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Optimality Conditions for Distributive Justice

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Abstract

This paper uses optimization theory to address a fundamental question of ethics: how to divide resources justly among individuals, groups, or organizations. It formulates utilitarian and Rawlsian criteria for distributive justice as optimization problems. The formulations recognize that some recipients are more productive than others, so that an inequitable distribution may create greater overall utility. Conditions are derived under which a distribution of resources is utility maximizing, and under which it achieves a lexicographic maximum, which we take as formulating the difference principle of John Rawls. It is found that utility maximization requires at least as much inequality as results from allotting resources in proportion to productivity, and typically a good deal more. Rawlsian justice requires a greater degree of equality than utilitarianism, particularly when the distribution of productivities across recipients has a short upper tail, although it is insensitive to the lower tail. It also requires greater equality when there are rapidly decreasing returns to investment in productivity, and ironically, when people have a stronger interest in getting rich.

1 Introduction

A fundamental issue for management ethics is the just distribution of resources. The issue is most naturally associated with the public sector, which must allot resources to individuals, institutions, and regions both productively and equitably. But it also arises in private sector management, which allocates salaries to individuals and budgets to divisions and subsidiaries. The ethical question is how one should allocate resources so that they are put to good use, while making sure that everyone receives a fair share.

Perhaps the two best-known criteria for just distribution are utilitarianism and the Rawlsian difference principle, one of which emphasizes efficiency and the other equity. Utilitarianism distributes resources to individuals in such a way that maximizes total utility, even if this results in inequality. The difference principle of John Rawls tolerates inequality only when it makes the least advantaged better off.

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Much has been said about the philosophical underpinnings of these two ethical theories [e.g. Daniels, 1989, Dworkin, 1977, Roemer, 1998, Stein, 1989, Williams and Cookson, 2000, Yaari and Bar-Hillel, 1984]. Yet there has been relatively little investigation of the actual distributions that result when these theories are applied. A particularly important issue is the extent to which an efficient distribution of resources requires inequality. It is sometimes argued that more utility is created when greater shares of resources are alloted to recipients that make more efficient use of it, perhaps because they are more talented, more productive, or work harder. (For brevity, we will frequently use the term "recipients" to denote the parties to whom resources are distributed, whether they be individuals, groups, or institutions.)

As it happens, both criteria for distributive justice pose optimization problems. Utilitarianism maximizes a social utility function whose arguments represent resources distributed to each recipient. The Rawlsian difference principle can be interpreted as calling for a lexicographic maximum (lexmax) of the utilities allotted to recipients. That is, the welfare of the least advantaged party should be maximized, after which the welfare of the second least advantaged party is maximized, and so forth. This suggests that the theory of optimization can provide some insight into the conditions under which a distribution of resources satisfies a utilitarian or a Rawlsian criterion. In fact, we will find that a fairly elementary analysis can lead to substantive and sometimes surprising conclusions about the characteristics of a just distribution, whether it be utilitarian or lexmax. It can reveal the extent to which a just distribution requires equality, and how its shape depends on utility functions (how people value wealth), productivity functions (how their productivity increases with investment), and the distribution of productivity among recipients. It can also allow one to calculate utilitarian and Rawlsian distributions.

Our assumption that recipients can have different productivities is a departure from axiomatic treatments in the social choice theory literature, which typically assume that recipients are indistinguishable [e.g. Blackorby et al., 2002, Roberts, 1980, Sen, 2004]. It also differs from related work in the emerging subfield of computational social choice theory, because our focus is primarily on structural properties of just distributions rather than computational techniques or complexity [see Bouveret and Lemaitre, 2006, Matt et al., 2006, Ogryczak, 2006].

A number of models for equitable distribution in specific application areas have appeared in the optimization literature [e.g. Betts et al., 1994, Brown, 1979, 1983, Daskin, 1995, Eiselt, 1986, Katoh and Ibaraki, 1998, Ogryczak, 1997], but these do not deal with the distributive justice problem in general. Luss [1999] discusses the lexmax (actually, lexmin) model as a general approach to equitable distribution and surveys applications, variations of the model, and algorithms that have been developed for them. (It studies lexmin rather than the lexmax solutions because it measures the performance of an activity to which resources are allocated by the shortfall from a target.) The applications include large-scale allocation problems with multiple knapsack resource constraints, multiperiod allocation problems for storable resources, and problems with substitutable resources. However, there are no structural results, of the sort provided here, that characterize the shape of the distribution, and none of the models account for different productivities of the recipients. In related work, Hall and Vohra [1993] describe an equity model based on proportionality constraints.

2 Overview of the Models

We use the modeling device of assigning to each recipient i a productivity function $u_i(\alpha)$ that measures the total utility eventually created when recipient i is initially alloted resources α. The utilitarian criterion can be interpreted as calling for a distribution of initially available resources that ultimately results in the greatest total utility. Utilitarians have historically argued that utility maximization favors a certain degree of equality, because there are decreasing returns to greater investments in the most productive recipients. We investigate the extent to which this is true, and we derive conditions under which a completely egalitarian distribution maximizes utility.

Another classical argument holds that excessive inequality leads to social disharmony and thereby reduces utility, so that, again, utility maximization pushes the distribution toward equality. We attempt to investigate this claim by adding a term in the objective function that measures the social disutility created by inequality as proportional to the gap between the largest and smallest allocations. We find that the optimality conditions have a relatively simple closed-form solution. It allows us to determine when the cost of inequality is high enough so that an egalitarian distribution of resources maximizes utility.

The Rawlsian difference principle states roughly that inequality should be tolerated only when it makes the least advantaged better off. The principle is based on a social contract argument, which begins with the premise that a rational allocation of primary goods must be one on which people can agree in an "original position"—that is, before they know which station in society they will occupy. Primary goods include such things as rights, opportunities, income, and power. To put Rawls' extended argument very briefly, a rational person would not agree to an unequal allocation in which he or she might end up in the least advantaged position, unless that position would be even worse in another allocation. A rational allocation must therefore maximize the welfare of the least advantaged, which results in a maximin solution. One might then consider all maximin solutions and argue along the same lines that the welfare of the second least advantaged party should be maximized, and so on recursively, resulting in a lexmax solution. We do not claim that the lexmax model is a precise interpretation of Rawls' own views, which, after all, evolved over his career, as witnessed by the differences between Rawls [1971] and Rawls [1993, 1999]. Our aim is to explore the mathematical implications of a lexmax criterion for its own sake.

To formulate a lexmax criterion, we suppose that the individual utility function $v(\alpha)$ measures the utility to any given recipient of possessing wealth α. The analysis is easily extended to allow each recipient i a different utility function $v_i(\alpha)$, but this complicates notation without adding much insight. We further suppose that the fraction of the total utility that is eventually enjoyed by a recipient is proportional to the utility of that recipient's initial resource allocation. Thus, for example, individuals who receive a better education or higher salary will have greater access to the fruits of those investments, or groups that receive a larger initial budget will enjoy a larger share of the wealth created.

We find that, remarkably, the optimality conditions for an egalitarian distribution in a lexmax problem have the same form as those for the utilitarian model that includes the social cost of inequality. However, the solution is different due to different right-hand sides. We derive conditions under which a distribution of initial resources satisfies the lexmax criterion, as well as conditions under which the lexmax distribution is completely egalitarian.

3 Utilitarian Model

The utilitarian thesis is that a just allocation of resources is one that maximizes total utility. Some of the classical utilitarians saw utility as reducible to pleasure, happiness, or some other universal end, but all that matters for our purposes that total utility can somehow be measured.

We wish to enrich the utilitarian model by accounting for the fact that some recipients have more productive potential than others. We suppose that each recipient i is given an initial allocation of resources x_i that enhance long-term productivity. For individuals, these might include education, salary, health care, tax breaks, or other benefits, and for groups or institutions they might include budget, personnel, or access to markets. Over some appropriate period of time, the recipient creates total utility given by the productivity function u_i. For a given investment α, $u_i(\alpha) > u_j(\alpha)$ when recipient i is more productive or more responsive to material incentives than recipient j. We will speak of the "productivity" of a recipient to mean its productive potential in this sense.

The utility created by a recipient reflects not only material wealth but other goods that are not easily assigned economic value, such as justice or environmental quality. Moreover, the utility eventually created by a recipient need not be identical with the utility enjoyed by that recipient. We suppose only that each recipient contributes to a pool of utility that is somehow distributed over all recipients. We do not describe this distribution in the utilitarian model, because only the aggregate utility ultimately matters. This is not to deny that the total utility of wealth and other goods depends on how they are distributed. As already noted, a highly skewed distribution of a fixed amount of resources may result in less utility than a more egalitarian distribution, due to the concavity of individual utility functions. However, we suppose that the distribution of wealth and other goods is already reflected in the productivity functions u_i. The Rawlsian model, on the other hand, will explicitly account for the distribution.

