The Circuit Polytope

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Abstract

The circuit constraint requires that a sequence of n vertices in a directed graph describe a hamiltonian cycle. The constraint is useful for the succinct formulation of sequencing problems, such as the traveling salesman problem. We analyze the circuit polytope as an alternative to the traveling salesman polytope as a means of obtaining linear relaxations for sequencing problems. We provide a nearly complete characterization of the polytope by showing how to generate, using a greedy algorithm, all facet-defining inequalities that contain at most $n-4$ terms. We suggest efficient separation heuristics. Finally, we show that proper choice of the numerical values that index the vertices can allow the resulting relaxation to exploit structure in the objective function.

1 The Circuit Constraint

The *circuit constraint* requires that a sequence of vertices in a directed graph define a hamiltonian circuit.

Let G be a directed graph on vertices $1, \ldots, n$, and let variable x_i denote the vertex that follows vertex i in the sequence. The *domain* D_i of each variable x_i (i.e, the set of values x_i can take) is the set of integers j for which (i, j) is an edge of G. The constraint

$$\text{circuit}(x_1, \ldots, x_n) \tag{1}$$

requires that $x = (x_1, \ldots, x_n)$ describe a hamiltonian circuit of G. For brevity, we will say that an x satisfying (1) is a *circuit*.

More precisely, x is a circuit if $\pi_1, \ldots, \pi_n$ is a permutation of $1, \ldots, n$, where $\pi_1 = 1$ and $\pi_{i+1} = x_{\pi_i}$ for $i = 1, \ldots, n-1$. Thus $\pi_1, \ldots, \pi_n$ indicates the order in which the vertices are visited. For example, if $\{1, 2, 3\}$ is the domain of each variable x_i, then $(x_1, x_2, x_3) = (3, 1, 2)$ is a circuit because $(\pi_1, \pi_2, \pi_3) = (1, 3, 2)$ is a permutation. The circuit goes from 1 to 3 to 2, and back to 1. However,

$(x_1, x_2, x_3) = (1, 2, 3)$ is not a circuit, because $(\pi_1, \pi_2, \pi_3) = (1, 1, 1)$ is not a permutation.

If x is a circuit, the sequence $x_1, \ldots, x_n$ is itself a permutation, but a given permutation x need not be a circuit. In fact, if the domain of each x_i is $\{1, \ldots, n\}$, then $n!$ values of x are permutations but only $(n-1)!$ of these are circuits. In the above example, there are six permutations but only two circuits, namely $(2, 3, 1)$ and $(3, 1, 2)$.

The circuit constraint is useful for formulating combinatorial problems that involve permutations or sequencing. One of the best known such problems is the traveling salesman problem, which may be very succinctly written

$$\begin{aligned} &\min \sum_{i=1}^{n} c_{ix_i} \\ &\text{circuit}(x_1, \ldots, x_n), \ \ x_i \in D_i, \ i = 1, \ldots n \end{aligned} \tag{2}$$

where c_{ij} is the distance from city i to city j. The objective is to visit each city once, and return to the starting city, in such a way as to minimize the total travel distance.

Domain filtering methods for the circuit constraint appear in [2, 6, 8]. These can be useful for eliminating infeasible values from the variable domains. The object of the present paper is to study the circuit polytope, so as to obtain a relaxation for the circuit constraint that can be combined with filtering to accelerate solution further.

2 The Circuit Polytope

The *circuit polytope* is the convex hull of the feasible solutions of (1) when G is a complete graph; that is, when each variable domain D_i is $\{1, \ldots, n\}$. To our knowledge, this polytope has not been studied. Rather, the circuit constraint is generally formulated by replacing the variables x_i with 0-1 variables y_{ij}, where $y_{ij} = 1$ if vertex j immediately follows vertex i in the hamiltonian circuit. The traveling salesman problem (2), for example, is typically written

$$\begin{aligned} &\min \sum_{ij} c_{ij} y_{ij} \\ &\sum_j x_{ij} = \sum_j y_{ji} = 1, \ \ i = 1, \ldots, n && (a) \\ &\sum_{\substack{i \in V \\ j \notin V}} y_{ij} \geq 1, \ \ \text{all } V \subset \{1, \ldots, n\} \text{ with } 2 \leq |V| \leq n-2 && (b) \\ &y_{ij} \in \{0, 1\}, \ \ \text{all } i, j && (c) \end{aligned} \tag{3}$$

The polyhedral structure of problem (3) has been intensively analyzed, and surveys of this work may be found in [1, 5, 7].

Rather than introduce 0-1 variables, we analyze the circuit polytope directly. In particular, we provide an almost complete description of the polytope, in the sense that we show how to identify almost all facets of the polytope by identifying *undominated* circuits. A subset of these facet-defining inequalities can be assembled to obtain a tight continuous relaxation of the circuit constraint.

This approach has four possible advantages. (a) The facet-defining inequalities are expressed in terms of n variables, rather than n^2 variables as in the conventional approach. (b) The inequalities are quite different from the traditional traveling salesman cuts and may have complementary strengths. (c) Because the variables can take arbitrary values (not just $1, \ldots, n$), these values can be chosen to exploit structure in the objective function coefficients. (d) We can give a nearly complete description of the circuit polytope, which does not appear to be possible for the 0-1 traveling salesman polytope.

We have not demonstrated these advantages computationally. The goal of this paper is to lay the theoretical groundwork by describing the circuit polytope, which is an interesting object of study in its own right.

3 Arbitrary Domains

A peculiar characteristic of the circuit constraint is that the values of its variables are indices of other variables. Because the vertex immediately after x_i is x_{x_i}, the value of x_i must index a variable. The numbers $1, \ldots, n$ are normally used as indices, but this is an arbitrary choice. One could just as well use any other set of distinct numbers, which would give rise to a different circuit polytope. Thus the circuit polytope cannot be fully understood unless it is characterized for general numerical domains, and not just for $1, \ldots, n$. This also provides more modeling flexibility that can be used to exploit problem structure (Section 11).

We therefore generalize the circuit constraint so that each domain D_i is drawn from an arbitrary set $\{v_0, \ldots, v_{n-1}\}$ of nonnegative real numbers. The constraint is written

$$\text{circuit}(x_{v_0}, \ldots, x_{v_{n-1}}) \tag{4}$$

It is convenient to assume $v_0 < \cdots < v_{n-1}$. Thus circuit$(x_0, x_{2.3}, x_{3.1})$ is a well-formed circuit constraint if the variable domains are subsets of $\{0, 2.3, 3.1\}$. The nonnegativity of the v_is does not sacrifice generality, since one can always translate the origin so that the feasible points lie in the nonnegative orthant.

To avoid an additional layer of subscripts, we will consistently abuse notation by writing x_{v_i} as x_i. We therefore write the constraint (4) as

$$\text{circuit}(x_0, \ldots, x_{n-1}) \tag{5}$$

Thus $x = (x_0, \ldots, x_{n-1})$ satisfies (5) if and only if $\pi_0, \ldots, \pi_{n-1}$ is a permutation of $0, \ldots, n-1$, where $\pi_0 = 0$ and $v_{\pi_i} = x_{\pi_{i-1}}$ for $i = 1, \ldots n-1$.

We define the circuit polytope $C_n(v)$ with respect to $v = (v_0, \ldots, v_{n-1})$ to be the convex hull of the feasible solutions of (5) for full domains; that is, each domain D_i is $\{v_0, \ldots, v_{n-1}\}$. All of the facet-defining inequalities we identify below for full domains are valid inequalities for smaller domains, even if they may not define facets of the convex hull.

The circuit polytope has a different character than most polytopes studied in combinatorial optimization. Normally the shape of the polytope does not depend on particular numerical values, but only on the structure of the problem. Because the structure of the circuit polytope depends on the domain values, the polytope is partly a discrete and partly a continuous object. This will be reflected in combinatorial and numerical phases of the method for generating facets.

4 Overview of the Results

We first examine the dimensionality of the circuit polytope (Theorem 1). We then prove the basic result (Theorems 4 and 5), which is the following. Consider any subset of at most $n-4$ variables, and let a partial solution of the circuit constraint be one that assigns values to these variables only. Then the facet-defining inequalities containing these variables are precisely the valid inequalities defined by affinely independent sets of *undominated* partial solutions. Furthermore, these inequalities are valid if and only if they are satisfied by all undominated partial solutions.

We can therefore identify all facet-defining inequalities with at most $n-4$ terms if we generate undominated partial solutions, which is a purely combinatorial problem that does not depend on the particular domain values $v_0, \ldots, v_{n-1}$. We solve this problem by describing a greedy algorithm that generates all undominated partial solutions for any given subset of variables (Theorems 6 and 7). We can now identify facet-defining inequalities by solving a continuous, numerical problem. We compute the inequalities defined by affinely independent sets of these partial solutions and check which ones are satisfied by the remaining partial solutions, given the particular numerical values of the domain elements. The inequalities that pass this test are facet-defining.

