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§0. INTRODUCTION

In recent years there has been some interest in trying to improve the behavior of maps by extending their domains. For example, in 1953 Whyburn showed that every map is the restriction of a compact map \[7\]. Similarly Kronele proved in 1967 that each locally perfect map can be extended to a perfect map \[12\] and in an as yet unpublished paper, Dickman obtained the same result for arbitrary maps \[4\]. Here we show that every map can be extended to an open map so that certain properties of the domain and range are preserved in the new domain. These results are then used to obtain analogues and improvements of recent theorems of Arhangel'skii, Coban, Hodel, and Proizvolov.

§1. OPEN EXTENSIONS

Let \( f : X \rightarrow Y \) be a function, not necessarily continuous, from a topological space \( X \) into a topological space \( Y \). We shall call a point \( x \in X \) and its image
f(x) \in Y \text{ singular points of } X \text{ and } Y \text{ respectively, if there is an open set } U \text{ of } X \text{ containing } x \text{ whose image } f(U) \text{ is not a neighbourhood of } f(x). \text{ The function } f \text{ is open just in case there are no singular points of } X \text{ (or equivalent of } Y).$

For each singular point \( x \) of \( X \), let \( Y_x \) be a copy of \( Y \). Let \( W = X \oplus (\oplus Y_x) \), where the second disjoint topological sum is taken over all singular points of \( X \). By identifying each singular point \( X \in X \) (with \( X \) thought of as a subset of \( W \)) with its image \( f(x) \) (as a point of \( Y \circ W \)) we arrive at a quotient space \( X^* \) of \( W \). The inclusion map \( i : X \to W \) composes with the quotient map \( q : W \to X^* \) to give an imbedding of \( X \) into \( X^* \). Hence we may think of \( X^* \) as an extension of \( X \).

Let \( f_* : W \to Y \) be the function whose restriction to \( X \) is \( f \), and whose restriction to each \( Y \) is the identity map \( 1 : Y \to Y \). We leave the reader to verify that the unique function \( f^* : X^* \to Y \) satisfying \( f^*oq = f \) is an open extension of \( f \). Recapitulating, we have that

1. There is an overspace \( X^* \) of \( X \) and an open function \( f^* : X^* \to Y \) whose restriction to \( X \) is \( f \); \( f^* \) is continuous just in case \( f \) is.

Since only sums and quotients were used in the construction of \( X^* \) it follows at once that
1.2 Any coreflexive property of $X$ and $Y$ is preserved in $X^*$. In particular, if $X$ and $Y$ are countably, sequentially or compactly generated so is $X^*$. If $X$ and $Y$ are locally connected, or are $P$-spaces, or are chain net spaces so is $X^*$. It is routine to verify that

1.3 $X^*$ satisfies the separation axioms $T_1$, $T_0$, $T_{0.5}$, and $T_3$ whenever $X$ and $Y$ do.

We shall prove only the last case, that of complete regularity. Suppose $F$ is a closed subset of $X^*$ and $p^F$. If $q^{-1}(p) \cap X = 0$ (where $q : W \rightarrow X^*$ is the quotient map), then $q^{-1}(p) \in Y$ for some $x$. Then there is a real valued function $*$ on $Y$ which is zero at $q^{-1}(p)$ and one at $x^* f(x) \in Y$ and on $q^{-1}(F) \cap Y$. Extend $*$ continuously to all of $W$ by taking it constantly one on $X$ and on each $Y_x$. This extended $*$ defines a real valued function on $X^*$ which separates $p$ and $F$. In the other case, if $x_0 = q^{-1}(p) \cap x$, let $*_0 : X \rightarrow R$ be zero at $X_0$ and one on $q^{-1}(F) \cap X$. For each singular $x$, choose $* : Y \rightarrow R$ which is one on $q^{-1}(F) \cap Y$ and such that $* (f(x)) = * (x)$. These functions combine to form one $* : W \rightarrow R$ which in turn induces a real valued function on $X^*$ separating $p$ and $F$.

$X^*$ can also be realized as an adjunction space. Let $F$ be the closed discrete subset of $\gamma Y$ whose intersection with each $Y_x$ is its singular point $f(x)$. The map $\gamma_x : \gamma_p \rightarrow X$ which sends each $f(x)$ to $x$, yields the adjunction space $\gamma Y \cup X$ which is homeomorphic to $X^*$. Using this representation we see that
1.4 \( X^* \) is normal, hereditarily normal, perfectly normal, collectionwise normal, or fully normal (i.e. paracompact) whenever \( X \) and \( Y \) are.

The first three of these properties are preserved under sums and adjunctions. For the other three, the assertion follows from a theorem of Tsuda [16].

