OSCILLATION AND BOUNDEDNESS CRITERIA FOR A CLASS OF NONLINEAR DIFFERENTIAL SYSTEMS

by

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1. We consider systems of the form

$$\frac{dx}{dt} = A(x,t)x, \tag{1}$$

where t varies in a closed real interval I, x is a real continuous n-vector, and $A(x,t)$ is a real $n \times n$ matrix whose elements are continuous functions of t (for $t \in I$) and of the components $x_1,\ldots,x_n$ of x (for $x_k \in (-\infty,\infty)$, $k = 1,\ldots,n$). A solution of (1) on I will be said to be nonoscillatory on this interval if at least one of its components does not vanish on I. While for a linear system (1) there always exists, for a given t_o, a value $t_1 \in (t_o,\infty)$ such that all the solutions of the system are nonoscillatory in $[t_o,t_1]$ [1], this is in general not the case if the system is nonlinear. An elementary example which exhibits this type of behavior is the system $x_1' = x_2$, $x_2' = -x_1^3$, which possesses solutions with arbitrarily small intervals of nonoscillation. In the nonlinear case it is therefore necessary to speak of the nonoscillation of specific solutions or, at best, of certain classes of solutions. We shall characterize an individual solution by its initial value at the left endpoint of the interval I. However, in order to formulate a meaningful criterion for the nonoscillation of such a solution it is also necessary to establish the existence of this solution throughout the interval I. The condition given in Theorem I guarantees both existence and nonoscillation. Theorem II gives a weaker condition which, though not sufficient for nonoscillation, still assures the existence in I of a solution with specified initial values and, moreover, yields an estimate for the norm of the solution vector.

We denote by $||B||$ the operator norm of the matrix B, i.e. $||B|| = \max_{a} ||BCX||$ _, where $||cc|| = 1$ and $||a|j$ is the euclidean norm of the vector a. For the matrix $A(x,t)$ we define the norms

(2) $$N_a[A(x,t)] = \max_{Hx||\leq Cf} ||A(x,t)||$$

which depend on a positive number a. While it may be difficult to compute N_σ for a given $A(x,t)$, it is generally very easy to find an upper bound for it, i.e., a function $R(cr,t)$ which increases with $a(cr \geq 0)$ and is such that

(3) $$l|A(x,t)|| \leq R(||x||,t).$$

We shall accordingly formulate our results in terms of such a bound $R(cr,t)$ rather than the norm (2) ; the best possible result will correspond to the choice $R(cr,t) = N_\sigma[A(x,t)]$.

For the existence proof in Theorem II we shall use the Lipschitz condition

(4) $$\|A(x,t) - A(y,t)\| \leq L_\sigma \|x-y\| \quad (\|x\| \leq \sigma, \|y\| \leq \sigma, t \in I).$$

Because of the continuity in t and the restrictions on x,y this condition will, for instance, be satisfied if the elements of $A(x,t)$ have continuous partial derivatives with respect to the components of x.

We now state our first result.

__Theorem I.__ __Let__ $A(x,t)$ __satisfy the inequality__ (3), __where__ $R(\sigma,t)$ __is nonnegative, continuous in both arguments, and non-decreasing in__ σ __for fixed__ t, __for__ $t \in [a,b]$ __and__ $\sigma \in [0,\infty)$. __Denote by__ τ __the solution of the ordinary differential equation__

(5) $$\tau' = \tau R(\tau,t)$$

__with the initial condition__ $\tau(a) = \gamma (\gamma > 0)$. __If__ τ __exists throughout__ $[a,b]$ __and__

(6) $$\tau(b) < \tau(a) e^{\frac{\pi}{2}}$$

__then all solutions__ x __of__ (1) __for which__ $\|x(a)\| \leq \gamma$ __exist and are nonoscillatory in__ $[a,b]$. __The constant__ $\exp(\pi/2)$ __in__ (6) __is the best possible; in fact, the conclusion does not necessarily follow if equality is permitted in__ (6).