We next contrast the circuit polytope with the permutation polytope, which contains the circuit polytope, and whose facial structure is well known. We identify a large class of permutation facets that are also circuit facets (Corollary 8). The circuit polytope is more complicated than the permutation polytope, however, and unlike the permutation polytope, its structure depends on the domain values. We also explicitly identify all two-term facets of the circuit polytope (Corollary 9).

We then address the separation problem, which is the problem of identifying facet-defining inequalities that are violated by a solution of the current relaxation of the problem. We describe two separation heuristics, one of which seeks separating inequalities with all positive coefficients, and one which seeks inequalities with arbitrary coefficients.

We conclude by showing how knowledge of the circuit polytope for arbitrary domains can allow one to exploit cost structure in the objective function of the problem.

5 Dimension of the Polytope

We begin by establishing the dimension of the circuit polytope.

Theorem 1 *The dimension of the circuit polytope $C_n(v)$ is $n-2$ for $n = 2, 3$ and $n-1$ for $n \geq 4$.*

Proof. The polytope $C_n(v)$ is a point (v_1, v_0) for $n = 2$ and the line segment from (v_1, v_2, v_0) to (v_2, v_0, v_1) for $n = 3$. In either case the dimension is $n-2$.

To prove the theorem for $n \geq 4$, note first that all feasible points for (5) satisfy

$$\sum_{i=0}^{n-1} x_i = \sum_{i=0}^{n-1} v_i \tag{6}$$

(Recall that x_i is shorthand for x_{v_i}.) Thus, $C_n(v)$ has dimension at most $n-1$. To show it has dimension exactly $n-1$, it suffices to exhibit n affinely independent points in $C_n(v)$. Consider the following n permutations of $v_0, \ldots, v_{n-1}$, where the first $n-1$ permutations consist of v_0 followed by cyclic permutations of $v_1, \ldots, v_{n-1}$. The last permutation is obtained by swapping v_{n-2} and v_{n-1} in the first permutation:

$$\begin{array}{llllll}
v_0, & v_1, & v_2, & \ldots, & v_{n-3}, & v_{n-2}, & v_{n-1} \\
v_0, & v_2, & v_3, & \ldots, & v_{n-2}, & v_{n-1}, & v_1 \\
v_0, & v_3, & v_4, & \ldots, & v_{n-1}, & v_1, & v_2 \\
 & & & \vdots & & & \\
v_0, & v_{n-2}, & v_{n-1}, & \ldots, & v_{n-5}, & v_{n-4}, & v_{n-3} \\
v_0, & v_{n-1}, & v_1, & \ldots, & v_{n-4}, & v_{n-3}, & v_{n-2} \\
v_0, & v_1, & v_2, & \ldots, & v_{n-3}, & v_{n-1}, & v_{n-2}
\end{array} \tag{7}$$

The rows of the following matrix correspond to circuit representations of the above permutations. Thus row i contains the values $x_0, \ldots, x_{n-1}$ for the ith permutation in (7).

$$\begin{bmatrix}
v_1 & v_2 & v_3 & \cdots & v_{n-2} & v_{n-1} & v_0 \\
v_2 & v_0 & v_3 & \cdots & v_{n-2} & v_{n_1} & v_1 \\
v_3 & v_2 & v_0 & \cdots & v_{n-2} & v_{n-1} & v_1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
v_{n-2} & v_2 & v_3 & \cdots & v_0 & v_{n-1} & v_1 \\
v_{n-1} & v_2 & v_3 & \cdots & v_{n-2} & v_0 & v_1 \\
v_1 & v_2 & v_3 & \cdots & v_{n-1} & v_0 & v_{n-2}
\end{bmatrix} \tag{8}$$

Since each row of (8) is a point in $C_n(v)$, it suffices to show that the rows are affinely independent. Subtract $[v_{n-1} \;\, v_2 \;\, v_3 \;\, \cdots \;\, v_{n-2} \;\, v_{n-1} \;\, v_1]$ from every row of (8) to obtain

$$\begin{bmatrix} v_1 - v_{n-1} & 0 & 0 & \cdots & 0 & 0 & v_0 - v_1 \\ v_2 - v_{n-1} & v_0 - v_2 & 0 & \cdots & 0 & 0 & 0 \\ v_3 - v_{n-1} & 0 & v_0 - v_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-2} - v_{n-1} & 0 & 0 & \cdots & v_0 - v_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_0 - v_{n-1} & 0 \\ v_1 - v_{n-1} & 0 & 0 & \cdots & v_{n-1} - v_{n-2} & v_0 - v_{n-1} & v_{n-2} - v_1 \end{bmatrix} \tag{9}$$

The rows of (8) are affinely independent if and only if the rows of (9) are. It now suffices to show that (9) is nonsingular, and we do so through a series of row operations. The first step is to subtract $(v_{n-2} - v_1)/(v_0 - v_1)$ times row 1, $(v_{n-1} - v_{n-2})/(v_0 - v_{n-2})$ times row $n-2$, and row $n-1$ from row n to obtain

$$\begin{bmatrix} v_1 - v_{n-1} & 0 & 0 & \cdots & 0 & 0 & v_0 - v_1 \\ v_2 - v_{n-1} & v_0 - v_2 & 0 & \cdots & 0 & 0 & 0 \\ v_3 - v_{n-1} & 0 & v_0 - v_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-2} - v_{n-1} & 0 & 0 & \cdots & v_0 - v_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_0 - v_{n-1} & 0 \\ E_n & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

where

$$E_n = -\frac{v_{n-1} - v_{n-2}}{v_{n-2} - v_0}(v_{n-1} - v_{n-2}) - (v_{n-1} - v_1)$$

Interchange the first and last rows of (10) to obtain

$$\begin{bmatrix} E_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ v_2 - v_{n-1} & v_0 - v_2 & 0 & \cdots & 0 & 0 & 0 \\ v_3 - v_{n-1} & 0 & v_0 - v_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-2} - v_{n-1} & 0 & 0 & \cdots & v_0 - v_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_0 - v_{n-1} & 0 \\ v_1 - v_{n-1} & 0 & 0 & \cdots & 0 & 0 & v_0 - v_1 \end{bmatrix} \tag{11}$$

Note that $E_n < 0$ since $v_0 < \cdots < v_{n-1}$. Thus (11) is a lower triangular matrix with nonzero diagonal elements and is therefore nonsingular. $\square$

As an example, consider

$$\text{circuit}(x_0, \ldots, x_6) \tag{12}$$

where each x_i has domain $\{v_0, \ldots, v_6\} = \{2, 5, 6, 7, 9, 10, 12\}$. The corresponding polytope $C_7(2, 5, 6, 7, 9, 10, 12)$ has dimension 6 and satisfies

$$x_0 + \cdots + x_6 = 51 \tag{13}$$

which describes its affine hull.

6 Facets of the Polytope

We now describe facets of the circuit polytope $C_n(v)$. The following lemma is key.

Lemma 2 *Suppose that the inequality*

$$\sum_{j \in J} a_j x_j \geq \alpha \tag{14}$$

is valid for circuit$(x_0, \ldots, x_{n-1})$ *and is satisfied as an equation by at least one circuit* x*. If* $|J| \leq n-4$ *and*

$$\sum_{j=0}^{n-1} d_j x_j = \delta \tag{15}$$

is satisfied by all circuits x *that satisfy (14) as an equation, then* $d_j = 0$ *for all* $j \notin J$.

Proof. It suffices to prove that $d_{j_0} = d_{j_1} = d_{j_3} = d_{j_4} = 0$ for any subset of four indices $j_0, \ldots, j_3 \notin J$. Note first that we can use (6) to eliminate any variable (say, x_{j_0}) from (15) and obtain an equation of the form (15) in which $d_{j_0} = 0$. We therefore assume without loss of generality that $d_{j_0} = 0$.

Now let x^0 be any circuit that satisfies (14) as an equation, and let the permutation described by x^0 be

$$v_0, \ldots, v_{j_0-1}, v_{j_0}, v_{j_0+1}, \ldots, v_{j_1-1}, v_{j_1}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}$$

Consider the circuits $x^1, \ldots, x^5$ that describe the following permutations, respectively:

$$v_0, \ldots, v_{j_0-1}, v_{j_0}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_0+1}, \ldots, v_{j_1-1}, v_{j_1}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}$$
$$v_0, \ldots, v_{j_0-1}, v_{j_0}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_0+1}, \ldots, v_{j_1-1}, v_{j_1}$$
$$v_0, \ldots, v_{j_0-1}, v_{j_0}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_0+1}, \ldots, v_{j_1-1}, v_{j_1}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}$$
$$v_0, \ldots, v_{j_0-1}, v_{j_0}, v_{j_0+1}, \ldots, v_{j_1-1}, v_{j_1}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}$$
$$v_0, \ldots, v_{j_0-1}, v_{j_0}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_0+1}, \ldots, v_{j_1-1}, v_{j_1}$$

We obtain $x^1, \ldots, x^5$ from x^0 by viewing the permutation represented by the latter as a concatenation of four subsequences, each ending in one of the values v_{j_i}. We fix the first subsequence and obtain x^1 and x^2 by cyclically permuting the remaining three subsequences. We obtain x^3, x^4 and x^5 by interchanging a pair of subsequences.