In the utilitarian model, the goal is to find an initial allocation of resources that maximizes $\sum_i u_i(x_i)$. If the total resource budget is 1, the problem becomes

$$\begin{aligned} &\max \sum_{i=1}^{n} u_i(x_i) \quad (a) \\ &\sum_{i=1}^{n} x_i = 1 \qquad (b) \\ &x_i \geq 0, \text{ all } i \qquad (c) \end{aligned} \tag{1}$$

If we associate Lagrange multiplier λ with the constraint (1b), any optimal solution of (1) in which each $x_i > 0$ must satisfy

$$u_i'(x_i) - \lambda = 0, \; i = 1, \ldots, n$$

Eliminating λ yields

$$u_1'(x_1) = \cdots = u_n'(x_n) \tag{2}$$

Thus a resource distribution is optimal only when the marginal productivity of resources is the same for everyone.

Assume that recipients $1, \ldots, n$ are indexed by increasing marginal productivity:

$$u_i'(\alpha) \leq u_{i+1}'(\alpha) \text{ for all } \alpha \geq 0 \text{ and } i = 1, \ldots, n-1 \tag{3}$$

In this case, (2) is satisfied only if $x_1 \leq \cdots \leq x_n$. Thus the less productive recipients receive less resources, as one might expect. Furthermore, a utilitarian distribution is completely egalitarian ($x_1 = \cdots = x_n = 1/n$) only when the marginal productivities are equal:

$$u_1'(1/n) = \cdots = u_n'(1/n) \tag{4}$$

The result (4) is illustrated in Fig. 1, which shows productivity functions for five recipients. The optimal distribution of resources is one at which the curves have equal slope. Note that less productive recipients receive fewer resources.

To obtain some idea of how skewed the resource distribution might be, it is helpful to assume a specific form

$$u_i(x_i) = c_i x_i^p \tag{5}$$

for the productivity functions, where $p \geq 0$ and each $c_i \geq 0$. Here c_i is a productivity coefficient for recipient i. When $p = 1$, recipient i produces utility in proportion to the resources received. When $0 < p < 1$, greater resources have decreasing marginal utility, and $p = 0$ indicates inability to use resources to create utility. If recipients are indexed in order of marginal productivity, we have that $c_1 \leq \cdots \leq c_n$.

Since an optimal solution of (1) in which each $x_i > 0$ must satisfy (1b) and (2), it is

$$x_i = c_i^{\frac{1}{1-p}} \left(\sum_{j=1}^{n} c_j^{\frac{1}{1-p}} \right)^{-1} \tag{6}$$

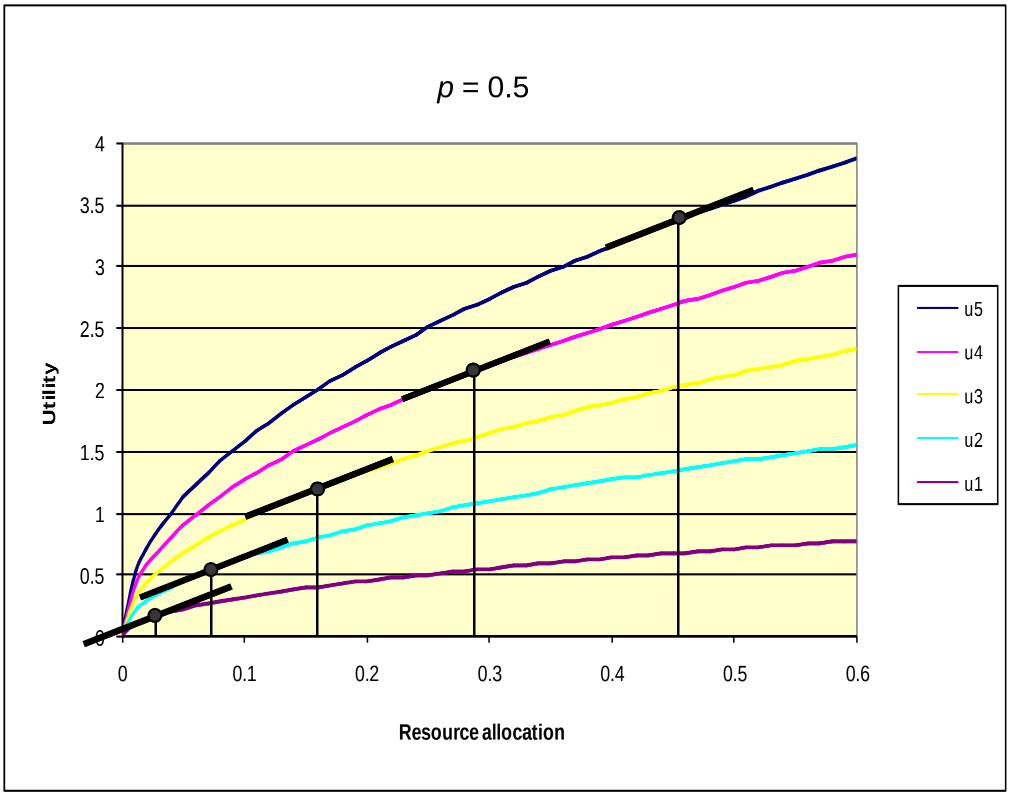

Figure 1: Optimal utilitarian resource allocation for five recipients. The curves show the productivity functions $u_i(x_i) = c_i x_i^p$ of the five recipients with $p = 0.5$ and $(c_1, \ldots, c_5) = (1, 2, 3, 4, 5)$. The slopes shown at the optimal allotments are equal.

when $0 \leq p < 1$. When $p \geq 1$, it is clear on inspection that an optimal solution sets $x_n = 1$ and $x_i = 0$ for $i = 1, \ldots, n-1$.

The optimal distribution is completely unequal when utility generated is proportional to resources allocated ($p = 1$). The most productive recipient receives all the resources. The distribution becomes increasingly egalitarian as p approaches zero, reaching in the limit a distribution in which each recipient i is allotted resources in proportion to c_i. Thus the most egalitarian distribution that is possible in this utilitarian model is one in which recipients are allocated resources in proportion to their productivity coefficient. Moreover, this occurs only in the limiting case when the utility generated becomes independent of the resources received ($p \to 0$). This is illustrated in Fig. 2, in which the straight line indicates the most egalitarian distribution of resources that can occur in a utilitarian solution.

When $0 < p \leq 1$, a utilitarian distribution can be completely egalitarian ($x_1 = \cdots = x_n$) only when $c_1 = \cdots = c_n$. When $p > 1$, one recipient must receive all the resources even when $c_1 = \cdots = c_n$.

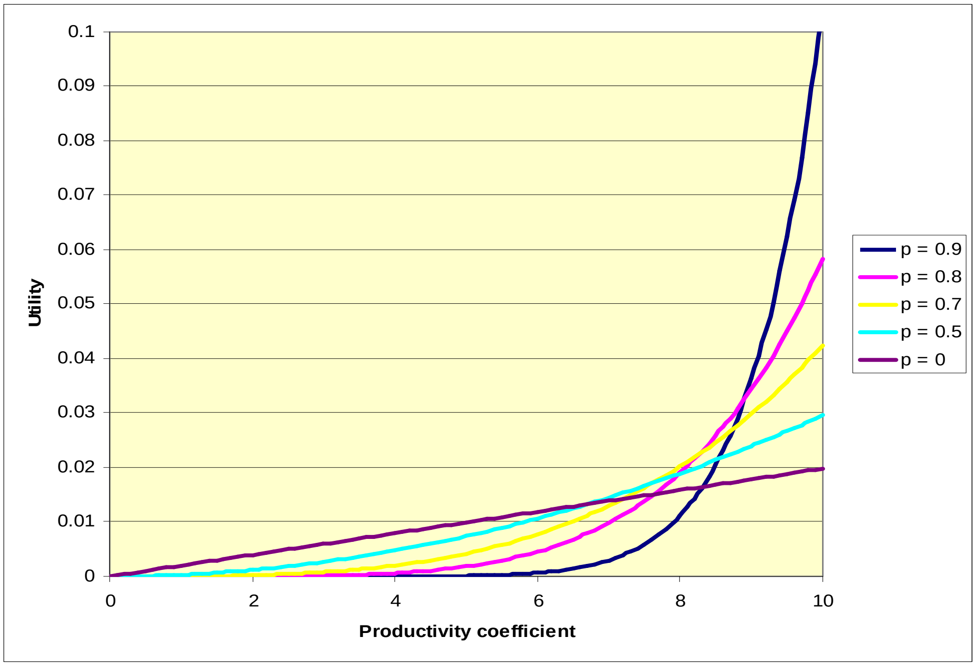

Figure 2: Optimal utilitarian resource allocation to a recipient, as a function of that recipient's productivity coefficient. Each curve corresponds to a different exponent p in the productivity function $c_i x^p$.