If the system (1) is linear -- i.e., if the coefficient matrix A does not depend on x -- (6) reduces to the condition

(7) $$\int_a^b \|A(t)\| dt < \frac{\pi}{2} .$$

[1], which evidently guarantees the nonoscillation of __all__ solutions of the equation on $[a,b]$. A similar conclusion can be

drawn if the system is sublinear, i.e., if for all positive a the norms (2) are dominated by a continuous function $R(t)$. If $\int_a^b R(t)dt < \frac{\pi}{2}$, all solutions will again be nonoscillatory.

2. The proof of Theorem I in the general case will be obtained by combining the result for linear systems [1] with a suitable majorizing procedure, and we therefore begin with a brief derivation of the linear result. If $||x|| = a$ and $u = ax$, the equation $x' = A(t)x$ transforms into the equation

$$u' = Au - \frac{a'}{a}u$$

for the unit vector u. Since $uu' = 0$, this takes the form

$$u' = Au - u(uAu).$$

If c is a constant unit vector, we have

$$(cu)' = [c - (cu)u]Au$$

and thus

$$|(cu)'| \leq ||c - (cu)u|| \, ||A|| = \sqrt{1 - (cu)^2} \, ||A||,$$

i.e.,

$$\left| \frac{(cu)'}{\sqrt{1-(cu)^2}} \right| \leq ||A||. \tag{8}$$

Integrating over $[a,b]$, we obtain

$$|\arcsin cu(b) - \arcsin cu(a)| \leq \int_a^b ||A|| dt. \tag{9}$$

If x is oscillatory on $[a,b]$, there exist values $t_1,\ldots,t_n$ on $[a,b]$ such that $u_k(t_k) = 0$ $(k=1,\ldots,n,$ $u = (u_1,\ldots,u_n))$. If we denote by $S(t)$ the diagonal matrix with the elements $s_{kk}(t) = \operatorname{sgn}(t_k - t)$ $(t \neq t_k)$, $s_{kk}(t_k) = 0$ and define the piecewise constant unit vector $c(t)$ by $c(t) = S(t)c$, the inner product $c(t)u(t)$ remains continuous throughout $[a,b]$. We may therefore replace c by $c(t)$ in (8) and integrate over $[a,b]$. Since $c(a) = c$ and $c(b) = -c$, this yields

$$|\arcsin cu(b) + \arcsin cu(a)| \leq \int_a^b \|A\|dt.$$

Combining this with (9), we have

$$|\arcsin cu(b)| + |\arcsin cu(a)| \leq \int_a^b \|A\|dx,$$

and thus, with the particular choice $c = u(a)$,

$$\frac{\pi}{2} + |\arcsin u(a)u(b)| \leq \int_a^b \|A\|dx.$$

Since this is incompatible with (7), the latter condition has thus been shown to be sufficient to guarantee the nonoscillation of all nontrivial solutions of $x' = A(t)x$ in $[a,b]$.

Turning now to the general case (and postponing the discussion of existence until the proof of Theorem II), we note that a nontrivial solution $\bar{x}(t)$ of (1) may be regarded as a solution of the linear system

$$x' = A(\bar{x}(t),t)x.$$

Since, as just shown, all nontrivial solutions of this system (including $\bar{x}$) will be nonoscillatory on $[a,b]$ if

(10) $$\int_a^b \|A(\bar{x}(t),t)\|dt < \frac{\pi}{2} ,$$

Theorem I will be proved if the assumptions made can be shown to imply (10).

By (1) and (3), we have

$$\bar{x}\bar{x}' = \bar{x}A(\bar{x},t)\bar{x} \leq \|\bar{x}\|^2\|A(\bar{x},t)\| \leq \|\bar{x}\|^2R(\|\bar{x}\|,t) .$$

Setting $\|\bar{x}\| = \sigma$ and noting that $\bar{x}\bar{x}' = \sigma\sigma'$, we obtain

