1970

A categorical characterization of CH

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A CATEGORICAL CHARACTERIZATION OF CH

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Report 70-33

September, 1970
An obviously appropriate concern of Categorical Topology is to characterize categorically various subcategories of TOP (the category of topological spaces and continuous functions). For example, the sequential spaces can be characterized extrinsically as the mono-coreflective hull of the subcategory of TOP whose objects are just the homeomorphs of $u)^0 + 1$ with its order topology. [F]

An example of a somewhat different nature is Hatcher's recent characterization of non-empty, complete subcategories of CH (the compact Hausdorff spaces and continuous functions) as precisely those which can be defined by a class of identities on some space in CH, i.e. given a subcategory of CH there is a compact Hausdorff space $X$ and a class of pairs of maps

$$a, b$$

$\star z=\star x$ (where $\star$ is the singleton space) such that a compact Hausdorff space $Y$ belongs to the subcategory just in case

1 Only full and replete subcategories will be considered.
Another remarkable example is the recent result of Herrlich and Strecker [H-S] which roughly asserts that CH is the only epi-reflective subcategory of T2 (the Hausdorff spaces and continuous maps) which is varietal [He].

The purpose of this note is to point out that a very elegant theorem of de Groot can be given a categorical formulation which yields an intrinsic characterization of CH in TOP.

De Groot's theorem asserts that the compact Hausdorff spaces are the only non-trivial, productive, and closed hereditary class of spaces which are absolutely closed and preserved under closed images [W].

For the categorical translation let S denote the Sierpinski two point space with 0 the open point and 1 the closed point, and c:*->S the closed map. Three easy lemmas will now lead us to the theorem.

2 If a subspace of a space in the class also belongs, then it is a closed subspace.
LEMMA 1. A subspace $F$ of a space $X$ is closed iff there is some $f: X \to S$ such that

$$
\begin{array}{c}
\ast \\
\downarrow \\
F \\
\downarrow \\
i \\
\downarrow \\
X,
\end{array}
$$

with $i$ an extremal mono, is the pullback of

$$
\begin{array}{c}
\ast \\
\downarrow \\
c \\
\downarrow \\
f \\
\downarrow \\
X.
\end{array}
$$

The proof is routine since any such $f$ must be the characteristic function of $F$, and this function will serve.

This lemma allows us to use the notion of a closed subspace of an object of $\text{TOP}$ in categorical characterizations of other concepts. The next lemma is part of the folklore ([F], p. 33).

LEMMA 2. In any category with the extremal epi - mono factorization property

i) factorization is unique and extremal epimorphisms compose iff

ii) for each commutative diagram
there exists a unique $D \overset{h}{\rightarrow} A$ making everything commute.

It can be readily seen that $\text{TOP}$ satisfies the dual of condition ii) and hence extremal monos compose in $\text{TOP}$. To show that the intersection of extremal monos is again an extremal mono consider the commutative diagram

where each $m_\alpha$ is an extremal mono, and $e$ is epi. The $i_\alpha$ are all mono by a standard unique factorization argument. Condition ii) of the lemma yields an $h : A \rightarrow D$ for each $y$ so that the system $\{A, h_\psi \}$ (with $h = f$) is a lower bound. Then the unique $f : A \rightarrow HD$ obtained because $\text{PID}$ is a limit is, in fact, a left inverse for the epimorphism $e$, so each $i_\alpha$ is extremal.
It is now meaningful to discuss the "smallest" extremal mono satisfying a given condition and Lemma 3 can be stated.

**LEMMA 3.** A morphism \( f: X \rightarrow Y \) in TOP is closed iff for each closed \( F; i \rightarrow X \) the smallest \( G \rightarrow Y \) through which \( f \) factors is closed.

Proof: Consider the following diagram

\[
\begin{array}{ccc}
S & \xleftarrow{g} & X \\
\downarrow{h} & & \downarrow{f} \\
Y & \xrightarrow{c} & X \\
\downarrow{i} & & \downarrow{j} \\
F & \xrightarrow{m} & G
\end{array}
\]

It is clear that \( G \) is in fact \( f(F) \) and that the lemma is merely a restatement of the definition of a closed map.

Lemma 3 provides the needed categorical translation of "closed epi". In terms of it we may now state the main result.

**THEOREM.** CH JLS the only nontrivial productive subcategory \( K \) of TOP which contains closed epimorphic images and satisfies

(1) for any extremal mono \( F \xrightarrow{m} X \) in TOP with

\( X \) an object of \( K \), \( F \) JLS an object of \( K \)

iff \( * \xrightarrow{c} S \xleftarrow{\alpha} X \) is a pullback for some

\( * \xrightarrow{c} S \xleftarrow{\alpha} X \).
The proof consists of verifying that our translation is "correct," i.e., that our conditions are equivalent to those of de Groot (only "imply" is needed). Thus one need only check that (1) is equivalent to saying that the class of objects of $K$ is hereditarily closed and absolutely closed, since the rest is obvious.

Note that (1) can be subdivided into two categorical conditions, the first a translation of "hereditarily closed" and the second of "absolutely closed". This makes the proof completely trivial but results in an ungainly statement of the Theorem.

Finally, each of the hypotheses of the Theorem is necessary and independent. There follows a list giving for each condition of the Theorem a subcategory which satisfies the hypotheses except for that condition.

Non-trivial: The category of singleton spaces

Productive: The category of compact and sequential-ly compact Hausdorff spaces

Contains closed epimorphic images: The category of zero-dimensional compact Hausdorff spaces
(1): The category of continua.

(De Groot's conditions are also independent: continua work for closed hereditary, and compact spaces or ultracompact³ Hausdorff spaces for absolutely closed.)

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³ Countable sets have compact closure [B].
REFERENCES


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