Figure 3 illustrates the optimal utilitarian distribution of initial resources in a multi-class society, as a function of the exponent p. Here the classes are defined by productivity rather than5 actual wealth, although they may correspond roughly to socioeconomic classes. In fact, differences in productivity may be due in part to social factors that allow individuals in historically privileged classes to respond more effectively to investment of resources. The dark bars indicate the relative population of each class, and the light bars the total resources allocated to each class. The distribution is inegalitarian even for small p, and extremely so as $p \to 1$. Classes that are only somewhat less productive than others may receive almost no investment. This extreme result may help to motivate subsequent models, which make inequality a more explicit factor in evaluating distributive justice.

One can measure the utility that is sacrificed, if any, by imposing a completely egalitarian distribution of resources. In a maximum-utility distribution with $0 \leq p < 1$, the total utility is

$$\left(\sum_{i=1}^{n} c_i^{\frac{1}{1-p}} \right)^{1-p} \tag{7}$$

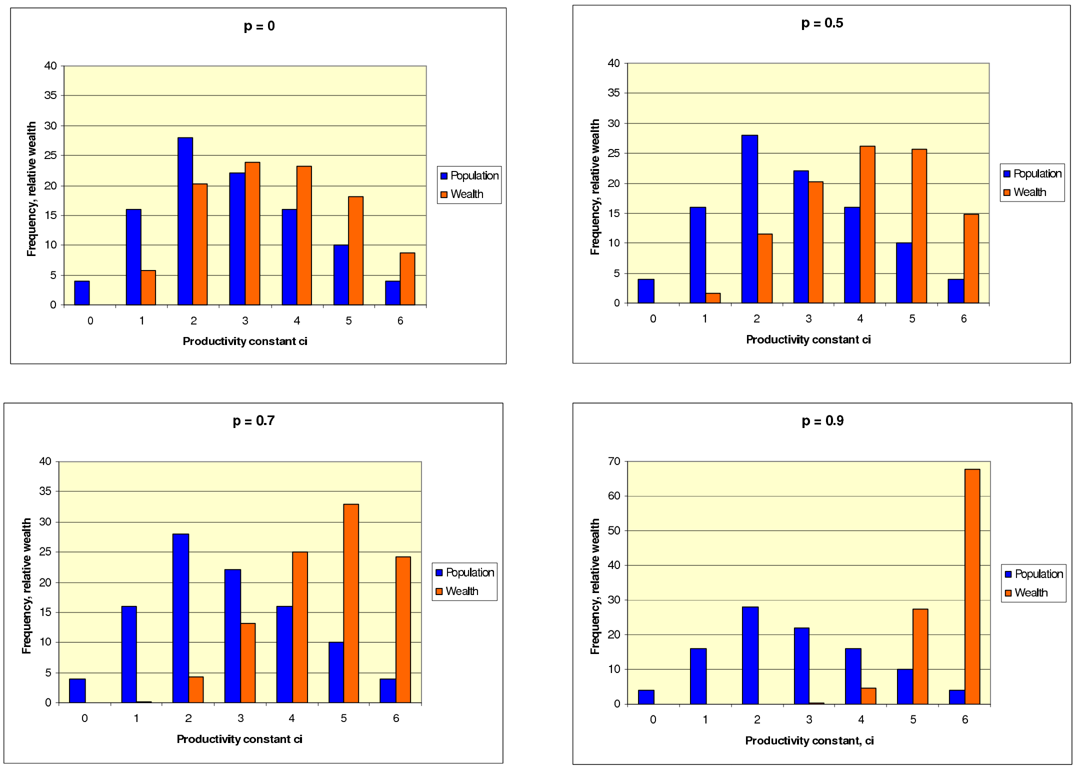

Figure 3: Utilitarian resource allocation in a multiclass society. The optimal distribution is shown for different values of the exponent p in the productivity function $c_i x^p$.

In a completely egalitarian distribution, each $x_j = 1/n$, and the total utility is

$$\left(\frac{1}{n}\right)^p \sum_{i=1}^{n} c_i \tag{8}$$

The ratio (8)/(7) indicates the relative efficiency of complete equality; that is, the ratio of total utility under complete equality to maximum utility.

Interestingly, this ratio tends to 1 as $p \to 0$ (Fig. 4). Thus if individual output is insensitive to investment, complete equality results in negligible utility loss, even though the optimal distribution of resources is substantially inegalitarian.

4 Modeling the Social Cost of Inequality

The rudimentary utilitarian model above implies that a utilitarian solution can result in considerable inequity when recipients differ in productivity. A classical defense of utilitarianism, however, is that excessive inequity generates disutility by contributing to social

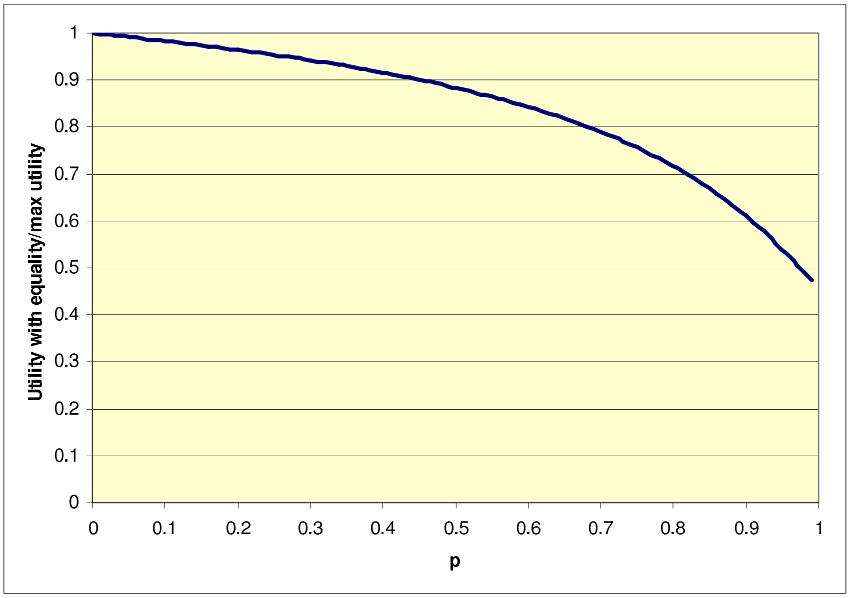

Figure 4: Efficiency of a completely egalitarian distribution of resources. The curve shows the ratio of utility under complete equality to maximum utility, as a function of the productivity exponent p, in the multiclass society of Fig. 3.

disharmony. In an institution context, inequity may reduce morale or create friction in the organization. A model with an additively separable productivity function, such as (1), does not account for any such cost of inequality. A more adequate model may result in utilitarian resource distributions that are more equitable.

A simple way to try to capture the cost of inequity is to model it as a proportional to the total range of resource allocations. The model (1) becomes

$$\begin{aligned} &\max \sum_{i=1}^{n} u_i(x_i) - \beta\left(\max_i\{x_i\} - \min_i\{x_i\}\right) \\ &\sum_{i=1}^{n} x_i = 1 \\ &x_i \geq 0, \text{ all } i \end{aligned} \tag{9}$$

Presumably, a positive cost factor β could result in utilitarian solutions that distribute resources more equally. It is also interesting to derive how large β must be to result in a completely egalitarian distribution.

The analysis is easier if we linearize problem (9) using the following lemma. We again assume that recipients are indexed by increasing marginal productivity, as in (3).

Lemma 1 *If the utility functions u_i satisfy (3), and (9) has an optimal solution, then the following problem has the same optimal value as (9):*

$$\begin{aligned} &\max \sum_{i=1}^{n} u_i(x_i) - \beta(x_n - x_1) && (a) \\ &\sum_{i=1}^{n} x_i = 1 && (b) \\ &x_i \leq x_{i+1},\ i = 1, \ldots, n-1 && (c) \\ &x_i \geq 0,\ \textit{all } i && (d) \end{aligned} \tag{10}$$

Proof. Let x^* be an optimal solution of (9) with optimal value U^*. If $x_j^* > x_k^*$ for some j, k with $j < k$, then create a new solution x^1 defined by $x_j^1 = x_k^*$, $x_k^1 = x_j^*$, and $x_i^1 = x_i^*$ for $i \neq j, k$. If U_1 is the objective function value of solution x^1 in (9), then

$$U_1 = U^* + u_j(x_k^*) - u_j(x_j^*) + u_k(x_j^*) - u_k(x_k^*) \tag{11}$$

But due to (3),

$$u_k(x_j^*) - u_k(x_k^*) \geq u_j(x_j^*) - u_j(x_k^*)$$

because $j < k$. This and (11) imply that $U_1 \geq U^*$. Now if $x_j^1 > x_k^1$ for some j, k with $j < k$, create a new solution x^2 in the same manner, and observe again that the objective function of (9) does not decrease. Continue with the sequence $x^1, \ldots, x^t$ until $x_1^t \leq \cdots \leq x_n^t$. Then x^t is feasible in the problem

$$\begin{aligned} &\max \sum_{i=1}^{n} u_i(x_i) - \beta\left(\max_i\{x_i\} - \min_i\{x_i\}\right) \\ &\sum_{i=1}^{n} x_i = 1 \\ &x_i \leq x_{i+1},\ i = 1, \ldots, n-1 \\ &x_i \geq 0,\ \text{all } i \end{aligned} \tag{12}$$

and has an objective function value no less than U^*. But (12) has an optimal value no greater than U^* because it is more highly constrained than (9). Thus (9) and (12) have the same optimal value. But (12) is obviously equivalent to (10), which implies that (9) and (10) have the same optimal value, as claimed.

