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James S. W. Wong

1. Mappings on metric spaces which shrink distances in some manner have been of interest for many years. Let (X,ρ) be a complete metric space, and f be a continuous mapping of X into itself. We list below some of these distance-diminishing transformations for which various fixed or periodic point theorems have been established.

(1) (Banach) f is called a <u>strict contraction</u> if there exists λ, $0 \leq \lambda < 1$ such that for all $x,y \in X$, $\rho(f(x),f(y)) \leq \lambda\rho(x,y)$.

(2) (Boyd and Wong) f is said to be a <u>nonlinear contraction</u> if there exists a continuous function Φ on non-negative reals R^+ satisfying $\Phi(\rho) < \rho$ for $\rho > 0$ such that for all $x,y \in X$, $\rho(f(x),f(y)) \leq \Phi(\rho(x,y))$.

(3) (Edelstein) f is said to be <u>contractive</u> if for all $x,y \in X$, $x \neq y$, $\rho(f(x),f(y)) < \rho(x,y)$.

(4) (Freudenthal and Hurewicz) f is said to be <u>non-expansive</u> if for all $x,y \in X$, $\rho(f(x),f(y)) \leq \rho(x,y)$.

(5) (Bailey) f is said to be <u>weakly contractive</u> if for every $x,y \in X$, $x \neq y$, there is a positive integer $n(x,y) \in I^+$, such that $\rho(f^n(x),f^n(y)) < \rho(x,y)$.

(6) (Kirk) f is said to have <u>diminishing orbital diameters</u> if for each $x \in X$, the diameter of the orbit $\mathcal{O}(x) = \{f^j(x), j=0,1,2,\ldots\}$, say $\delta(\mathcal{O}(x))$, satisfies the property that $0 < \delta(\mathcal{O}(x)) < \infty$ implies $\delta(\mathcal{O}(x)) > \lim_{n\to\infty} \delta(\mathcal{O}(f^n(x)))$. (Note that for any subset $A \subseteq X$, the diameter of A is defined by $\delta(A) = \sup\{\rho(x,y) : x,y \in A\}$).

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(7) (Cacciopoli) f is said to be <u>iteratedly contractive</u> if for each integer k there exists constants C_k such that $p(f^k(x), f^k(y)) \leq C_k p(x,y)$ for all $x, y \in X$ and $\sum_{k=1}^{\infty} C_k < \infty$.

(8) (Belluce-Kirk) f is said to have ~~shrinking orbits~~ if for each $x \in X$ ~~with~~ $0 < \delta(\delta(x)) < \infty$, there exists an integer n such that $\delta(f^n(x)) < \delta(x)$.

(9,10) (Edelstein) f is said to be ~~ϵ-contractive~~ (ϵ-~~nonexpansive~~) for some positive $\epsilon > 0$, if for all $x, y \in X$, $p(x,y) < \epsilon$ implies $p(f(x), f(y)) < p(x,y)$ ($p(f(x), f(y)) \leq p(x,y)$).

(11) (Bailey) f is said to be ϵ-~~weakly contractive~~ for some positive $\epsilon > 0$, if for all $x, y \in X$, $p(x,y) < \epsilon$ implies there exists $n(x,y) \in I$ such that $p(f^n(x), f^n(y)) < p(x,y)$.

(12) (Krasnoselskii) f is said to be ~~asymptotically regular~~ if for each $x \in X$, $\lim_{n \to \infty} p(f^n(x), f^{n+1}(x)) = 0$.

Without any additional assumptions on X or the mapping f, the following sequences of implications hold between the various distance diminishing properties given above:

$$\begin{tikzcd}
(7) \Leftrightarrow (12) & & (5) \arrow[r, Rightarrow] & (11) \\
(1) \arrow[r, Rightarrow] & (2) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (3) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (4) \arrow[r, Rightarrow] & (10) \\
& (6) \Leftrightarrow (8) & & (9) &
\end{tikzcd}$$

In proving fixed point theorems for these distance diminishing mappings, various techniques involving iterates of the mappings were used, often at the first sight one approach is disimilar from the others. We observe, however, these results, among others, may be proved by considering a suitable continuous or lower semi-continuous function on X. In the following sections, we show in several different settings how results concerning fixed or periodic points may be derived from judicious choices of continuous or lower semi-continuous functions on X.

Let $\varphi : X \to R^+$ be a lower semi-continuous function on X. We say that φ is <u>invariant under</u> f if $\varphi(x) = 0$ implies $f(x) = x$ and φ is <u>regular with respect to</u> f if $\lim_{n\to\infty} \varphi(f^n(x))$ exists for each $x \in X$. Moreover, we say that φ is <u>weakly contractive on</u> f if for each $x \in X$, $\varphi(x) > 0$, there exists a positive integer $n(x) \in I^+$ such that $\varphi(f^n(x)) < \varphi(x)$. Similarly, φ is said to be ϵ-<u>weakly contractive</u> if for every $x \in X$ with $\varphi(x) < \epsilon$ there exists $n(x) \in I^+$ such that $\varphi(f^n(x)) < \varphi(x)$. Also, we call φ ϵ-<u>regular with respect to</u> f if for every $x \in X$ with $\varphi(x) < \epsilon$ the limit $\lim_{n\to\infty} \varphi(f^n(x))$ exists.

In section 2, X is assumed to be compact, and in section 3, we consider 'φ-regular' mappings on arbitrary complete metric space. Periodic points and local distance diminishing mappings are discussed in section 4. In the final section, we consider problems concerning non-expansive and ϵ-non-expansive mappings

on compact metric spaces and in particular isometries and e-isometries. There we also examine a question related to the converse problem on compact metric space.