Note that variables $x_{j_0}, \ldots, x_{j_3}$ have the values shown below in each circuit x^i:

x_{j_0}	x_{j_1}	x_{j_2}	x_{j_3}	
v_{j_0+1}	v_{j_1+1}	v_{j_2+1}	v_0	(x^0)
v_{j_2+1}	v_{j_1+1}	v_0	v_{j_0+1}	(x^1)
v_{j_1+1}	v_0	v_{j_2+1}	v_{j_0+1}	(x^2)
v_{j_1+1}	v_{j_2+1}	v_{j_0+1}	v_0	(x^3)
v_{j_0+1}	v_{j_2+1}	v_0	v_{j_1+1}	(x^4)
v_{j_2+1}	v_0	v_{j_0+1}	v_{j_1+1}	(x^5)

and all other variables x_j have value x_j^0 in each circuit x^i. Thus all six circuits $x^0, \ldots, x^5$ satisfy (14) as an equation, so that $dx^i = \delta$ for $i = 0, \ldots, 5$. This implies

$$\frac{1}{2}\begin{bmatrix} (dx^0+dx^1+dx^5)-(dx^2+dx^3+dx^4) \\ (dx^0+dx^2+dx^5)-(dx^1+dx^3+dx^4) \\ (dx^0+dx^3+dx^5)-(dx^1+dx^2+dx^4) \\ (dx^0+dx^2+dx^4)-(dx^1+dx^3+dx^5) \\ (dx^0+dx^4+dx^5)-(dx^1+dx^2+dx^3) \\ (dx^0+dx^1+dx^3)-(dx^2+dx^4+dx^5) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting the values of $x^0, \ldots, x^5$, we obtain

$$\begin{bmatrix} v_{j_2+1}-v_{j_1+1} & v_{j_1+1}-v_{j_2+1} & 0 & 0 \\ 0 & v_0-v_{j_2+1} & v_{j_2+1}-v_0 & 0 \\ 0 & 0 & v_{j_0+1}-v_0 & v_0-v_{j_0+1} \\ v_{j_0+1}-v_{j_2+1} & 0 & v_{j_2+1}-v_{j_0+1} & 0 \\ v_{j_0+1}-v_{j_1+1} & 0 & 0 & v_{j_1+1}-v_{j_0+1} \\ 0 & v_{j_1+1}-v_0 & 0 & v_0-v_{j_1+1} \end{bmatrix} \begin{bmatrix} d_{j_0} \\ d_{j_1} \\ d_{j_2} \\ d_{j_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from which we can conclude that $d_{j_0} = d_{j_1} = d_{j_2} = d_{j_3}$. But since $d_{j_0} = 0$, this proves the lemma. $\square$

It will be convenient denote by $x(J)$ the tuple $(x_{j_0}, \ldots, x_{j_m})$ when $J = \{j_0, \ldots, j_m\}$. We say that $x(J)$ is a *J-circuit* if it creates no cycles and is therefore a partial solution of the circuit constraint. That is, $x(J)$ is a J-circuit if $\pi_0, \ldots, \pi_m$ are all distinct, where $\pi_0 = j_0$ and $v_{\pi_i} = x_{\pi_{i-1}}$ for $i = 1, \ldots, m$. We will need the following lemma.

Lemma 3 *If $\bar{x}(J)$ is a J-circuit, then there is a circuit x such that $x(J) = \bar{x}(J)$.*

Proof. Let $J = \{j_0, \ldots, j_m\}$, and let $\{v_{i_0}, \ldots, v_{i_r}\}$ be the subset of domain values $v_0, \ldots, v_{n-1}$ that occur in neither $\{v_{j_0}, \ldots, v_{j_m}\}$ nor $\{\bar{x}_{j_0}, \ldots, \bar{x}_{j_m}\}$. Consider the directed graph $G_{\bar{x}(J)}$ that contains a vertex v_i for each $i \in \{0, \ldots, n-1\}$, a directed edge $(v_{j_k}, \bar{x}_{j_k})$ for $k = 0, \ldots, m$, and a directed edge $(v_{i_k}, v_{i_{k+1}})$ for each $k = 0, \ldots, r-1$. The maximal subchains of $G_{\bar{x}(J)}$ have the form

$$\begin{aligned} & v_{k_1} \to \cdots \to v_{k'_1} \to \bar{x}_{k'_1} \\ & v_{k_2} \to \cdots \to v_{k'_2} \to \bar{x}_{k'_2} \\ & \vdots \\ & v_{k_p} \to \cdots \to v_{k'_p} \to \bar{x}_{k'_p} \\ & v_{i_0} \to \cdots \to v_{i_r} \end{aligned}$$

where possibly $k_t = k'_t$ for some values of t. Because maximal subchains are disjoint, we can form a hamiltonian circuit in $G_{\bar{x}(J)}$ by linking the last element of each subchain to the first element of the next, and linking v_{i_r} to v_{k_1}. Let

$v_{s_0}, \ldots, v_{s_{n-1}}$ be the resulting circuit. Then if x is given by $x_i = v_{s_{(i+1) \bmod n}}$ for $j = 0, \ldots, n-1$, then x is a circuit and $x(J) = \bar{x}(J)$. $\square$

The idea of domination is central to characterizing facets of $C_n(v)$. Let $J = J_+ \cup J_-$ (with $J_+ \cap J_- = \emptyset$) be a subset of variable indices. For $i \in J$ we say that $x_i \preceq y_i$ if $i \in J_+$ and $x_i \leq y_i$, or $i \in J_-$ and $x_i \geq y_i$. Also $x_i \prec y_i$ if $x_i \preceq y_i$ and $x_i \neq y_i$. We say that $x'(J)$ *dominates* $x(J)$ with respect to $J = J_+ \cup J_-$ when $x'_j \preceq x_j$ for all $j \in J$. A J-circuit $x(J)$ is *undominated* if no other J-circuit dominates it.

The following theorem provides the basis for generating facets of $C_n(v)$ by generating undominated J-circuits.

Theorem 4 *Let S be the set of J-circuits that are undominated with respect to $J = J_+ \cup J_-$, where $1 \leq |J| \leq n-4$. Consider any subset of $|J|$ affinely independent J-circuits in S. If these J-circuits satisfy*

$$\sum_{j \in J} a_j x_j = \alpha, \quad \textit{where } a_j > 0 \textit{ for } j \in J_+ \textit{ and } a_j < 0 \textit{ for } j \in J_- \tag{16}$$

and the remaining J-circuits in S satisfy (14), then (14) defines a facet of $C_n(v)$.

Proof. Let $S = \{x^0(J), \ldots, x^m(J)\}$, and suppose $S' \subset S$ is a set of $|J|$ affinely independent circuits. We first show that (14) is valid; that is, satisfied by any circuit x. Because S contains all undominated J-circuits, $x(J)$ is dominated by some $x^i(J) \in S$ with respect to $J = J_+ \cup J_-$, which means that $a_j(x_j - x^i_j) \geq 0$ for all $j \in J$. Thus we have

$$\sum_{j \in J} a_j x_j \geq \sum_{j \in J} a_j x^i_j \geq \alpha$$

because $x^i(J)$ satisfies (14), and so x satisfies (14).

Now let (15) be any equation satisfied by all circuits x that satisfy (14) as an equation. Because $|J| \geq 1$ and S is therefore nonempty, at least one J-circuit $x^i(J) \in S$ satisfies (14) as an equation. Lemma 3 now implies that at least one circuit x^i satisfies (14) as an equation. Thus since $|J| \leq n-4$, we have from Lemma 2 that $d_j = 0$ for all $j \notin J$, so that

$$\sum_{j \in J} d_j x_j = \delta \tag{17}$$

Because the J-circuits in S' are affinely independent and satisfy (16) and (17), these two equations are the same up to a scalar multiple. Therefore, any equation satisfied by all circuits that satisfy (14) as an equation has the form (17). This means that (14) defines a facet of the circuit polytope. $\square$

We show now that the previous theorem completely characterizes facet-defining inequalities having no more than $n-4$ terms.