To characterize optimal solutions of (10), we associate Lagrange multiplier λ with (10b) and multipliers $\mu_1, \ldots, \mu_{n-1}$ with the constraints in (10c). The Karush-Kuhn-Tucker (KKT) optimality conditions imply that x is optimal in (10) only if there are a value of λ and nonnegative values of $\mu_1, \ldots, \mu_{n-1}$ such that

$$\begin{aligned} &u_1'(x_1) + \beta - \lambda - \mu_1 = 0 \\ &u_i'(x_i) - \lambda + \mu_{i-1} - \mu_i = 0,\ i = 2, \ldots, n-1 \\ &u_n'(x_n) - \beta - \lambda + \mu_{n-1} = 0 \end{aligned} \tag{13}$$

where $\mu_i = 0$ if $x_i < x_{i+1}$ in the solution.

We first examine the case in which each recipient has a different resource allotment x_i. In this case each $\mu_i = 0$, and we can eliminate λ from (13) to obtain

$$\begin{aligned} u_2'(x_2) &= \cdots = u_{n-1}'(x_{n-1}) \\ u_1'(x_1) &= u_i'(x_2) - \beta, \quad i = 2, \ldots, n-1 \\ u_n'(x_n) &= u_i'(x_2) + \beta, \quad i = 2, \ldots, n-1 \end{aligned}$$

Thus all recipients who are not at the extremes of the distribution have equal marginal productivity in a utilitarian distribution, just as they do in the solution of the original model (1). The recipient at the bottom of the distribution, however, has marginal productivity that is β smaller than that of those in the middle, while the recipient at the top has marginal productivity that is β larger than that of those in the middle. This tends to result in somewhat larger allotment for the recipient at the bottom, and a smaller allotment for the one at the top. Since the remaining recipients are forced to lie between these extremes, the net result is a distribution that is less skewed than in the original model.

This is illustrated in Fig. 5, in which $p = 0.5$ and $\beta = 0.5$. When β is raised to 0.87, the two most productive recipients begin with the same allotment, and as β is further increased, the differences continue to collapse until complete equality occurs (Fig. 6).

5 Equality in the Social Cost Model

We can determine what value of β results in a completely egalitarian solution. In this case the multipliers μ_i can be nonzero. Again eliminating λ from the KKT conditions (13), we get

$$\begin{aligned} &2\mu_1 - \mu_2 = d_1 \\ &\mu_1 + \mu_i - \mu_{i+1} = d_i, \; i = 2, \ldots n-2 \\ &\mu_1 + \mu_{n-1} = d_{n-1} \end{aligned} \tag{14}$$

where

$$\begin{aligned} d_i &= u_1'(x_1) - u_{i+1}'(x_{i+1}) + \beta, \; i = 1, \ldots, n-2 \\ d_{n-1} &= u_1'(x_1) - u_n'(x_n) + 2\beta \end{aligned} \tag{15}$$

Remarkably, (14) has the following relatively simple closed form solution:

$$\mu_k = \frac{k}{n} \sum_{i=k}^{n-1} d_i - \left(1 - \frac{k}{n}\right) \sum_{i=1}^{k-1} d_i \tag{16}$$

for $k = 1, \ldots, n-1$. Substituting (15) into (16), we get

$$\mu_k = \beta - \frac{k(n-k)}{n} \left(\frac{1}{n-k} \sum_{i=k+1}^{n} u_i'(x_i) - \frac{1}{k} \sum_{i=1}^{k} u_i'(x_i) \right) \tag{17}$$

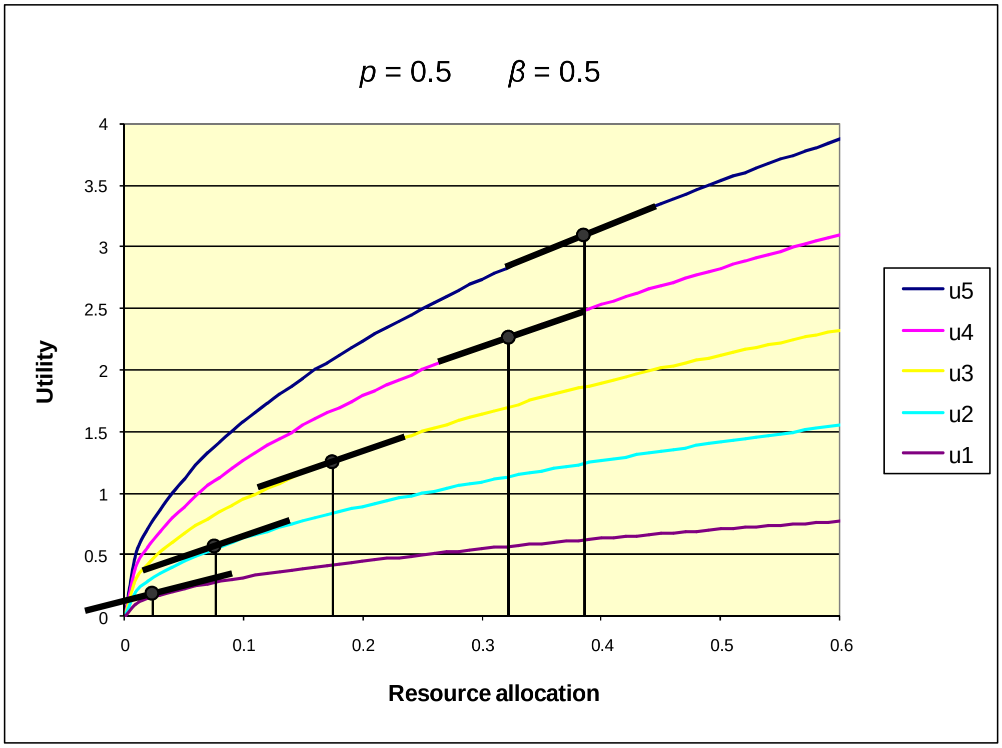

Figure 5: Optimal resource allocation for five recipients in the social cost model, based on the productivity functions of Fig. 1. The three slopes in the middle are equal to 3.52. The rightmost slope is $3.52 + \beta = 4.02$, and the leftmost slope is $3.52 - \beta = 3.02$. This reduces the largest allotment and increases the smallest relative to Fig. 1, and the remaining allotments are somewhat compressed as a result.

for $k = 1, \ldots, n-1$.

We now consider an egalitarian solution, in which each $x_i = 1/n$. Since each $\mu_i \geq 0$ in an optimal solution, we obtain the following from (17).

Theorem 1 *Suppose that recipients are indexed in order of increasing marginal productivity. Then an utilitarian distribution in the model (9) is egalitarian ($x_1 = \cdots = x_n$) only if*

$$\beta \geq \frac{k(n-k)}{n} \left(\frac{1}{n-k} \sum_{i=k+1}^{n} u_i'(1/n) - \frac{1}{k} \sum_{i=1}^{k} u_i'(1/n) \right) \tag{18}$$

for $k = 1, \ldots, n-1$.