(11) $$\sigma' \leq \sigma R(\sigma,t) , \qquad \sigma(a) = \|\bar{x}(a)\|.$$

If we compare this with the differential equation

(12) $$\tau' = \tau R(\tau,t) , \qquad \tau(a) = \gamma \geq \|\bar{x}(a)\| = \sigma(a) ,$$

and take account of the fact that $R(\tau,t)$ is a nonnegative nondecreasing function of τ, an elementary argument shows that $\sigma \leq \tau$ as long as the solution τ of (12) exists. Using again the monotonicity of $R(\tau,t)$, and observing (11), we find that

$$\int_a^b R(\sigma,t)\,dt \leq \int_a^b R(\tau,t)\,dt = \log \frac{\tau(b)}{\tau(a)} < \frac{\pi}{2} ,$$

where the last inequality follows from (6). By (3), and because of $\|\bar{x}\| \leq \tau$, the solution $\bar{x}$ of (1) is thus subject to condition (10). Since this was shown to imply to nonoscillation of $\bar{x}$ on $[a,b]$, the main assertion of Theorem I is proved.

As mentioned before, (6) reduces to (7) if the system is linear. Since (7) is the best possible condition of its kind [1], this establishes the assertion regarding the sharpness of condition (6).

3. We illustrate the use of Theorem I by two examples. Our first example concerns the n-th order differential equation

$$(13)\qquad x^{(n)} + x^{(r)}F(t,x,x',\dots,x^{(n-1)}) + x = 0,\quad n \ge 2,\quad 1 \le r \le n-1,$$

which is equivalent to a system (1) if the non-zero elements $A_{\mu\nu}$ of the matrix A are $A_{\nu,\nu+1} = 1$ $(\nu=1,\dots,n-1)$, $A_{n1} = -1$, $A_{n,r+1} = -F$. If $a = (a_1,\dots,a_n)$ is a unit vector, we have,

$$(Aa)^2 = 1 + 2a_1a_{r+1}F + a_{r+1}^2F^2,$$

and therefore

$$\|A\|_2^2 = 1 + \frac{1}{2}F^2 + \frac{1}{2}|F|\left(F^2+4\right)^{1/2}.$$

An application of Theorem I thus yields the following result.

Let $|F(t,x,x',\dots,x^{(n-1)})| \le R(\rho,t)$, *where* $\rho^2 = x^2 + x'^2 + \dots + [x^{(n-1)}]^2$ *and* $R(\rho,t)$ *is nondecreasing in* ρ *and continuous in both arguments, and let the differential equation* $r' = rR(r,t)$ *have a continuous solution in* $[a,b]$ *with a positive initial value* $r(a)$. *If* $r(b) < r(a)e^{A}$ *and if* x *is a solution of* (13) *in* $[a,b]$ *for which* $x^2(a) + x'^2(a) + \dots + [x^{(n-1)}(a)]^2 \le r^2(a)$, *then at least one of the functions* $x, x', \dots, x^{(n-1)}$ *does not vanish on* $[a,b]$.

It may be noted that, because of Rolle's theorem, it also follows that such a solution cannot have more than $n-1$ zeros in $[a,b]$.

To illustrate the last statement in a specific instance, we consider the equation

$$(14)\qquad x'' + p(t)F(x,x') + x = 0,$$

with $|F(x,x')| \le R(\sqrt{x^2+x'^2})$, where $R(\alpha)$ increases with α. If x is a solution of (14) with $x(a) = 0$, $x'(a) = \gamma > 0$, and if b is subject to

$$\int_{\gamma}^{\gamma e^{\frac{\pi}{2}}} \frac{ds}{sR(s)} > \int_a^b |p(t)|\,dt,$$

then x is positive and increasing in $[a,b]$.

As another application of Theorem I we consider the autonomous equation

$$x^{(n)} + xF(x,x',\ldots,x^{(n-1)}) = 0,$$

which may be replaced by the system $x_1' = cx_2$, $x_2' = cx_3, \ldots, x_{n-1}' = cx_n$, $x_n' = -c^{1-n}x_1F$, where c is an arbitrary positive constant. If this system is written in the form (1), the norm of the matrix A is easily found to be

$$\|A\| = \max\,[c,c^{1-n}|F|].$$

Hence, if $|F(x,\ldots,x^{(n-1)})| \le R(\sigma)$ $(\sigma^2 = x^2 + \ldots + [x^{(n-1)}]^2)$ and $R(\sigma)$ is an increasing function, the right endpoint of the interval $[a,b]$ in which nonoscillation can be guaranteed is given by $r(b) = r(a)e^{\frac{\pi}{2}}$ where r is the solution of