Theorem 5 *Consider any inequality (14) that is facet-defining for a circuit polytope $C_n(v)$, and let $J_+ = \{j \mid a_j > 0\}$ and $J_- = \{j \mid a_j < 0\}$. Then there are affinely independent J-circuits $x^0(J), \ldots, x^{|J|-1}(J)$ that are undominated with respect to $J = J_+ \cup J_-$ and satisfy (16).*

Proof. Any facet-defining inequality (14) is satisfied as an equation by $n-1$ affinely independent circuits $\bar{x}^0, \ldots, \bar{x}^{n-1}$. Then $\{\bar{x}^0(J), \ldots, \bar{x}^{n-1}(J)\}$ has some subset $\{\bar{x}^{j_0}(J), \ldots, \bar{x}^{j_{|J|-1}}(J)\}$ of $|J|$ affinely independent J-circuits. These are undominated with respect to $J = J_+ \cup J_-$, because otherwise, some J-circuit $\hat{x}(J)$ strictly dominates some $\bar{x}^{j_i}(J)$ with respect to $J = J_+ \cup J_-$. Also by Lemma 3, $\hat{x}(J)$ is part of some circuit $\hat{x}$. This means

$$\sum_{j \in J} a_j \hat{x}_j < \sum_{j \in J} a_j \bar{x}_j^{j_i} = \alpha$$

and $\hat{x}$ violates (14). This implies that (14) is not valid and therefore is not facet-defining as assumed. □

7 Facet Generation

The results of the previous section indicate how to generate all facet-defining inequalities for $C_n(v)$ having at most $n-4$ terms. To generate all such facet-defining inequalities (14) in which $a_j > 0$ for $j \in J_+$ and $a_j < 0$ for $j \in J_-$, first generate the set S of all J-circuits that are undominated with respect to $J = J_+ \cup J_-$. Then consider all affinely independent subsets of $|J|$ J-circuits in S. Each subset uniquely defines an equation (16) up to scalar multiple. If the remaining J-circuits in S satisfy (14), then list (14) as a facet-defining inequality.

Note that we do not identify a facet by generating $n-1$ affinely independent circuits that define the facet, as this would be a difficult task in general. Rather, we generate $|J|$ affinely independent J-circuits that define the coefficients of an inequality containing $|J|$ terms. This inequality defines a facet if it is valid, which we can easily check. In the next section we will show how to generate the undominated partial solutions efficiently with a greedy procedure.

As an example, we identify all facet-defining inequalities of the form

$$a_0 x_0 + a_2 x_2 + a_3 x_3 \geq \alpha, \quad \text{with } a_0, a_2, a_3 > 0 \tag{18}$$

for example (12). Four J-circuits $\bar{x}^i(J)$ are undominated with respect to $J = J_+ = \{0, 2, 3\}$. They are independent of the particular domain values $v_0, \ldots, v_6$ and can therefore be written

$$\begin{aligned} \bar{x}^1(J) &= (v_1, v_0, v_2) \\ \bar{x}^2(J) &= (v_1, v_3, v_0) \\ \bar{x}^3(J) &= (v_2, v_1, v_0) \\ \bar{x}^4(J) &= (v_3, v_0, v_1) \end{aligned} \tag{19}$$

Table 1: Hyperplanes determined by undominated J-circuits for example (12).

Defining J-circuits	Uniquely defined hyperplane $a(J)x(J) = \alpha$	Is $a(J)x(J) \geq \alpha$ valid?
$\bar{x}^1(J), \bar{x}^2(J), \bar{x}^3(J)$	$8x_0 + 4x_2 + 5x_3 = 78$	Yes, violated by $\bar{x}^4(J)$
$\bar{x}^1(J), \bar{x}^2(J), \bar{x}^4(J)$	$5x_0 + 8x_2 + 10x_3 = 101$	No, violated by $\bar{x}^3(J)$
$\bar{x}^1(J), \bar{x}^3(J), \bar{x}^4(J)$	$3x_0 + 7x_2 + 6x_3 = 65$	Yes, satisfied by $\bar{x}^2(J)$
$\bar{x}^2(J), \bar{x}^3(J), \bar{x}^4(J)$	$6x_0 + 3x_2 + x_3 = 53$	No, violated by $\bar{x}^1(J)$

(We will show how to obtain these J-circuits using a greedy algorithm in the next section.) There are four subsets of three J-circuits ($|J| = 3$), shown in Table 1, and each uniquely defines a hyperplane and a corresponding inequality. The first inequality, defined by $\bar{x}^1(J)$, $\bar{x}^2(J)$, and $\bar{x}^3(J)$, is satisfied by the remaining J-circuit $\bar{x}^4(J)$, and similarly for the third inequality. The second and fourth inequalities, however, are violated by the remaining J-circuit and are not valid. This means there are exactly two facets defined by inequalities of the form (18), namely those defined by

$$\begin{aligned} 8x_0 + 4x_2 + 5x_3 &\geq 78 \\ 3x_0 + 7x_2 + 6x_3 &\geq 65 \end{aligned}$$

Now we find all facet-defining inequalities of the form (18) but with $a_0, a_2 > 0$ and $a_3 < 0$, so that $J_+ = \{0, 2\}$ and $J_- = \{3\}$. In this case, there is only one undominated circuit, $x(J) = (v_1, v_0, v_6)$. Because we do not have three undominated circuits to define a hyperplane, there are no facet-defining inequalities of this form.

8 Generation of Undominated Circuits

A simple greedy procedure can be used to generate all J-circuits $\bar{x}(J)$ that are undominated with respect to $J = J_+ \cup J_-$. It is applied for each ordering $j_0, \ldots, j_m$ of the elements of J. First, let $\bar{x}_{j_0}$ be the smallest domain value v_i if $j_0 \in J_+$, or the largest if $j_o \in J_-$. Then let $\bar{x}_{j_1}$ be the smallest (or largest) remaining domain value that does not create a cycle. Continue until all $\bar{x}_j$ for $j \in J$ are defined. The precise algorithm appears in Fig. 1.

Theorem 6 *The greedy procedure of Fig. 1 generates J-circuits that are undominated with respect to $J = J_+ \cup J_-$.*

Proof. Let $\bar{x}(J)$ be a J-circuit generated by the procedure for a given ordering $j_0, \ldots, j_m$. To see that $\bar{x}(J)$ is undominated with respect to $J = J_+ \cup J_-$, assume otherwise. Then there exists a J-circuit $\bar{y}(J)$ such that $\bar{x}(J) \succeq \bar{y}(J)$ and $\bar{x}_{j_t} \succ \bar{y}_{j_t}$ for some $t \in \{0, \ldots, m\}$. Let t be the smallest such index, so that $\bar{x}_{j_k} = \bar{y}_{j_k}$ for $k = 0, \ldots, t-1$. This contradicts the greedy construction of $\bar{x}$, because $\bar{y}_{j_t}$ is available when $\bar{x}_{j_t}$ is assigned to x_{j_t}. $\square$

For each ordering $j_0, \ldots, j_m$ of the elements of J:
  Let $\bar{J} = \{0, \ldots, n-1\}$ and $J' = \emptyset$.
  For $i = 0, \ldots, m$:
    Add j_i to J'.
    If $j_i \in J_+$ then let $\bar{x}_{j_i}$ be the minimum value v_k in $\{v_i \mid i \in \bar{J}\}$
      such that $\bar{x}(J')$ is a J'-circuit.
    Else let $\bar{x}_{j_i}$ be the maximum value v_k in $\{v_i \mid i \in \bar{J}\}$
      such that $\bar{x}(J')$ is a J'-circuit.
    Remove k from $\bar{J}$.
  Add $\bar{x}(J)$ to the list of undominated J-circuits.

Figure 1: Greedy procedure for generating J-circuits that are undominated with respect to $J = J_+ \cup J_-$.

For example, the undominated circuits (19) for circuit constraint (12) can be generated by considering the six orderings of $J = J_+ = \{0, 2, 3\}$, listed on the left below. The resulting undominated J-circuits appear on the right.

$$
\begin{array}{ll}
0,2,3 & (v_1, v_0, v_2) = \bar{x}^1(J) \\
0,3,2 & (v_1, v_3, v_0) = \bar{x}^2(J) \\
2,0,3 & (v_1, v_0, v_2) = \bar{x}^1(J) \\
2,3,0 & (v_3, v_0, v_1) = \bar{x}^4(J) \\
3,0,2 & (v_1, v_3, v_0) = \bar{x}^2(J) \\
3,2,0 & (v_2, v_1, v_0) = \bar{x}^3(J)
\end{array}
$$

When $J_+ = \{0, 2\}$ and $J_- = \{3\}$, all six orderings result in the same J-circuit (v_1, v_0, v_6).

The greedy procedure not only generates undominated J-circuits, but generates all of them.

Theorem 7 *Any undominated circuit with respect to $J = J_+ \cup J_-$ can be generated in a greedy fashion for some ordering of the indices in J.*

Proof. Let $\bar{x}$ be a circuit that is undominated with respect to $J = J_+ \cup J_-$, where $|J| = m$, $J_+ = \{i_0, \ldots, i_p\}$ and $J_- = \{j_0, \ldots, j_q\}$. Suppose the variables are indexed so that $\bar{x}_{i_\ell} < \bar{x}_{i_{\ell'}}$ when $\ell < \ell'$ and $i_\ell, i_{\ell'} \in J_+$, and $\bar{x}_{j_\ell} > \bar{x}_{j_{\ell'}}$ when $\ell < \ell'$ and $j_\ell, j_{\ell'} \in J_-$.