The quantity in parentheses is the difference between the average marginal productivity of the $n-k$ most productive recipients and that of the k least productive recipients, when

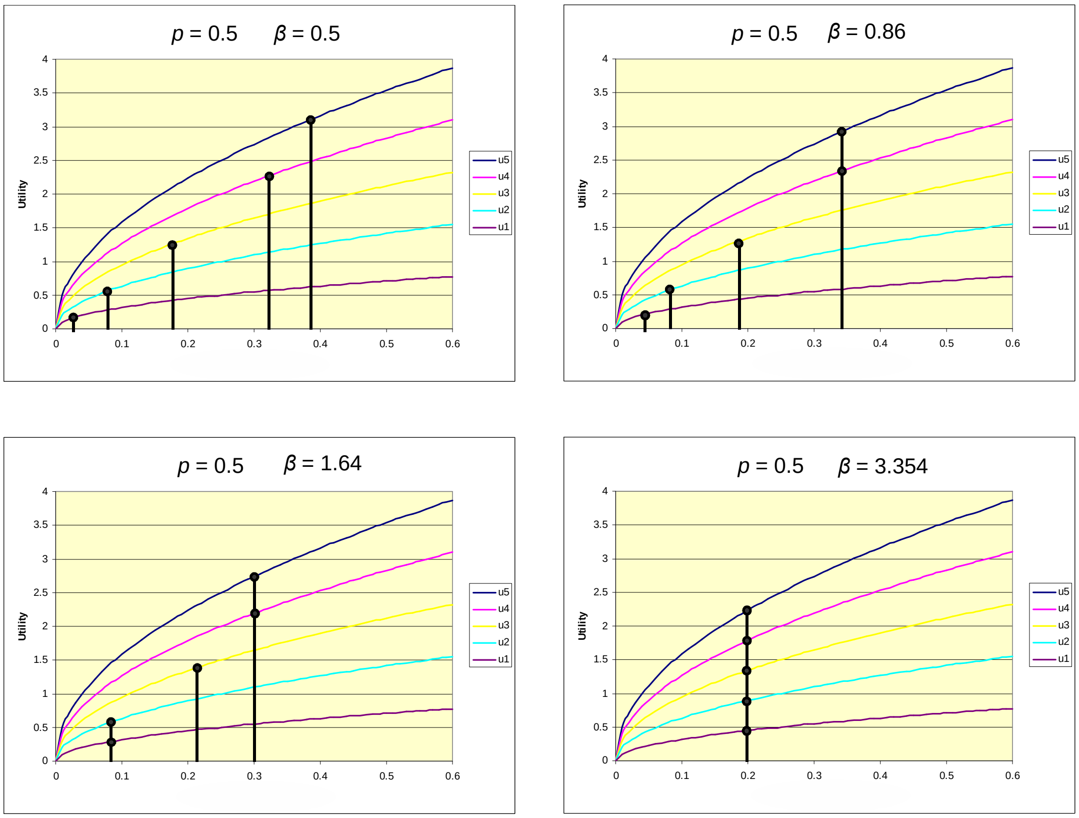

Figure 6: Optimal resource allocation to five recipients for several values of the social cost coefficient β. The productivity functions are as in Fig. 3.

initial resources are distributed equally. This may be easier to interpret for the specific productivity functions defined earlier.

Corollary 1 *If the productivity function u_i are given by (5), a utilitarian distribution in the model (9) is egalitarian only if*

$$\beta \geq \frac{p}{n^p} k(n-k) \left(\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \right)$$

for $k = 1, \ldots, n-1$.

Thus to determine the minimum β required to ensure equality, we examine each group of k smallest coefficients $c_1, \ldots, c_k$. The value of β depends on the difference between the average of these coefficients and the average of the remaining coefficients. Thus if there is a group of recipients who are much less productive on the average than the remaining

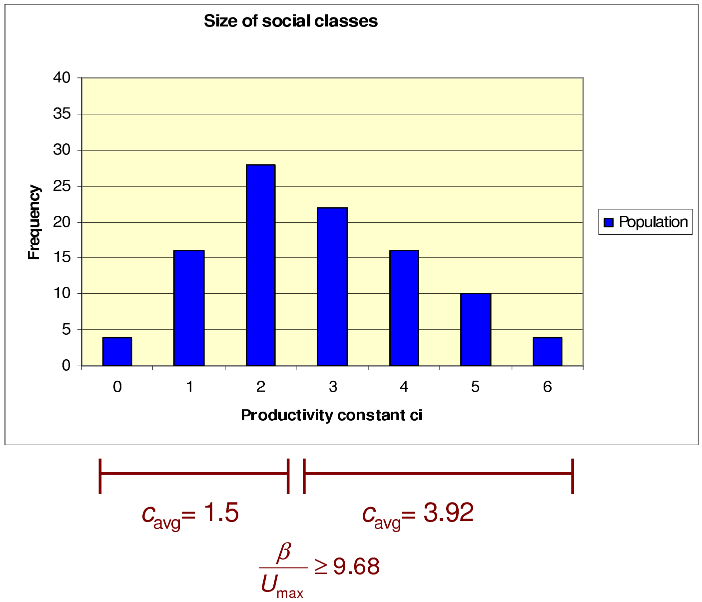

Figure 7: Minimum β for complete equality in a unimodal productivity distribution. The brackets show the population split that results in the minimum β for which complete equality is optimal in a multiclass society with a unimodal productivity distribution. The quantity c_{avg} on the left is the average productivity coefficient of individuals in the lowest three classes, and c_{avg} on the right is the average coefficient in the remaining classes.

recipients, relative to the overall range of productivities, a larger β is required to ensure equality. This could occur in a two-class society with a relatively homogeneous underclass and relatively homogenous elites, for example.

The variance of the productivity distribution does not capture this measure. If, for example, there are three recipients, the productivity coefficients $(c_1, c_2, c_3) = (0, 1, 2)$ have the same variance as $(c_1, c_2, c_3) = (\alpha, \alpha, 2)$, where $\alpha = 2 - 3^{1/2} \approx 0.268$. But if $k = 2$, the difference of means is 1.5 for the first distribution and $3^{1/2} \approx 1.732$ for the second. The second imposes a stricter condition for equality because there is a greater productivity gap between one homogeneous group of recipients (the first two) and the rest (the third).

The lower bound in Corollary 1 tends to be largest when the value of k is chosen to maximize $k(n-k)$, or when $k \approx n/2$. For the five recipients of Fig. 6, the largest bound of 3.354 is achieved when $k = 2, 3$. Thus we have equality when $\beta \geq 3.354$. In the multiclass society of Fig. 3, the lower bound on β is largest when k is the number of individuals in the three lowest classes (Fig. 7). In this case the bound is $9.68U_{\max}$, where $U_{\max} = 31.2$ is the maximum possible utility, ignoring the social cost of inequality. Thus equality is required only when β is quite large.

Figure 8 shows a similar analysis in a multiclass society with a bimodal productivity

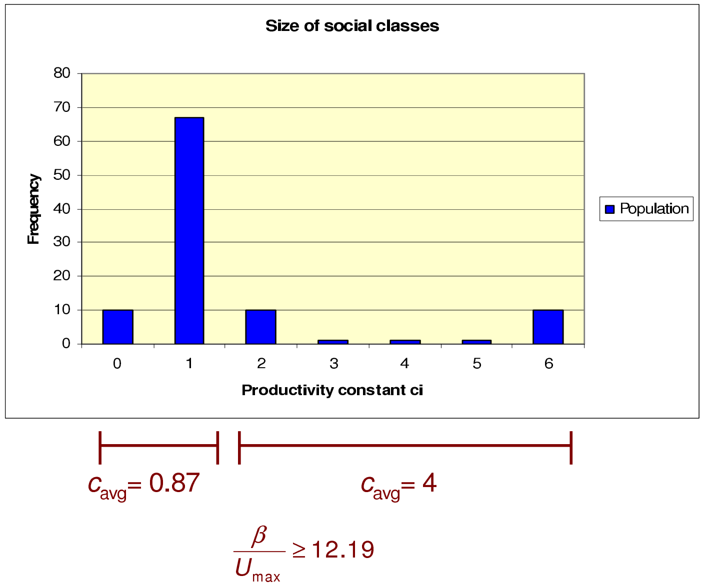

Figure 8: Minimum β for complete equality in a bimodal productivity distribution. The notation is the same as in Fig. 7.

distribution. Most people occupy the three lowest classes, and small minority occupy the highest class, with very few in between. In this case the critical cut point is between the second and third social classes. Due to the large gap between the least and most productive classes, a larger $\beta = 12.19U_{\max}$ (relative to $U_{\max} = 22.7$) is necessary to enforce equality.

6 Rawlsian Distribution

A lexmax (lexicographic maximum) model can be used to represent a resource distribution that satisfies the Rawlsian difference principle. As before we let $u_i(x_i)$ be a productivity function that measures the utility generated by a recipient i who begins with resources x_i. We also suppose that the fraction of total utility enjoyed by recipient i is proportional to the utility $v(x_i)$ of recipient i's initial resource allocation. Thus every recipient has the same utility function, even though different recipients may have different productivity functions.