If \( x \) is a singular point of \( X \), then \( f(x) \in Y \) and \( q(x) = q(f(x)) \). If \( X \) and \( Y \) are (pathwise) connected, so are their continuous images \( q(X) \) and \( q(Y) \). Since each \( q(X) \cap q(Y) \neq \emptyset \), \( U(q(x) \cup q(Y)) \) is (pathwise) connected. Thus we have that

1.5 \( X^* \) is (pathwise) connected whenever \( X \) and \( Y \) are.

Let \( X \) be the plane set consisting of the union of the closed intervals \([-1, 1]\) on the two axes. Let \( Y = [-1, 1] \) on one axis and let \( f \) be the restriction to \( X \) of the projection onto this same axis. Each point on one axis, except the origin is a singular point of \( X \). This example shows that

1.6 \( X^* \) need not preserve metrizability, either axiom of countability, weight or local weight, separability or density, the Lindelöf property, or (countable, sequential, pseudo) compactness.

(Another open extension of a map \( f \) can be given whose domain will preserve many of these properties. Since the family \( \{ l_x, f \} \) separates points and also separates points from closed sets, the evaluation map \( e : X \to X \times Y \) (given by \( e(x) = (x, f(x)) \)) is an
embedding. Hence $X$ is homeomorphic to $e(X)$, the graph of $\ell$, and the projection $i_r : X \times Y \to Y$ restricted to $e(X)$ is essentially $\ell$. Thus each map is the restriction of a projection map. If $X$ is compact, $f$ if the restriction of a clopen map.

This is also the case if $X$ is countably compact and $Y$ a subspace of a sequential space $[5J$. Clearly any finitely productive property of $X$ and $Y$ is preserved in the domain of the projection. Hence most of the properties mentioned in 1.6 are preserved.)

By imposing restrictions on the set $S$ of singular points of $X$, $X^*$ may be induced to preserve many other properties.

For example

1.7 If the singular points of $X \not\in D$ not accumulate, metrizability, local compactness and local weight are preserved in $X^*$. Further if the cardinality of $S$ is not more than the larger of the weights of $X$ and $Y$, then neither is the weight of $X^*$.

(Here we assume that $X$ and $Y$ are $T_1$ spaces.)

We first show that the quotient map $q : W \to X^*$ is a closed mapping under the given hypothesis. Suppose $F$ is a closed subset of $W$ and $\{pg\}$ is a net in $q(F)$ converging to a point $p$ in $X^*$. Now $q(F) = q(F \cap X) \cup (\bigcup q(F \cap Y) \cap x)$ and the restriction of $q$ to $X$ and to each $Y^*$ is an embedding. Since $q(X)$ and each $q(Y)$ is closed in $X^*$ we conclude that $q(F \cap X)$ and each $q(F \cap Y)$ is closed in $X^*$ also. Thus if $(p^*)$ is frequently in $q(F \cap X)$ or in some $q(F \cap Y)$, $p$ must belong to $q(F)$ and we are done. If $q^* (p) = \{y\}$ with $y \in Y^*$, let $U = Y^* \times x$.
If $q^{-1}(p) = \{x\}$ where $x$ is a non-singular point of $X$, let $U$ be a neighborhood of $x$ in $X$ which is free of singular points. If $q^{-1}(p) = (x, f(x))$ let $U$ be the union of $Y_x$ and a neighborhood of $x$ in $X$ which is free of singular points of $X$ other than $x$. In any case, $q(U)$ is a neighborhood of $p$ which has a non-empty intersection with at most one $q(Y \cap F)$. But $\{p^x\}$ is eventually in $q(U)$ and hence is frequently in either $q(U) \cap q(F \cap X)$ or $q(U) \cap q(F \cap Y)$. Hence $q$ is a closed map. Since each $q(p)$ is at most a doubleton, $q$ is a perfect map. Since metrizability and local compactness are preserved under sums and perfect maps, the first two assertions of 1.7 are proved.

Suppose $m$ is the larger of the local weights of $X$ and $Y$. If $\langle \sim (p)^-= fy \rangle$ where $y \in Y \{f(x)\}$ the image of a base at $y$ under $q$ is a base at $p$. If $q^{-1}(p) = \{x\}$ where $x$ is a non-singular point of $X$, a base can be chosen at $x$ whose members contain no singular points. The image of this base under $q$ is a base at $p$. If $q^{-1}(p) = \{x, f(x)\}$, choose a base $IB$ at $x$ whose members contain only one singular point, and choose a base $IB_1$ at $f(x) \in Y$, with the cardinality of $U$ and $IB$ no larger than $m$. The images under $q$ of sets of the form $B \cup V$ with $B \in B$ and $V \in V$, form a base at $p$ of cardinality no larger than $m$.

