrafilter of the form U_x** is alv< sy distinct f^fom one of the form li, .» **There** are also a few typographical errors,, Here is a corrected proof-

*f?% Suppose (S(K₉.xv)> c R-(p, n). Than for each R x, *E{X, :-I>:- we 'have R , c R, . and so R f fl R -, - , 4 , 0 for ~~sease~~ j (2...1,3); furthermore^ since R₀ c R-(p, n) (2.1,3), we **have** i'**(J: »E I ^ n 4- 1 because s* contains CrP^ ~~tfifch~~ 'c^j. «« j^i: while all P_n points in R₇^ must ha-^e x-coordxnates of Ie_frç:h at least [p f + n •+ l> This implies R » • * 3 R , for exactly one j (2olol;

If R...f: R... -t*ix| < 'p_vi' + n (2..1.1) , Pick x^j such that x^{rf}(i) « x^{'i}(i) for i » 0^o, o, -p_i^: 4 n, x⁹ {j p_x j * n ^ 1; * x¹(!p_xl ^ n -^ 1) and Py?r{K; x+), If n > i we have q(i; p_xl + n + 1) ≈ Pⁿ - I) for all q-^ n i< (p, nj', New let q^f p^ H R_x« and q^f-SPj H R_x - qⁿ and q cannot bcth he ii<. K/(Pi-n) « contradicting f{K, X4-) c R_(p, n)

If n « 1 we distinguish three cases; . x' » j p j * i_r ix| = |p-ic |xj < j p i* If :;: « |p-j 4^ 1 then we take q^{*1} and q in Cs(K^x4-)) * n P₁ With q⁹C; p j - 2/ ^ q' d P i ^ 2/ Since the ClP_x^ ^ D-coordinates of q^{c2} and q are the sam-c. and since any P₁^points in R_(p, 3) with identical CiP_x! ^ li-coordinates also have icientxcal (ip^f-^- 2)-coordinates (1, 3, 9-10)

q^{f1} and q^* cannot both be in R^* . If $jxf = \{p_vj\}$ the same argument works since $|K| \geq 2$ and so we can find q^M and q^A that differ in the $(|p_v| + 2)$ -coordinate. The last case is disposed of by finding $q \in (\mathcal{L}(K^*xf)) \cap P_j^*$ such that $q(\{p_x(\cdot) \geq P_k(|p_xf)\}$.

Now the comment in brackets on page 12, lines 9-10 becomes superfluous* as does "with $n > 1^n$ " on page 11, four lines from the bottom*