Because the lexmax objective function is sensitive to how utility is distributed across recipients, as well as the total utility, we let y_i be the utility enjoyed by recipient i. Any solution of the following problem is a Rawlsian distribution:

$$\begin{array}{ll} \text{lexmax } y & (a) \\ \dfrac{y_i}{y_1} = \dfrac{v(x_i)}{v(x_1)},\ i = 2, \ldots, n & (b) \\ \displaystyle\sum_{i=1}^n y_i = \sum_{i=1}^n u_i(x_i) & (c) \\ \displaystyle\sum_{i=1}^n x_i = 1 & (d) \\ x_i \geq 0,\ i = 1, \ldots, n & (e) \end{array} \tag{19}$$

where $y = (y_1, \ldots, y_n)$. By definition, y^* solves (19) if and only if y_k^* solves problem L_k for $k = 1, \ldots, n$, where L_k is

$$\begin{array}{l} \max\ \min\ \{y_k, \ldots, y_n\} \\ (y_1, \ldots, y_{k-1}) = (y_1^*, \ldots, y_{k-1}^*) \\ (19b)\text{–}(19e) \end{array} \tag{20}$$

The lexmax solution is frequently defined with respect to a particular ordering $y_1, \ldots, y_n$ of the variables (e.g., in Isermann [1982]), in which case L_1 maximizes y_1 rather than maximizing $\min\{y_1, \ldots, y_n\}$. This is inappropriate for the Rawlsian problem because we do not know in advance how the solution values y_k^* will rank in size.

Suppose, however, that recipients $1, \ldots, n$ are indexed by increasing marginal productivity as in (3). Then we can assume without loss of generality that recipients with less marginal productivity are nearer the bottom of the distribution.

Lemma 2 *Suppose that (3) holds and that $v(\alpha)$ is monotone nondecreasing for $\alpha \geq 0$. Then if (19) has a solution, it has a solution in which $y_1 \leq \cdots \leq y_n$.*

Proof. Since v is monotone, it suffices to show that (19) has a solution $(\bar{x}, \bar{y})$ in which $\bar{x}_1 \leq \cdots \leq \bar{x}_n$. For this it suffices to exhibit a solution $(\bar{x}, \bar{y})$ that solves L_k for $k = 1, \ldots, n$ and for which $\bar{x}_1 \leq \cdots \leq \bar{x}_n$.

Let (x^*, y^*) be a solution of (19), and let $(x^0, y^0) = (x^*, y^*)$. If $x_1^0 \leq x_i^0$ for $i = 2, \ldots, n$, then x^0 solves L_1 and we let $x^1 = x^0$. Otherwise we suppose $x_k^0 = \min_i\{x_i^0\}$ and define x^1 by $x_1^1 = x_k^0$, $x_k^1 = x_1^0$, and $x_i^1 = x_i^0$ for $i \neq 1, k$. We define y^1 to satisfy (19b)-(19c). We can see as follows that (x^1, y^1) solves L_1. If $U_0 = \sum_i u_i(x_i)$ is the total utility for solution (x^0, y^0), then the total utility for solution (x^1, y^1) is

$$U_1 = U_0 + u_k(x_1^0) - u_k(x_k^0) + u_1(x_k^0) - u_1(x_1^0)$$

But we have from (3) that

$$u_k(x_1^0) - u_1(x_k^0) \geq u_1(x_1^0) - u_1(x_k^0)$$

Thus $U_1 \geq U_0$, and x^1 generates no less total utility than x^0. Since utility is allotted to the y_i^1s in proportion to $v(x_i^1)$, and v is monotone nonincreasing, we get $y_1^1 \leq y_1^0$. Thus (x^1, y^1) solves L_1.

Now if $x_1^1 \leq x_i^1$ for $i = 2, \ldots, n$, then (x^1, y^1) solves L_1, L_2 and we let $(x^2, y^2) = (x^1, y^1)$. Otherwise we suppose $x_k^1 = \min_{i\geq 2}\{x_i^1\}$ and define x^2 by $x_1^2 = x_k^1$, $x_k^2 = x_1^1$, and $x_i^2 = x_i^1$ for $i > 2$ and $i \neq k$. We can show as above that (x^2, y^2) solves L_1, L_2. In this fashion we construct the sequence $(x^1, y^1), \ldots, (x^n, y^n)$ and let $(\bar{x}, \bar{y}) = (x^n, y^n)$. By construction, $\bar{x}_1 \leq \cdots \leq \bar{x}_n$. Since $(\bar{x}, \bar{y})$ solves $L_1, \ldots, L_n$, it solves (19).

To analyze solutions of (19), it is convenient to eliminate the variables y_i from each L_k. Using constraints (19b)–(19c), we get

$$y_i = v(x_i)\frac{\sum_{j=1}^n u_j(x_j)}{\sum_{j=1}^n v(x_j)}, \; i = 1, \ldots, n$$

Using Lemma 2, L_k can be written

$$\begin{aligned}
&\max \; v(x_k)\frac{\sum_{i=1}^n u_i(x_i)}{\sum_{i=1}^n v(x_i)} && (a)\\
&(x_1, \ldots, x_{k-1}) = (x_1^*, \ldots, x_{k-1}^*) && (b)\\
&\sum_{i=1}^n x_i = 1 && (c)\\
&x_k \leq \cdots \leq x_n && (d)\\
&x_k \geq 0 && (e)
\end{aligned} \qquad (21)$$

where $x_1^*, \ldots, x_{k-1}^*$ are previously computed solutions of $L_1, \ldots, L_{k-1}$, respectively.

We focus first on L_1. Associating Lagrange multipliers $\mu_1, \ldots, \mu_{n-1}$ with the constraints in (21d), the KKT optimality conditions imply that a solution x with each $x_i > 0$ is optimal in (21) only if there are nonnegative values of $\mu_1, \ldots, \mu_{n-1}$ such that

$$\begin{aligned}
&v'(x_1)\frac{\Sigma u}{\Sigma v} + v(x_1)\frac{u_1'(x_1)\Sigma v - v'(x_1)\Sigma u}{(\Sigma v)^2} - \lambda - \mu_1 = 0\\
&v(x_1)\frac{u_i'(x_i)\Sigma v - v'(x_i)\Sigma u}{(\Sigma v)^2} - \lambda + \mu_{i-1} - \mu_i = 0, \quad i = 2, \ldots, n-1\\
&v(x_1)\frac{u_n'(x_n)\Sigma v - v'(x_n)\Sigma u}{(\Sigma v)^2} - \lambda + \mu_{n-1} = 0
\end{aligned} \qquad (22)$$

where

$$\Sigma u = \sum_{i=1}^{n} c_i u_i(x_i), \quad \Sigma v = \sum_{i=1}^{n} v(x_i)$$

and where $\mu_i = 0$ if $x_i < x_{i+1}$ in the solution.

We begin by examining the case in which each recipient has a different allotment x_i. Here each $\mu_i = 0$, and (22) implies

$$\frac{v'(x_1)}{v(x_1)} + \frac{u_1'(x_1)}{\Sigma u} - \frac{v_1'(x_1)}{\Sigma v} = \frac{u_i'(x_i)}{\Sigma u} - \frac{v_i'(x_i)}{\Sigma v}$$

for $i = 1, \ldots, n-1$, assuming $v(x_1) > 0$. This says that the marginal difference between productivity and utility is the same for every recipient except the lowest ranked recipient, for whom the difference is somewhat less. This tends to increase the allotment to the lowest recipient, reducing the gap between this recipient and the others. The optimality conditions for L_2 are similar and likewise move the second closest recipient closer to those who are more highly ranked. Thus in general, the lexmax solution results in a distribution that is more egalitarian than one in which the marginal difference between productivity and utility is the same for every recipient.

We will find that the lexmax distribution for the five recipients of Fig. 1 is completely egalitarian, with each recipient allotted 0.2. If the individual productivities are more diverse, however, the Rawlsian distribution may not be completely egalitarian. This is the case for the five recipients of Fig. 9, which shows the very skewed utility maximizing distribution. Fig. 10 shows the Rawlsian distribution, which is much more egalitarian, if not completely so.

7 Equality in a Rawlsian Distribution

We now examine conditions under which a Rawlsian distribution can be egalitarian. We found earlier that a utilitarian distribution with productivity functions $u_i(x_i) = c_i x_i^p$ cannot be egalitarian unless recipients are identical in their productivity. We will show that a Rawlsian distribution can, under certain conditions, be egalitarian in a more diverse population.

We will suppose that the utility function has the form $v(\alpha) = \alpha^q$. This means that if one posses wealth α, the marginal value of wealth is $q\alpha^{q-1}$. Normally $q < 1$, indicating a concave utility function, because the marginal value of wealth tends to decrease as one becomes richer. A q that is close to one indicates that the marginal value of wealth remains nearly constant as one accumulates wealth. Thus getting rich is important, because one continues to value additional wealth even after having achieved riches. A small q indicates that the marginal value of wealth is more or less inversely proportional to current wealth. This can be interpreted as a relative lack of interest in getting rich, because the importance of acquiring additional wealth rapidly falls off as one becomes more comfortable.