Let $\bar{y}$ be a J-circuit that is generated in greedy fashion with respect to an ordering $k_0, \ldots, k_m$ determined in the following way. Let r and s index the elements of J_+ and J_-, respectively, with $r = 0$ and $s = 0$ initially. Also let $V = \{v_0, \ldots, v_{n-1}\}$ initially. At each step of the procedure, we assign the greedy value to x_j for the next $j \in J_+$ unless we can avoid deviating from $\bar{x}$ by assigning the greedy value to x_j for the next $j \in J_-$, or unless we have already assigned values to x_j for all $j \in J_+$. That is, for $\ell = 0, \ldots, m$, do the following. Let $v_{\min}$ be the smallest value in V such that setting $x_{i_r} = v_{\min}$ does not create a cycle.

Let $v_{\max}$ be the largest value in V such that setting $x_{j_s} = v_{\max}$ does not create a cycle. If $r \le p$, and if $\bar{x}_{i_r} = v_{\min}$ or $\bar{x}_{j_s} < v_{\max}$ or $s > q$, then let $k_\ell = i_r$, let $\bar{y}_{i_r} = v_{\min}$, set $r = r + 1$, and remove $v_{\min}$ from V. Otherwise, let $k_\ell = j_s$, let $\bar{y}_{j_s} = v_{\max}$, set $s = s + 1$, and remove $v_{\max}$ from V. Then $(\bar{y}_0, \ldots, \bar{y}_m)$ is the greedy solution with respect to the ordering $k_0, \ldots, k_m$.

We claim that $\bar{x}_{j_\ell} = \bar{y}_{j_\ell}$ for $\ell = 0, \ldots, m$, which suffices to prove the theorem. Supposing to the contrary, let $\bar{\ell}$ be the smallest index for which $\bar{x}_{k_{\bar{\ell}}} \ne \bar{y}_{k_{\bar{\ell}}}$. Clearly $\bar{x}_{k_{\bar{\ell}}} \prec \bar{y}_{k_{\bar{\ell}}}$ is inconsistent with the greedy choice, because $\bar{x}_{k_{\bar{\ell}}}$ is available when $\bar{y}_{k_{\bar{\ell}}}$ is assigned to $x_{k_{\bar{\ell}}}$. Thus we have $\bar{x}_{k_{\bar{\ell}}} \succ \bar{y}_{k_{\bar{\ell}}}$

By hypothesis, $\bar{x}$ is undominated with respect to $J = J_+ \cup J_-$. We therefore have $\bar{x}_{k_\ell} \prec \bar{y}_{k_\ell}$ for some $\ell \in \{\bar{\ell} + 1, \ldots, m\}$. Let $\hat{\ell}$ be the smallest such index. Then there are two cases: (1) $k_{\bar{\ell}}$ and $k_{\hat{\ell}}$ are both in J_+ or both in J_-, or (2) they are in different sets.

Case 1: $k_{\bar{\ell}}$ and $k_{\hat{\ell}}$ are both in J_+ or both in J_-. We will suppose that both are in J_+. The argument is symmetric if both are in J_-.

Let t be the index such that $i_t = k_{\bar{\ell}}$, and u the index such that $i_u = k_{\hat{\ell}}$. Then $\bar{x}_{j_t} > \bar{y}_{j_t}$ because $\bar{x}_{j_t} \succ \bar{y}_{j_t}$ and $j_t \in J_+$. Let t' be the largest index in $\{t, \ldots, u - 1\}$ such that $\bar{x}_{i_{t'}} > \bar{y}_{i_{t'}}$. We know that t' exists because $\bar{x}_{i_t} > \bar{y}_{i_t}$. Thus we have two sequences of values related as follows:

$$\begin{array}{ccccccccccccccc}
\bar{x}_{i_0} & < \cdots < & \bar{x}_{i_{t-1}} & < & \bar{x}_{i_t} & < \cdots < & \bar{x}_{i_{t'-1}} & < & \bar{x}_{i_{t'}} & < \cdots < & \bar{x}_{i_{u-1}} & < & \bar{x}_{i_u} \\
= & & = & & > & & \ge & & > & & \ge & & < \\
\bar{y}_{i_0} & \cdots & \bar{y}_{i_{t-1}} & & \bar{y}_{i_t} & \cdots & \bar{y}_{i_{t'-1}} & & \bar{y}_{i_{t'}} & \cdots & \bar{y}_{i_{u-1}} & & \bar{y}_{i_u}
\end{array}$$

Let u' be the largest index for which $x_{j_{u'}}$ has been assigned a value at the time $\bar{y}_{i_u}$ is assigned to x_{i_u}. We have the two sequences of values

$$\begin{array}{ccccc}
\bar{x}_{j_0} & > \cdots > & \bar{x}_{j_{u'-1}} & > & \bar{x}_{j_{u'}} \\
\bar{y}_{j_0} & \cdots & \bar{y}_{j_{u'-1}} & & \bar{y}_{j_{u'}}
\end{array}$$

We first show that value $\bar{x}_{i_u}$ has not yet been assigned in the greedy algorithm when $\bar{y}_{i_u}$ is assigned to x_{i_u}. That is, we show that $\bar{x}_{i_u} \notin \{\bar{y}_{i_0}, \ldots, \bar{y}_{i_{u-1}}\}$ and $\bar{x}_{i_u} \notin \{\bar{y}_{j_0}, \ldots, \bar{y}_{j_{u'}}\}$. To see that $\bar{x}_{i_u} \notin \{\bar{y}_{i_0}, \ldots, \bar{y}_{i_{u-1}}\}$, suppose to the contrary that $\bar{x}_{i_u} = \bar{y}_{i_w}$ for some $w \in \{0, \ldots, u-1\}$. This is impossible, because $\bar{x}_{i_u} > \bar{x}_{i_w} \ge \bar{y}_{i_w}$. Also $\bar{x}_{i_u} \notin \{\bar{y}_{j_0}, \ldots, \bar{y}_{j_{u'}}\}$, because assigning value $\bar{x}_{i_u}$ to x_{j_w} for some $w \in \{0, \ldots, u'\}$ contradicts the greedy construction of $\bar{y}$, due to the fact that value $\bar{y}_{i_u}$ was available at that time and is a superior choice.

We next show that value $\bar{x}_{i_{t'}}$ has not yet been assigned in the greedy algorithm when $\bar{y}_{i_u}$ is assigned to x_{i_u}. That is, we show that $\bar{x}_{i_{t'}} \notin \{\bar{y}_{i_0}, \ldots, \bar{y}_{i_{u-1}}\}$ and $\bar{x}_{i_{t'}} \notin \{\bar{y}_{j_0}, \ldots, \bar{y}_{j_{u'}}\}$. To begin with, we have that $\bar{x}_{i_{t'}} \notin \{\bar{y}_{i_0}, \ldots, \bar{y}_{i_{t'-1}}\}$, by virtue of the same reasoning just applied to $\bar{x}_{i_u}$. Also $\bar{x}_{i_{t'}} \ne \bar{y}_{i_{t'}}$, since by hypothesis $\bar{x}_{i_{t'}} > \bar{y}_{i_{t'}}$. To show that $\bar{x}_{i_{t'}} \notin \{\bar{y}_{i_{t'+1}}, \ldots, \bar{y}_{i_{u-1}}\}$, suppose to the contrary that $\bar{x}_{i_{t'}} = \bar{y}_{i_w}$ for some $w \in \{t'+1, \ldots, u-1\}$. Then since $\bar{x}_{i_{t'}} < \bar{x}_{i_w}$, we must have $\bar{x}_{i_w} > \bar{y}_{i_w}$. But this contradicts the definition of t' $(< w)$ as the largest index in $\{0, \ldots, u-1\}$ such that $\bar{x}_{i_{t'}} > \bar{y}_{i_{t'}}$. Thus $\bar{x}_{i_{t'}} \ne \bar{y}_{i_w}$. Finally, $\bar{x}_{i_{t'}} \notin \{\bar{y}_{j_0}, \ldots, \bar{y}_{j_{u'}}\}$ because assigning value $\bar{x}_{i_{t'}}$ to x_{j_w} for some $w \in \{0, \ldots, u'\}$

contradicts the greedy construction of $\bar{y}$, due to the fact that $\bar{y}_{i_u}$ was available at the time and $\bar{y}_{i_u} > \bar{x}_{i_u} > \bar{x}_{i_{t'}}$.