For what remains we need only note that weight $w = \text{weight } X + \text{card } S$. weight $Y <^+ m + m = m$ and that perfect maps do not increase weight.
Example 1.6 shows that the restrictions imposed on the set $S$ of singular points in 1.7 are not superfluous. By replacing one of the intervals $[-1, 1]$ in 1.6 by a sequence converging to zero, one can easily see that these cannot be weakened even to $S$ being a countable discrete subset.

§2. FINITE-TO-ONE MAPPINGS

In 1966, Proizvolov [15] showed that weight and metrizability are inversely preserved in locally compact spaces under open finite-to-one maps. Later that year, Arhangelskii [1] [2] showed they were always inversely preserved under clopen finite-to-one maps. In 1967, Coban [3] proved that hereditary paracompactness (metacompactness, Lindelof) are inversely preserved under open finite-to-one maps. (Some separation axioms are required for all these results.)

If $f$ is a finite-to-one mapping and the set $S$ of singular points of $X$ is finite (i.e. $f$ is open except at finitely many points) then $f^*: X^* \to Y$ is an open, finite-to-one map and hence $X^*$ will inherit properties from $Y$ by the results quoted in the paragraph above. (1.3 shows that the needed separation axioms (see below) also lift properly.) Since these properties are all hereditary, $X$ must also enjoy them. Thus we see that it is sufficient to require that $f$ be open except at finitely many points to arrive at the desired conclusion. For the convenience of the reader, we list precise statements of the improved theorems. (Assume all spaces to be Hausdorff, and that $f$ is continuous and onto.)
2.1 (Proizvolov) If X and Y are locally compact, and f is finite-to-one and open except at finitely many points, then weight $X \leq$ weight $Y$. If $Y$ is metrizable, so is $X$. In the proof, 1.7 as well as Proizvolov's original theorem must be used.

2.2 (Arhangelskiy) If $X$ and $Y$ are completely regular and $f$ is a finite-to-one closed map which is open except at finitely many points, then weight $X <$ weight $Y$. If $Y$ is metrizable, so is $X$.

Here, in addition to 1.3 we need to only note that with $S$ finite, $f^*$ is closed iff $f$ is closed.

2.3 (Coban) If $f$ is finite-to-one and open except at finitely many points, then $X$ is hereditarily paracompact (metacompact, Lindelöf) whenever $Y$ is. (For paracompactness $X$ is required to be regular.)

Simple examples can be given to show that the conditions on the singular points of $f$ cannot simply be omitted in 2.1, 2.2, and 2.3. First let $X = \text{fcty } Y = \{0\} \cup \{1/n|n \in \mathbb{N}\} \subset \mathbb{R}$ and $f : X \rightarrow Y$ arise from $n - 1/n$. $Y$ is locally compact, second countable and metrizable. $X$ is locally compact, has weight $c$ and is not metrizable, even though $f$ is a perfect map. This covers 2.1 and 2.2.

For 2.3 let $Y$ be as before and let $X^f$ be the ordinal compactification of $U$ recently constructed by Franklin and Rajagopalan [7], i.e. $X^f = IT U u^+ + 1$ with $U^f$ embedded as an open dense subspace, $u^+ + 1$ embedded as a closed subspace, $ITf 1 a) + 1 = 0$
and $X^1$ compact Hausdorff. Let $X = X^1 \setminus \{^\wedge\}$ and define $f : X \to Y$ by $f(n) = 1/n$ and $f(x) = 0$ otherwise. $Y$ is hereditarily Lindelöf (and much more) while $X$ fails to be metacompact.

§3. DIMENSION

(in this section all spaces are assumed to be metric, and $f$ is continuous and onto. In 1963 Hodel [11] showed that dimension cannot be lowered by open maps $f$ such that each $f^n(1)$ is discrete. The technique of the last section can be used to improve this result also.

3.1 (Hodel) if the singular points of $X$ do not accumulate and if each $f^n(1)$ is discrete, then $\dim X \leq \dim Y$.

For the proof we use 1.7, Hodel's original theorem and that $X$ is a closed subspace of $X^*$.

Hodel's theorem (in both the original and the improved version) holds true for not necessarily continuous $f$ if $Y$ is taken to be locally compact and separable.

To show that some hypothesis is needed on $f$, one need only look at Paeno's map of the interval onto the square.

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References


Footnotes

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(1) See Herrlich [10] for the definition and properties of coreflective subcategories.

(2) A countably generated space is one determined by countable subsets in the sense of Moore and Mrowka [14]. Sequentially generated spaces are the sequential spaces [6] and compactly generated spaces the k-spaces.

(3) For the coreflexivity of local connectedness, see Gleason [9]. A P-space is one in which each $G_\delta$ is open (see Gilman and Jerison [8]). A chain net is one whose underlying directed set is a chain. Chain nets are said to suffice if any set containing the limits of all its convergent chain nets is closed (see Misra [13]).