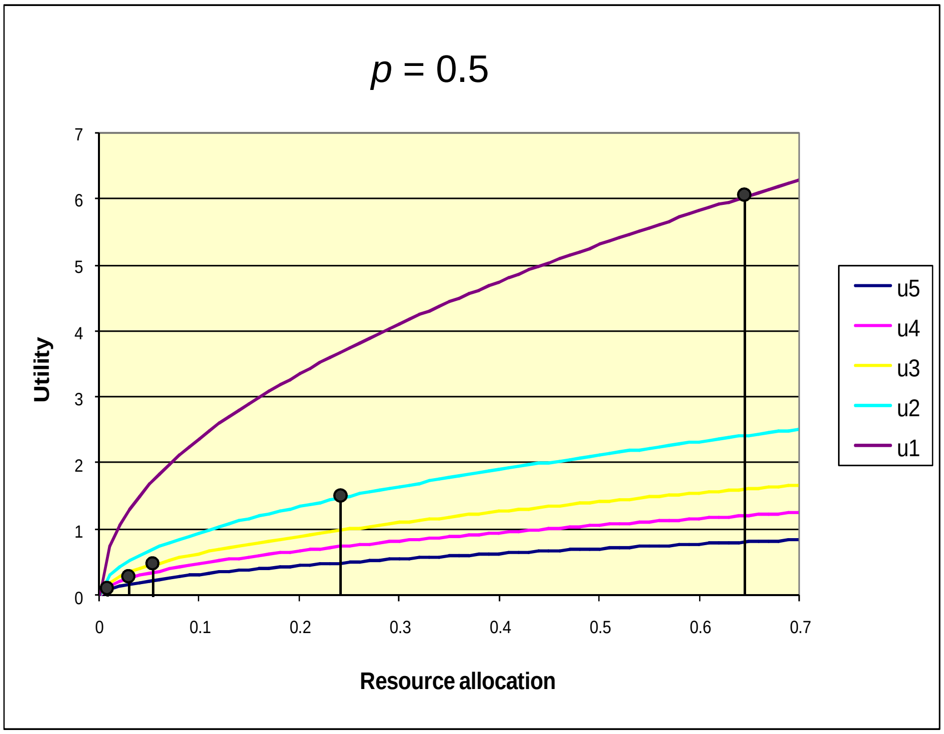

Figure 9: Utility maximizing allocation for five recipients with diverse productivities. The productivity functions are $u_i(x_i) = c_i x_i^p$, where $(c_1, \ldots, c_n) = (1, 1.5, 2, 4, 6.5)$.

In an egalitarian distribution any μ_i can be nonzero. We eliminate λ from the optimality conditions (22) for L_1 to obtain

$$\frac{v'(x_1)}{v(x_1)} + \frac{u_1'(x_1)}{\Sigma u} - \frac{v'(x_1)}{\Sigma v} - \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}\mu_1 = \frac{u_i'(x_i)}{\Sigma u} - \frac{v'(x_i)}{\Sigma v} + \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}(\mu_{i-1} - \mu_i) \quad (23)$$

for $i = 2, \ldots, n-1$, and

$$\frac{v'(x_1)}{v(x_1)} + \frac{u_1'(x_1)}{\Sigma u} - \frac{v'(x_1)}{\Sigma v} - \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}\mu_1 = \frac{u_n'(x_n)}{\Sigma u} - \frac{v'(x_n)}{\Sigma v} + \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}\mu_{n-1} \quad (24)$$

Remarkably, these equations have the same form as the optimality conditions (14) for the social cost model, but with different right-hand sides:

$$d_i = v(x_1)\frac{\Sigma u}{\Sigma v}\left(\frac{v'(x_1)}{v(x_1)} - \frac{u_{i+1}'(x_{i+1}) - u_1'(x_1)}{\Sigma u} + \frac{v'(x_{i+1}) - v'(x_1)}{\Sigma v}\right), \quad i = 1, \ldots, n-1 \quad (25)$$

This yields the following.

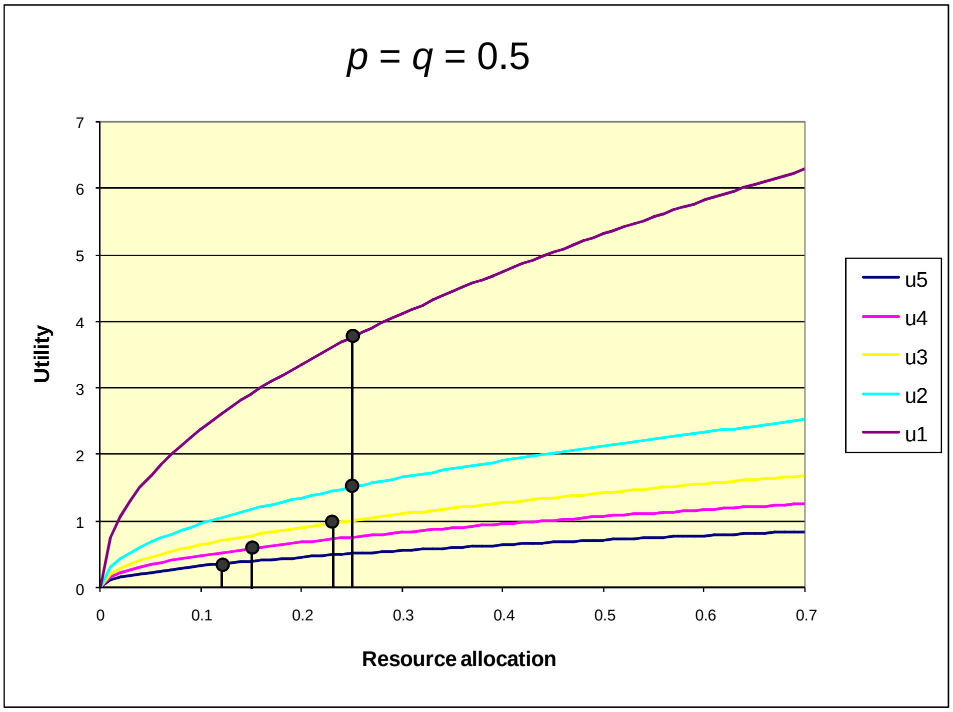

Figure 10: Lexmax distribution for the five recipients of Fig. 9. The individual utility function is $v(x_i) = x_i^q$.

Theorem 2 *Suppose the productivity functions are given by $u_i(\alpha) = c_i\alpha^p$ and the utility function by $v(\alpha) = \alpha^q$. Then L_1 has an egalitarian solution ($x_1 = \cdots = x_n$) only if*

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \leq \left(k \cdot \frac{p}{q}\right)^{-1} \sum_{i=1}^{n} c_i \tag{26}$$

for $k = 1, \ldots, n-1$.

Proof. The equations (23)–(24) can be written as (14) where the d_is are given by (25). Substituting $x_1 = \cdots = x_n = 1/n$ and the functions u_i, v as given above, we obtain

$$d_i = qn^{-p}\sum_{j=1}^{n} c_j - pn^{-p}\left(c_{i+1} - c_1\right) \tag{27}$$

Since (16) solves (14), we can substitute (27) into (16) and get

$$\mu_k = p\frac{k(n-k)}{n^{1+p}}\left(\frac{q}{pk}\sum_{i=1}^{n} c_i + \frac{1}{k}\sum_{i=1}^{k} c_i - \frac{1}{n-k}\sum_{i=k+1}^{n} c_i\right)$$

for $k = 1, \ldots, n-1$. The KKT conditions imply that $x_k = \cdots = x_n = 1/n$ can be an optimal solution only if $\mu_k \geq 0$ for $k = 1, \ldots, n-1$, which implies (26).

An egalitarian solution ($x_1 = \cdots = x_n$) solves L_1 if and only if it solves the lexmax problem (19). If it solves L_1, then a lexmax solution must have $x_1 = 1/n$, which implies by (19d) that $x_2 = \cdots = x_n = 1/n$. If an egalitarian solution does not solve L_1, then some distribution with $x_1 < 1/n$ solves L_1, which implies that $x_1 < 1/n$ in any lexmax solution. Thus we have

Corollary 2 *If the productivity functions are given by $u_i(\alpha) = c_i\alpha^p$ and the utility function by $v(\alpha) = \alpha^q$, then a lexmax distribution is egalitarian ($x_1 = \cdots = x_n$) only if (26) holds.*

Thus a Rawlsian distribution is completely egalitarian when the gap between the average productivity of the k least productive recipients and that of the remaining recipients is not too great for any k. The maximum gap is inversely proportional to k and p/q. This means that a smaller gap is required when the marginal utility of wealth increases rapidly with the level of wealth (p is large), and when the opposite is true of marginal productivity (q is small). Thus an inegalitarian distribution is more likely when allocating greater advantages to talented or industrious recipients reaps consistently greater rewards. Inequality is also more likely when recipients do not care very much about getting rich and are satisfied with a moderate level of prosperity.

An egalitarian distribution is also consistent with a much smaller productivity gap between the highest class and the remaining population (i.e., when $k = n-1$) than between the lowest class and the remaining population ($k = 1$). Thus if the distribution of talents and industry has a long tail at the upper end, as is commonly supposed, the condition for equality could be hard to meet. However, a long tail at the lower end has little effect on whether the productivity distribution meets the condition for equality.