Since $\bar{x}_{i_u} < \bar{y}_{i_u}$ and value $\bar{x}_{i_u}$ has not yet been assigned, setting $x_{i_u} = \bar{x}_{i_u}$ must create a cycle in $\bar{y}$, because otherwise $x_{i_u} = \bar{x}_{i_u}$ would have been the greedy choice. Also, setting $x_{i_u} = \bar{x}_{i_{t'}}$ was not the greedy choice because $\bar{y}_{i_u} > \bar{x}_{i_u} > \bar{x}_{i_{t'}}$. Thus setting $x_{i_u} = \bar{x}_{i_{t'}}$ must likewise create a cycle in $\bar{y}$, because $\bar{x}_{i_{t'}}$ has not yet been assigned. Now define $G_{\bar{y}(J)}$ as before and consider the maximal subchain in $G_{\bar{y}(J)}$ that contains $\bar{y}_{i_u}$. Let the segment of the subchain up to $\bar{y}_{i_u}$ be

$$v_{i_w} \to \cdots \to v_{i_u} \to \bar{y}_{i_u}$$

Because setting $x_{i_u} = \bar{x}_{i_u}$ creates a cycle in $\bar{y}$, we must have $\bar{x}_{i_u} = v_{i_w}$. Similarly, because setting $x_{i_u} = \bar{x}_{i_{t'}}$ creates a cycle in $\bar{y}$, we must have $\bar{x}_{i_{t'}} = v_{i_w}$. This implies $\bar{x}_{i_u} = \bar{x}_{i_{t'}}$, which is impossible because $\bar{x}_{i_u} > \bar{x}_{i_{t'}}$.

Case 2: $k_{\bar{\ell}} \in J_+$ and $k_{\hat{\ell}} \in J_-$, or $k_{\bar{\ell}} \in J_-$ and $k_{\hat{\ell}} \in J_+$. We can rule out the latter subcase immediately, because $k_{\bar{\ell}}$ can be in J_- only if $r > p$ when $\bar{y}_{k_{\bar{\ell}}}$ is assigned to $x_{k_{\bar{\ell}}}$. This means $k_{\hat{\ell}}$ must be in J_- as well, since $x_{k_{\hat{\ell}}}$ is assigned after $x_{k_{\bar{\ell}}}$, and the situation reverts to Case 1. We therefore suppose $k_{\bar{\ell}} \in J_+$ and $k_{\hat{\ell}} \in J_-$.

Let t be the index such that $i_t = k_{\bar{\ell}}$, and u the index such that $j_u = k_{\hat{\ell}}$. Again $\bar{x}_{i_t} > \bar{y}_{i_t}$ because $\bar{x}_{i_t} \succ \bar{y}_{i_t}$ and $j_t \in J_+$. Thus, at the time value $\bar{y}_{i_t}$ was assigned to x_{i_t}, we had $\bar{x}_{j_s} < v_{\max}$ for the current value of s. So we have two sequences of values related as follows:

$$\begin{array}{ccccccccc} \bar{x}_{j_0} > \cdots > & \bar{x}_{j_{s-1}} & > & \bar{x}_{j_s} & > \cdots & \bar{x}_{j_{u-1}} & > & \bar{x}_{j_u} \\ = & = & & \leq & & \leq & & > \\ \bar{y}_{j_0} \quad \cdots & \bar{y}_{j_{s-1}} & & \bar{y}_{j_s} & \cdots & \bar{y}_{j_{u-1}} & & \bar{y}_{j_u} \end{array} \tag{20}$$

where $v_{\max} > \bar{x}_{j_s}$. Let t' be the largest index for which $x_{i_{t'}}$ has been assigned a value at the time $\bar{y}_{j_u}$ is assigned to x_{j_u}. We have two sequences of values related as follows:

$$\begin{array}{ccccccc} \bar{x}_{i_0} < \cdots < & \bar{x}_{i_{t-1}} & < & \bar{x}_{i_t} & < \cdots < & \bar{x}_{i_{t'}} \\ = & = & & > & & \\ \bar{y}_{i_0} \quad \cdots & \bar{y}_{i_{t-1}} & & \bar{y}_{i_t} & \cdots & \bar{y}_{i_{t'}} \end{array}$$

We first show that a cycle must be created if value $\bar{x}_{j_u}$ rather than $\bar{y}_{j_u}$ is assigned to x_{j_u}. Because $\bar{y}_{j_u} < \bar{x}_{j_u}$, it suffices to show that value $\bar{x}_{j_u}$ has not yet been assigned in the greedy algorithm when $\bar{y}_{j_u}$ is assigned to x_{j_u}. That is, we show that $\bar{x}_{j_u} \notin \{\bar{y}_{j_0}, \ldots, \bar{y}_{j_{u-1}}\}$ and $\bar{x}_{j_u} \notin \{\bar{y}_{i_0}, \ldots, \bar{y}_{i_{t'}}\}$. If $\bar{x}_{j_u} = \bar{y}_{j_w}$ for some $w \in \{0, \ldots, u-1\}$, then $\bar{x}_{j_u} < \bar{x}_{j_w} \leq \bar{y}_{j_w}$, which is impossible. Thus $\bar{x}_{j_u} \notin \{\bar{y}_{j_0}, \ldots, \bar{y}_{j_{u-1}}\}$. Also $\bar{x}_{j_u} \notin \{\bar{y}_{i_0}, \ldots, \bar{y}_{i_{t'}}\}$, because assigning value $\bar{x}_{j_u}$ to x_{i_w} for some $w \in \{0, \ldots, t'\}$ contradicts the greedy construction of $\bar{y}$, due to the fact that value $\bar{y}_{j_u}$ was available at that time and is a superior choice.

We next show that a cycle must be created if value $v_{\max}$ rather than $\bar{y}_{j_u}$ is assigned to x_{j_u}. Note that $v_{\max} \notin \{\bar{y}_{i_0}, \ldots, \bar{y}_{i_{t'}}\}$, because assigning value $v_{\max}$ to x_{i_w} for some $w \in \{0, \ldots, t'\}$ contradicts the greedy construction of $\bar{y}$, due to the fact that value $\bar{y}_{j_u}$ was available at that time and is a superior choice because $v_{\max} > \bar{x}_{j_s} > \bar{x}_{j_u}$. Now suppose, contrary to the claim, that assigning $v_{\max}$ to x_{j_u} does not create a cycle. Then since $v_{\max} > \bar{y}_{j_u}$, the value $v_{\max}$ must have already been assigned in the greedy algorithm at the time $\bar{y}_{j_u}$ is assigned to x_{j_u}. This implies $v_{\max} \in \{\bar{y}_{j_s}, \ldots, \bar{y}_{j_{u-1}}\}$. But in this case we must have $\bar{y}_{j_s} = v_{\max}$, because assigning $v_{\max}$ to x_{j_s} does not create a cycle and, by definition, is the most attractive choice at the time. Thus (20) becomes

$$\begin{array}{ccccccccccccccc}
\bar{x}_{j_0} & > \cdots > & \bar{x}_{j_{s-1}} & > & \bar{x}_{j_s} & > \cdots > & \bar{x}_{j_{s'-1}} & > & \bar{x}_{j_{s'}} & > \cdots > & \bar{x}_{j_{u-1}} & > & \bar{x}_{j_u} \\
= & & = & & < & & \leq & & < & & \geq & & > \\
\bar{y}_{j_0} & \cdots & \bar{y}_{j_{s-1}} & & \bar{y}_{j_s} & \cdots & \bar{y}_{j_{s'-1}} & & \bar{y}_{j_{s'}} & \cdots & \bar{y}_{j_{u-1}} & & \bar{y}_{j_u}
\end{array}$$

where $\bar{y}_{j_s} = v_{\max}$ and where s' is the largest index in $\{s, \ldots, u-1\}$ such that $\bar{y}_{j_{s'}} < \bar{x}_{j_{s'}}$. Now we can argue as in Case 1 that assigning $\bar{x}_{j_u}$ to x_{j_u} creates a cycle, and assigning $\bar{x}_{j_{s'}}$ to x_{j_u} creates a cycle, which implies $\bar{x}_{j_{s'}} = \bar{x}_{j_u}$, a contradiction because $\bar{x}_{j_{s'}} > \bar{x}_{j_u}$. We conclude that assigning $v_{\max}$ to x_{j_u} creates a cycle.

Having shown that assigning $\bar{x}_{j_u}$ to x_{j_u} creates a cycle, and assigning $v_{\max}$ to x_{j_u} creates a cycle, we derive as in Case 1 that $v_{\max} = \bar{x}_{j_u}$, a contradiction because $v_{\max} \geq \bar{x}_{j_s} > \bar{x}_{j_u}$. The theorem follows. $\square$.

9 Permutation and Two-term Facets

In this section we examine two special classes of facets of $C_n(v)$—permutation facets and two-term facets.