$$r' = r\max\,[c,c^{1-n}R(r)],\quad r(a) = \sigma(a).$$

Since r is increasing for any choice of c, we will have $R(r) \ge c^n$ if c is so chosen that $R[\sigma(a)] = c^n$. With this value of c, the equation is therefore of the form $r' = c^{-n+1}rR(r)$, and an application of Theorem I shows that the existence and non-oscillation of a solution x for which $x^2(a) + x'^2(a) + \ldots + [x^{(n-1)}(a)]^2 = \gamma^2\ (\gamma > 0)$ can be guaranteed on the interval $[a,b]$, where

$$b - a = [R(\gamma)]^{\frac{n-1}{n}} \int_{\gamma}^{\gamma e} \frac{ds}{sR(s)} .$$

4. So far we have not shown that the hypotheses of Theorem I are sufficient to guarantee the existence throughout $[a,b]$ of the solutions under consideration. This question will be settled by the following theorem. In fact, it will be shown that for the purpose of establishing existence we may replace the norm (2) by the maximum of the quadratic form $\alpha A(x,t)\alpha$ $(\|\alpha\| = 1)$ for $\|x\| \leq \epsilon_a$. Since this maximum cannot exceed the norm *(2)*, our result will be stronger than that required for the purposes of Theorem I.

Theorem II. Let $A(x,t)$ *satisfy the Lipschitz condition* (4) *and the inequality*

$$(15) \qquad \alpha A(x,t)\alpha \leq S(\|x\|,t) \qquad (\|\alpha\| = 1),$$

where the function $S(\sigma,t)$ *is continuous in both arguments, and nondecreasing in* σ *for fixed* t, *for* $t \in [a,b]$ *and* $\sigma \in [0,\infty)$. *Let* r *be a solution of the ordinary differential equation*

$$(16) \qquad r' = rS(r,t)$$

whose interval of existence includes $[a,b]$ *and for which* $r(a) > 0$. *If* c *is a constant vector satisfying* $\|c\| \leq r(a)$, *then the system* (1) *has a unique solution* $x(t)$ *in* $[a,b]$ *with the initial condition* $x(a) = c$, *and we have*

$$(17) \qquad \|x(t)\| \leq r(t).$$

We note that for sufficiently small (positive) initial values equation (16) will have solutions which exist throughout [a,b]. Indeed, if we take $0 < \tau(a) \leq \gamma$, where

$$(18) \qquad \gamma = M \exp[-\int_a^b S(M,t)dt], \quad M > 0,$$

(M constant) and consider the sequence of functions $\tau_0, \tau_1, \ldots$ defined by

$$(19) \qquad \tau_{m+1}(t) = \tau(a) \exp[\int_a^t S(\tau_m, s)ds], \quad \tau_0 = \tau(a),$$

it follows from (18) and the fact that $\tau(a) \leq \gamma$ that

$$\tau_{m+1}(t) \leq M \exp\{\int_a^t [S(\tau_m, s) - S(M,s)]ds - \int_t^b S(M,s)ds\}.$$

An induction argument (using the monotonicity of S and the fact that $\tau_0 \leq M$) then shows that $\tau_m \leq M$ for all m. Moreover $\tau_1 \geq \tau_0$, and the application of a standard argument to (19) shows that $\tau_{m+1} \geq \tau_m$. Because of the equicontinuity of the sequence $\{\tau_m\}$ (which follows from (19) and the uniform boundedness of the τ_m) it thus has a uniform limit, which clearly is a solution of (16) (and is bounded by M). Theorem II will therefore have the following:

Corollary. *If $\|c\| \leq \gamma$, where γ is the number defined in (18), the solution $x(t)$ of (1) with the initial value $x(a) = c$ exists, and satisfies $\|x(t)\| \leq M$, throughout $[a,b]$.*

Turning now to the proof of Theorem II, we remark that it is easy to obtain the estimate (17) for a solution of (1) whose existence in [a,b] is known beforehand. Indeed, if we set