The ratio p/q requires some interpretation. Recall that the productivity functions $u_i(x_i) = c_i x_i^p$ measure the *utility* of wealth created (as opposed to the wealth itself) and therefore already reflect the concavity of the utility function $v(x_i) = x_i^q$. Thus we would expect p to be somewhat smaller than q when the wealth created by a recipient is less than proportional to investment x_i. The wealth created could be more than proportional to investment in some cases, as when one's education level passes a critical level at which productivity jumps, but probably not a great deal more in general. This suggests that p/q will normally be less than 1, or in any event not much greater than 1, in most situations familiar to us.

One can check that the conditions of Corollary 2 hold for the five recipients of Fig. 1 when $p/q \leq 1.5$, which is very likely to hold in practice. The lexmax distribution is therefore completely egalitarian.

In the unimodal productivity distribution of Fig. 7, Rawlsian justice requires complete equality when p is somewhat smaller than q, in particular when $p/q \leq 0.852$. Because we do not expect p/q to be much larger than 1 in any case, this indicates that equality is required for a rather large range of utility and productivity functions. However, equality is required in the bimodal distribution of Fig. 8 only when $p/q \leq 0.361$, a much stronger condition that is perhaps unlikely to be met in reality.

8 Conclusion

We addressed the ethical question of how one can best distribute resources under utilitarian and Rawlsian models of justice. The resources allocated to an individual, group, or institution are understood to be an investment in its productivity, perhaps in the form of education, salary, incentives, health care, budget, market access, or tax breaks.

We find that a utilitarian distribution of resources can result in substantial inequality when some recipients are more productive than others. The optimal distribution is completely egalitarian only when every recipient has the same marginal productivity. When marginal productivities are unequal, the most egalitarian distribution that is possible is one in which recipients are allocated wealth in proportion to their productivity, and this occurs only when there are rapidly decreasing marginal returns for greater allocations of wealth. However, if individual production is not very sensitive to investment, relatively little utility is sacrificed by a distribution that is more egalitarian than one that maximizes utility.

A more egalitarian optimal distribution results when the utility function includes a penalty to account for social dysfunction that inequality may cause. In particular, if the penalty is proportional to the gap between the richest and poorest recipients, we can calculate a constant of proportionality that results in a completely egalitarian distribution. This constant tends to be larger when there is large gap in average productivity between two segments of the population. That is, there a group of recipients that have a much smaller average marginal productivity than the remaining recipients, relative to the overall range of productivities. This may occur, for example, in a society where elites and common people form fairly homogenous groups separated by a large gap in average productivity.

Finally, the Rawlsian difference principle can result in a significantly more egalitarian distribution than utility maximization. It can require a completely egalitarian distribution when no two segments of the population are separated by a large gap in average productivity. Equality is more likely to be required when there are decreasing returns for placing greater investment in more productive recipients. Somewhat surprisingly, equality is also more likely when people are nearly as concerned about getting rich as about living a minimally comfortable lifestyle. When people want riches more, a privileged class or a highly paid executive corps is less likely to be consistent with Rawlsian justice.

Equality is less likely to be required when the most productive recipients are substantially more productive than the average, as when a society has an elite intelligentsia or a company has a few managerial superstars. One the other hand, the relative productivity of the least productive recipients is much less relevant. The existence of an underclass of unproductive or disabled citizens, or of unskilled workers in a company, does not reduce the degree to which equality must be achieved. This differs from the situation in the utility-maximizing social cost model, where a long lower tail on the productivity distribution has the same effect as a long upper tail, but gaps in the middle of the distribution are somewhat more important.

Rawlsian justice may well require a completely egalitarian distribution of resources, for example in a society that has a large middle class in the sense that most people are near the average in productivity. In particular, it requires equality when individual productivity functions level off somewhat more quickly than individual utility functions. A Rawlsian criterion is substantially less likely to require equality when there is a large productivity gap between the most productive recipients and the rest.

The range of individual productivities are themselves likely to depend on the scale and depth of the resource allocation. If society allocates resources only at the margin, for instance by subsidizing higher education or providing tax breaks, then the response to social investment will remain highly dependent on socioeconomic factors. Historically underprivileged individuals will remain much less productive that those in the upper classes. Because the productivity functions are very different across the population, it is unlikely that a Rawlsian model will call for an equal distribution of resources. However, if society controls distribution of a wide range of resources, including community services, early childhood care, housing, health care, education, and a guaranteed income, then productivity functions will show less variation, depending primarily on such individual traits as talent and industry. In this case, the Rawlsian solution is more likely to be egalitarian.

A similar principle applies in an institutional context. When managers take greater control of resources that determine the productivity of employees or divisions, such as training programs and incentive systems, then the allocations must be more nearly equal to be fair in a Rawlsian sense. Conversely, when employees or divisions fund their productivity development primarily from their own resources, as from commissions or internally generated revenue, rather than through allocations from headquarters, then those resources that are allocated need not be so nearly equal to achieve Rawlsian fairness. In general, managers who take greater responsibility for building productivity incur stricter obligations for achieving distributive justice.

References

L. M. Betts, J. R. Brown, and H. Luss. Minimax resource allocation problems with ordering constraints. *Naval Research Logistics*, 41:719–738, 1994.

C. Blackorby, W. Bossert, and D. Donaldson. Utilitarianism and the theory of justice. In K. Arrow, A. Sen, and K. Suzumura, editors, *Handbook of Social Choice and Welfare, Vol. 1*, volume 19 of *Handbooks in Economics*, pages 543–596. Elsevier, Amsterdam, 2002.

S. Bouveret and M. Lemaitre. Finding leximin-optimal solutions using constraint programming: New algorithms and their application to combinatorial auctions. In U. Endriss and J. Lang, editors, *1st International Workshop on Computational Social Choice*, Amsterdam, 2006.

J. R. Brown. The knapsack sharing problem. *Operations Research*, 27:341–355, 1979.

J. R. Brown. The flow circulation sharing problem. *Mathematical Programming*, 25: 199–227, 1983.

N. Daniels. *Reading Rawls: Critical Studies on Rawls' "A Theory of Justice"*. Stanford University Press, 1989.

M. S. Daskin. *Network and Discrete Location: Models, Algorithms, and Applications.* Wiley, New York, 1995.

R. Dworkin. *Taking Rights Seriously.* Harvard University Press, Cambridge, MA, 1977.

H. A. Eiselt. Continuous maximin knapsack problems with GLB constraints. *Mathematical Programming*, 13:114–121, 1986.

N. G. Hall and R. V. Vohra. Towards equitable distribution via proportional equity constraints. *Mathematical Programming*, 58:287–294, 1993.

H. Isermann. Linear lexicographic optimization. *OR Spektrum*, 123:223–228, 1982.

N. Katoh and T. Ibaraki. Resource allocation problems. In D.-Z. Du and P. M. Pardalos, editors, *Handbook of Combinatorial Optimization, Vol. 2*, pages 159–260. Kluwer, Dordrecht, 1998.

H. Luss. On equitable resource allocation problems: A lexicographic minimax approach. *Operations Research*, 47:361–378, 1999.

P.-A. Matt, F. Toni, and D. Dionysiou. The distributed negotiation of egalitarian allocations. In U. Endriss and J. Lang, editors, *1st International Workshop on Computational Social Choice*, Amsterdam, 2006.

W. Ogryczak. On the lexicographic minimax approach to location problems. *European Journal of Operational Research*, 100:218–223, 1997.

W. Ogryczak. Bicriteria models for fair resource allocation. In U. Endriss and J. Lang, editors, *1st International Workshop on Computational Social Choice*, Amsterdam, 2006.

J. Rawls. *A Theory of Justice.* Harvard University Press, Cambridge, MA, 1971.

J. Rawls. *Political Liberalism*, volume 4 of *The John Dewey Essays in Philosophy.* Columbia University Press, New York, 1993.

J. Rawls. *The Law of Peoples.* Harvard University Press, Cambridge, MA, 1999.

K. Roberts. Interpersonal comparability and social choice theory. *Review of Economic Studies*, 47:421–439, 1980.

J. E. Roemer. *Theories of Distributive Justice*. Harvard University Press, Cambridge, MA, 1998.

A. Sen. *Rationality and Freedom*. Belknap Press, Cambridge, MA, 2004.

M. S. Stein. *Distributive Justice and Disability: Utilitarianism against Egalitarianism*. Stanford University Press, 1989.

A. Williams and R. Cookson. Equity in health. In A. J. Culyer and J. P. Newhouse, editors, *Handbook of Health Economics, Vol. 1*, pages 1863–1910. Elsevier, Amsterdam, 2000.

M. E. Yaari and M. Bar-Hillel. On dividing justly. *Social Choice and Welfare*, 1:1–24, 1984.