The *permutation polytope* $P_n(v)$ for a given domain $\{v_0, \ldots, v_{n-1}\}$ is the convex hull of all points whose coordinates are permutations of $v_0, \ldots, v_{n-1}$. The circuit polytope $C_n(v)$ is contained in $P_n(v)$ because every circuit $(x_0, \ldots, x_{n-1})$ is a permutation of $v_0, \ldots, v_{n-1}$. This means that every facet-defining inequality for $P_n(v)$ is valid for circuit but not necessarily facet defining. This raises the question as to which permutation facets are also circuit facets. We will identify a large family of permutation facets that can be immediately recognized as circuit facets.

The permutation polytope $P_n(v)$ has dimension $n-1$. The facets of $P_n(v)$ are identified in [3, 9], and they are defined by

$$\sum_{j \in J} x_j \geq \sum_{j=0}^{|J|-1} v_j \tag{21}$$

for all $J \subset \{0, \ldots, n-1\}$ with $1 \leq |J| \leq n-1$. (Recall that $v_0 < \cdots < v_{n-1}$.) This result is generalized in [4] to domains with more than n elements.

For example, the permutation polytope $P_3(v)$ with $v = (2, 4, 5)$ is defined by

$$\begin{aligned}
&x_0 + x_1 + x_2 = 11 \\
&x_i \geq 2, \text{ for } i = 0, 1, 2 \\
&x_i + x_j \geq 6, \text{ for distinct } i, j \in \{0, 1, 2\}
\end{aligned}$$

We can see at this point that a facet-defining inequality for $P_n(v)$ need not be facet-defining for $C_n(v)$. The inequality $x_0 + x_1 \geq 6$ is facet-defining for $P_3(v)$ but not for $C_3(v)$, which is the line segment from $(4,5,2)$ to $(5,2,4)$.

Theorems 4, 6, and 7 allow us to identify a family of permutation facets that are also circuit facets.

Corollary 8 *The inequality (21) defines a facet of* $C_n(v)$ *if* $1 \leq |J| \leq n-4$ *and* $j \geq |J|$ *for all* $j \in J$.

Proof. Let $J = \{j_0, \ldots, j_m\}$. Due to Theorem 6 and the fact that $j \geq m$ for all $j \in J$, the following are undominated J-circuits with respect to $J = J_+$:

$$\text{all } \bar{x}(J) \text{ for which } \bar{x}_{j_0}, \ldots, \bar{x}_{j_m} \text{ is a permutation of } v_0, \ldots, v_m \tag{22}$$

Theorem 7 tells us that (22) is the complete set of J-circuits that are undominated with respect to $J = J_+$. Consider the following J-circuits from (22):

$$\begin{aligned}
\bar{x}^0(J) &= (v_0, v_1, v_2, v_3, \ldots, v_{n-2}, v_{n-1}) \\
\bar{x}^1(J) &= (v_1, v_0, v_2, v_3, \ldots, v_{n-2}, v_{n-1}) \\
\bar{x}^2(J) &= (v_0, v_2, v_1, v_3, \ldots, v_{n-2}, v_{n-1}) \\
&\vdots \\
\bar{x}^m(J) &= (v_0, v_1, v_2, v_3, \ldots, v_{n-1}, v_{n-2})
\end{aligned} \tag{23}$$

where $\bar{x}^i(J)$ is obtained for $i > 0$ by swapping v_{i-1} and v_i in $\bar{x}^0(J)$. These circuits are affinely independent, as can be seen by subtracting $\bar{x}^0(J)$ from each. By construction, all the J-circuits (23) satisfy (21) as an equation. Thus the affinely independent J-circuits (22) satisfy (21) as an equation, and all the remaining J-circuits in (23) satisfy (21). So by Theorem 4, (21) is facet-defining. □

We can check on a case-by-case basis whether permutation facets other than those mentioned in Corollary 8 are circuit facets. For example, if $J = J_+ = \{2,3,4\}$, then application of the greedy procedure in Fig. 1 yields the undominated J-circuits

$$\begin{aligned}
\bar{x}^0 &= (v_0, v_1, v_2) & \quad \bar{x}^3 &= (v_3, v_0, v_1) \\
\bar{x}^1 &= (v_0, v_2, v_1) & \quad \bar{x}^4 &= (v_1, v_2, v_0) \\
\bar{x}^2 &= (v_1, v_0, v_2) & \quad \bar{x}^5 &= (v_3, v_1, v_0)
\end{aligned}$$

Some subsets of three J-circuits, such as $\{\bar{x}^0, \bar{x}^1, \bar{x}^2\}$, satisfy (21) as an equation. Because the remaining J-circuits clearly satisfy (21), the permutation facet (21) is also a circuit facet.

Another special class of facet-defining inequalities are those containing two terms. Because a set of two undominated J-circuits (where $|J| = 2$) defines exactly one facet, the two-term facets can be exhaustively listed in closed form.

Corollary 9 *If $n \geq 6$, the two-term facets of $C_n(v)$ are precisely those defined by*

$$\begin{aligned}
&x_i + x_j \geq v_0 + v_1, \ \ \textit{for distinct } i, j \in \{2, \ldots, n-1\} \\
&(v_2 - v_0)x_0 + (v_2 - v_1)x_1 \geq v_2^2 - v_0 v_1 \\
&(v_1 - v_0)x_1 + (v_2 - v_0)x_i \geq v_1 v_2 - v_0^2, \ \ \textit{for } i \in \{2, \ldots, n-1\} \\
&x_i + x_j \leq v_{n-2} + v_{n-1}, \ \ \textit{for distinct } i, j \in \{0, \ldots, n-3\} \\
&(v_{n-2} - v_{n-3})x_{n-2} + (v_{n-1} - v_{n-3})x_{n-1} \leq v_{n-1}v_{n-2} - v_{n-3}^2 \\
&(v_{n-1} - v_{n-3})x_i + (v_{n-1} - v_{n-2})x_{n-2} \leq v_{n-1}^2 - v_{n-2}v_{n-3}, \\
&\hspace{12em} \textit{for } i \in \{0, \ldots, n-3\}
\end{aligned}$$

The proof is straightforward.

10 Separation Heuristics

The greedy procedure described above for generating undominated J-circuits suggests some simple separation heuristics. Suppose we have a solution $\hat{x}$ of the current relaxation of the problem, and that $\hat{x}$ violates the circuit constraint. The *separation problem* is to find one or more facet-defining inequalities that separate $\hat{x}$ from the circuit polytope in the sense that $\hat{x}$ violates the inequalities. Separating inequalities can then be added to the relaxation to tighten it.

Suppose first that we seek separating inequalities with all positive coefficients, so that $J = J_+$. Given a point $\hat{x}$ to be separated, let $j_0, \ldots, j_{n-1}$ be an ordering of variable indices for which $\hat{x}_{j_0} \leq \cdots \leq \hat{x}_{j_{n-1}}$. We consider the sequence of subsets $J^0, J^1, \ldots, J^{n-1}$ where $J^i = \{j_0, \ldots, j_i\}$. Beginning with J^0, we try to generate facet-defining inequalities corresponding to each J^i, until we find a separating inequality. For each J^i we use the greedy procedure of Fig. 1 to generate all undominated J^i-circuits with respect to $J^i = J^i_+$ and use these J^i-circuits to generate facet-defining inequalities as described earlier. Any of the resulting inequalities violated by $\hat{x}$ are separating. If none are separating, we move to J^{i+1} and repeat. The precise algorithm appears in Fig. 2. A similar algorithm is shown in [4] to be a complete separation procedure for the permutation polytope.

In practice, the algorithm would not continue all the way to J^{n-1} when no separating inequalities are found, because it is impractical to generate all undominated J^k-circuits when k is large. Rather, the algorithm would stop at some predetermined maximum $k = k_{\max}$.

As an illustration, suppose that $(\hat{x}_0, \ldots, \hat{x}_6) = (6, 2, 5.5, 7, 5.7, 8, 9)$ in example (12). This is not a feasible solution, if only because it does not consist of values from the domain $\{2, 5, 6, 7, 9, 10, 12\}$. Here $(j_0, \ldots, j_6) = (1, 2, 4, 0, 3, 5, 6)$. For $J^0 = \{1\}$ we have the single facet-defining inequality $x_1 \geq 2$, but it does not separate $\hat{x}$. For $J^1 = \{1, 2\}$ we have the facet-defining inequality $3x_1 + 4x_2 \geq 26$, which again does not separate $\hat{x}$. But for $J^2 = \{1, 2, 4\}$ we have three facet-

Let $S = \emptyset$.
Order $j_0, \ldots, j_n$ so that $\hat{x}_{j_0} \leq \cdots \leq \hat{x}_{j_{n-1}}$.
For $k = 0, \ldots, k_{\max}$ while $S = \emptyset$:
  Let $J^k = \{j_0, \ldots, j_k\}$.
  Let $\bar{x}^0(J^k), \ldots, \bar{x}^m(J^k)$ be the undominated J^k-circuits generated
    by the greedy procedure of Fig. 1 with $J = J_+ = J^k$.
  For each $\{t_0, \ldots, t_k\} \subset \{0, \ldots, m\}$:
    Let $\sum_{i=0}^k a_{j_i} x_{j_i} = \alpha$ be an equation satisfied by $\bar{x}^{t_0}(J^k), \ldots, \bar{x}^{t_k}(J^k)$.
    If $\sum_{i=1}^k a_{j_i} \hat{x}_{j_i} < \alpha$ then add $\sum_{i=1}^k a_{j_i} x_{j_i} \geq \alpha$ to S.