$\sigma = ||x||$ we have, by (1) and (15), $\sigma\sigma' = xx' = xA(x,t)x \leq \sigma^2 S(\sigma,t)$, i.e., $\sigma' \leq \sigma S(\sigma,t)$. If T is the solution of $T' = TS(T,t)$ for the initial condition $T(a) = \sigma(a)$, it follows from the monotonicity of S (with respect to σ) that $\sigma \leq T$, and this establishes (17). It may be noted that for a linear system (1), (17) reduces to the well-known inequality

$$||x|| \leq ||x(a)||\exp\{\int_a^t \Lambda(s)ds\},$$

where $\Lambda(s) = \max \alpha A(s)\alpha$ for $||\alpha|| = 1$ [2].

To prove the existence part of Theorem II, we set up the iteration scheme

(20) $$x_{m+1}' = A(x_m,t)x_{m+1}, \qquad x_{m+1}(a) = c,$$

where it may be noted that, in contrast to the customary successive approximation procedure, every step requires obtaining the solution of a linear differential system. If we set $||x_m|| = \sigma_m$ we have, by (20) and (15),

$$\sigma_{m+1}\sigma_{m+1}' = x_{m+1}x_{m+1}' = x_{m+1}A(x_m,t)x_{m+1} \leq \sigma_{m+1}^2 S(\sigma_m,t).$$

Hence,

$$\sigma_{m+1}(t) \leq ||c||\exp\{\int_a^t S(\sigma_m,s)ds\}.$$

If T is a solution of (16) with an initial value $T(a) \geq ||c||$ and if $\sigma_m \leq T$, it follows that

$$\sigma_{m+1}(t) \leq T(a)\exp\{\int_a^t S(T,s)ds\}$$

and therefore, by (16), $\tau_{m+1} \le T$. If we begin the iteration with $x_0(t) = c$, we thus have $\tau_m \le T$ for all m.

Accordingly, if $T \le M$ in $[a,b]$, we have $\|x_m\| \le M$ for all m, and it follows from (4) that

$$\|A(x_m,s) - A(x_{m-1},s)\| \le L\|x_m - x_{m-1}\|, \quad t \in [a,b],$$

where $L = L_M$ is a constant independent of t. Similarly, because of $\|x_m\| \le M$ and the continuity of $A(x,t)$ in x and t there exists a constant K such that $\|A(x_m,t)\| \le K$ for $t \in [a,b]$ and all m. Noting that, by (20),

$$x_{m+1}(t) - x_m(t) = \int_a^t \{[A(x_m,s) - A(x_{m-1},s)]x_{m+1}$$

$$+ A(x_{m-1},s)(x_{m+1} - x_m)\}ds,$$

and utilizing these estimates, we are led to the inequality

(21) $$\|x_{m+1}-x_m\| \le ML\int_a^t \|x_m - x_{m-1}\|ds + K\int_a^t \|x_{m+1}-x_m\|ds.$$

With the abbreviation

$$P_m(t) = e^{-Kt}\int_a^t \|x_m - x_{m-1}\|ds,$$

this may be written

$$P_{m+1}(t) \le ML\int_a^t P_m(s)ds,$$

and a standard argument shows that this inequality implies the uniform convergence of $\sum_m P_m(t)$ in $[a,b]$. Since, by (21),

$$\|x_{m+1}-x_m\| \le e^{Kt}[MLP_m(t) + KP_{m+1}(t)],$$

this proves that the sequence $\{x_m\}$ converges uniformly to a solution of (1). The uniqueness of this solution is then shown by a similar modification of the usual argument. This completes the proof of Theorem II.

We finally note that the conditions imposed on the matrix $A(x,t)$ in Theorem II are in general not sufficient to make the system (1) nonoscillatory. For example, in the case of a skew-symmetric matrix $A(x,t)$ the function S in (15) may be taken to be identically zero, and the interval of existence of all solutions of (1) will therefore coincide with the interval in which $A(x,t)$ is continuous in t. An elementary example of an oscillatory system of this type is the linear system corresponding to the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Its general solution is the vector $(\gamma \cos(t-t_o), \gamma \sin(t-t_o))$, where γ_o and t_o are constants, and the system is thus oscillatory in any closed interval of length $\pi/2$.

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