Figure 2: Separation heuristic for finding a set S of facet-defining inequalities with positive coefficients violated by a given point $\hat{x}$.

Let $S = J_+ = J_- = \emptyset$.
Order $j_0, \ldots, j_n$ so that
  $\min\{\hat{x}_{j_0} - v_0, v_{n-1} - \hat{x}_{j_0}\} \leq \cdots \leq \min\{\hat{x}_{j_0} - v_0, v_{n-1} - \hat{x}_{j_0}\}$.
For $j = 0, \ldots, k_{\max}$:
  If $\hat{x}_{j_0} - v_0 \leq v_{n-1} - \hat{x}_{j_0}$ then add j to J_+.
  Else add j to J_-.
For $k = 0, \ldots, k_{\max}$ while $S = \emptyset$:
  Let $J^k = \{j_0, \ldots, j_k\}$, $J^k_+ = J^k \cap J_+$, $J^k_- = J^k \cap J_-$.
  Let $\bar{x}^0(J^k), \ldots, \bar{x}^m(J^k)$ be the undominated J^k-circuits generated
    by the greedy procedure of Fig. 1 with $J_+ = J^k_+$, $J_- = J^k_-$.
  For each $\{t_0, \ldots, t_k\} \subset \{0, \ldots, m\}$:
    Let $\sum_{i=0}^k a_{j_i} x_{j_i} = \alpha$ be an equation satisfied by $\bar{x}^{t_0}(J^k), \ldots, \bar{x}^{t_k}(J^k)$.
    If $\sum_{i=1}^k a_{j_i} \hat{x}_{j_i} < \alpha$ then add $\sum_{i=1}^k a_{j_i} x_{j_i} \geq \alpha$ to S.

Figure 3: Separation heuristic for finding a set S of facet-defining inequalities with arbitrary coefficients violated by a given point $\hat{x}$.

defining inequalities

$$\begin{aligned} &8x_1 + 5x_2 + 10x_4 \geq 101 \\ &12x_1 + 11x_2 + 15x_4 \geq 169 \\ &6x_1 + 3x_2 + 8x_4 \geq 73 \end{aligned}$$

Because the first and third are violated by $\hat{x}$, they are separating cuts.

The above heuristic can be modified slightly to generate separating inequalities with arbitrary signs. Rather than order the variables by nondecreasing size of $\hat{x}_j$, we can order them by nondecreasing size of $\min\{\hat{x}_j - v_0, v_{n-1} - \hat{x}_j\}$. Then we put $j \in J_+$ if $\hat{x}_j - v_0 \leq v_{n-1} - \hat{x}_j$ and $j \in J_-$ otherwise. The heuristic appears in Fig. 3.

11 Exploiting Cost Structure

One motivation for studying the circuit polytope for arbitrary domains is that it may allow us to exploit structure in a cost function that appears in the problem. A careful choice of the domain values can result in a tighter relaxation.

Suppose, for example, that the problem contains the cost function $\sum_i c_{ix_i}$ that appears in the traveling salesman problem (2). Associate each index i with a value v_i, and suppose that the costs c_{ij} have the property that, when the values v_i are properly chosen, $g(v_i, v_j) = c_{ij}$ is close to the value of an affine function $h(v_i, v_j)$ for $i < j$, and it is close to the value of an affine function $h'(v_i, v_j)$ when $j < i$. The v_is can be set to any nonnegative value, and the variables can be reordered if desired, to obtain a good affine fit. Then one can use computational geometry techniques to compute the convex hull of $S = \{(z, x_i, x_j) \mid z = g(x_i, x_j),\ x_i, x_j \in \{v_0, \ldots, v_{n-1}\}\}$. Consider all facets of the convex hull that are described by inequalities of the form

$$z \geq \beta_{0k} + \beta_{1k}x_i + \beta_{2k}x_j, \quad k \in K \tag{24}$$

Then all of the points of S are close to the facets described by (24).

Now let $Ax \geq b$ be a system of valid inequalities for the circuit polytope $C_n(v)$, where v is the vector of values just chosen. We can write a linear relaxation of the traveling salesman problem (2) that exploits the cost structure:

$$\begin{aligned} &\min \sum_{ij} z_{ij} \\ &z_{ij} \geq \beta_{0k} + \beta_{1k}x_i + \beta_{2k}x_j, \text{ for all } i, j \text{ and all } k \in K \\ &Ax \geq b \end{aligned} \tag{25}$$

For example, suppose the cost data c_{ij} are as in Table 2. If we let $(v_1, v_2, v_3) = (0, 1.5, 3.5)$, the values $g(v_i, v_j) = c_{ij}$ are close to the values of the affine function $h(v_i, v_j) = 4(v_i - v_j)$ for $i < j$ and close to $h'(v_i, v_j) = 4(v_j - v_i)$ for $j < i$. The convex hull of S has two facets of the form (24), namely

$$\begin{aligned} z &\geq \tfrac{26}{7}x_i - \tfrac{26}{7}x_j \\ z &\geq -\tfrac{26}{7}x_i + \tfrac{26}{7}x_j \end{aligned}$$

So if $Ax \geq b$ is a set of valid inequalities for $C_n(v)$, the relaxation (26) therefore becomes

$$\begin{aligned} &\min \sum_{i=0}^{2} \sum_{j=0}^{2} z_{ij} \\ &z_{ij} \geq \tfrac{26}{7}x_i - \tfrac{26}{7}x_j \text{ for all } i, j \in \{0, 1, 2\} \\ &z_{ij} \geq -\tfrac{26}{7}x_i + \tfrac{26}{7}x_j \text{ for all } i, j \in \{0, 1, 2\} \\ &Ax \geq b \end{aligned} \tag{26}$$

If c_{ij} is a distance, it may be possible to exploit the structure of the distance metric, particularly if it is rectilinear. Further details, along with an application to the quadratic assignment problem, may be found in [4].

Table 2: (a) Cost data c_{ij}. (b) Values of $h(x_i, x_j)$ when $x_i \le x_j$ and $h'(x_i, x_j)$ when $x_j \le x_i$.

(a)

	$i \backslash j$	0	1	2
i	0	0	6	13
	1	6	0	9
	2	13	9	0

(b)

	$x_i \backslash x_j$	0	1.5	3.5
x_i	0	0	6	14
	1.5	6	0	8
	3.5	14	8	0

12 Conclusions and Future Research

We provided a nearly complete characterization of the circuit polytope that identifies all facet-defining inequalities with at most $n-4$ terms. In particular, we showed that the facet-defining inequalities with a specified sign pattern are precisely those valid inequalities that are defined by subsets of J-circuits that are undominated with respect to that sign pattern. Inequalities of this sort are valid when they are satisfied by all undominated J-circuits.

This allows us to identify all facet-defining inequalities with a two-phase procedure. A combinatorial phase generates all undominated J-circuits with respect to a desired sign pattern $J = J_+ \cup J_-$, using a greedy algorithm. A numerical phase then computes equations that are satisfied by affinely independent subsets of the undominated J-circuits and checks them for validity. The first phase is independent of the domain values $v_0, \ldots, v_{n-1}$, but the second is not. This two-phase procedure can be viewed as isolating the discrete and continuous aspects of the circuit polytope.

We also identified a family of permutation facets that are circuit facets and explicitly described all two-term circuit facets. We presented two separation heuristics based on the greedy procedure, and we showed how the circuit constraint with arbitrary variable domains can exploit cost structure in the objective function.

These results presented here lay the theoretical groundwork for the solution of sequencing problems with the help of linear relaxations comprised of circuit inequalities. Computational testing is the next step, together with investigation of how the separation heuristics can be tuned or altered to achieve best results. The cost matrices of typical problems can be examined to determine the extent to which cost can be approximated as an affine function or a rectilinear metric, to allow an effective choice of domain values.

An interesting research question is whether circuit inequalities can be profitably converted to 0-1 inequalities and combined with known traveling salesman inequalities. For a given domain $\{v_0, \ldots, v_{n-1}\}$, the conversion could be based on the identity $x_i = \sum_j v_j y_{ij}$, where y_{ij} is the 0-1 variable that appears in the traveling salesman model (3).

Our primary goal, however, has been to explore the structure of the circuit polytope in the original space, as an alternative to the conventional 0-1 representation.

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