2. Throughout this section, the metric space X is assumed to be compact. We are now ready to state our first result:

THEOREM 1. *Let X be a compact metric space and f a continuous mapping from X into itself. Suppose that there exists a non-negative lower semi-continuous function φ which is invariant under f and weakly contractive on f, then f has a fixed point in X.*

PROOF. Since φ is lower semi-continuous on the compact space X, it must attain its minimum at some point $z \in X$, [7, p. 227]. If $\varphi(z) = 0$, then z is a fixed point of f, otherwise there exists $n(z) \in I^+$ so that $\varphi(f^n(z)) < \varphi(z)$ contradicting that φ is minimum at z.

Corollary 1. (Edelstein [8]) *Let X be compact metric and f be contractive on X. Then f has a fixed point in X.*

Corollary 2. (Bailey [3]) *Let X be compact metric and f be weakly contractive on X. Then f has a fixed point in X.*

PROOF. Let $\varphi(x) = \rho(x, f(x))$ which is continuous on X and invariant under f. Since f is weakly contractive on X, φ is weakly contractive on f. Hence, it follows from the theorem that f has a fixed point.

Corollary 3, (Kirk [14]) Let X be compact metric and f b£ a continuous mapping from X into itself which has diminishing orbital diameters. Then f has j* fixed point in X.

Corollary 4, (Belluce and Kirk [2]) Let X be compact metric and f be asymptotically regular. Then f has a fixed point in X.

Corollary 5. (Belluce and Kirk [2]) Let X b£ compact metric and f have shrinking orbits. Then f has j* fixed point,

PROOF. Let $\varphi(x) = \sup_{i,j} p(f^{i}(x), f^{j}(x)) = \delta(\mathcal{O}(x))$, which is clearly invariant under f. Since for each pair of integers $i, j, p(f^{i}(x), f^{j}(x))$ is a continuous function of x, $\varphi(x)$ is lower semi-continuous ([7, p. 85]). By hypothesis, there exists for each $x \in X$ a positive integer $n(x) \in \mathbb{Z}^{+}$ such that $\overline{\mathcal{O}(f^{n}(x))} \cap \overline{\mathcal{O}(x)}$ which implies that $\varphi(f^{n}(x)) < \varphi(x)$. Hence the existence of a fixed point of f follows from Theorem 1.

Similarly, we can also introduce the notion that f has diminishing orbital radii if $r(x) = \sup_{j} p(x, f^{j}(x)) > \lim_{n \to \infty} r(f^{n}(x))$ for each $x \in X$. More generally, we say f has weakly contractive orbital radii if for each $x \in X$ there exists a positive integer $n(x) \in \mathbb{Z}^{+}$ so that $r(f^{n}(x)) < r(x)$. Consider $\varphi(x) = \sup_{j} p(x, f^{j}(x))$ and apply a similar argument to get:

Corollary 6. Let X be compact metric and f is continuous from X into itself which has diminishing orbital radii. Then f has a fixed point in X.

Corollary 7. *Let* X *be compact metric and* f *is continuous from* X *into itself which has weakly contractive orbital radii. Then* f *has a fixed point*.

We remark that in addition to the simplicity of the present proofs as compared to those given earlier, there are two distinct features of this approach. First, no explicit manipulation with sequences or subsequences of iterates of f is required. Next, in contrast to earlier proofs of Corollaries 3, 4, 5, in which Zorn's lemma were used, no form of Axiom of Choice is necessary here.

3. In this section, we consider fixed points of continuous mappings on arbitrary complete metric spaces. By strengthening hypothesis on f and hence the associated function φ, we may establish a similar result as Theorem 1 without compactness of X, provided that certain sequence of iterates possesses convergent subsequences; specifically, we have:

THEOREM 2. *Let X be a complete metric space and f be a continuous mapping from X into itself. Suppose that there exists a non-negative continuous function φ which is invariant under f, regular with respect to f, and weakly contractive on f. If there exists an element $x \in X$ for which the sequence of iterates $\{f^n(x)\}$ possesses a convergent subsequence, then f has a fixed point in X.*

PROOF. Let the convergent subsequence $\{f^{n_k}(x)\}$ have a limit $z \in X$. Since φ is weakly contractive on f, there exists a positive integer $N(z)$ such that $\varphi(f^N(z)) < \varphi(z)$. Using the continuity of φ and the regularity of φ with respect to f, we have

$$\begin{aligned}\varphi(z) &= \lim_{k\to\infty} \varphi(f^{n_k}(x)) = \lim_{k\to\infty} \varphi(f^{n_k+N}(x)) \\ &= \varphi(\lim_{k\to\infty} f^{n_k+N}(x)) = \varphi(f^N(\lim_{k\to\infty} f^{n_k}(x))) \\ &= \varphi(f^N(z)) < \varphi(z),\end{aligned}$$

which is possible only if $\varphi(z) = 0$. In this case, f has a fixed point.

Corollary 8. (Belluce and Kirk [1]) *Let* X *be a complete metric space and* f *be a non-expansive mapping of* X *into itself which has diminishing orbital diameters. Suppose that for some* $x \in X$ *the sequence of its iterates has a convergent subsequence. Then* f *has a fixed point in* X.

Corollary 9. (Kirk [14]) *Let* X *be a complete metric space and* f *be a continuous mapping on* X *which in addition satisfies a uniform Lipschitz condition for all of its iterates*, i.e. *there exists a constant* C *such that for each positive integer* k *and for each* $x, y \in X$, $\rho(f^k(x), f^k(y)) \leq C\rho(x,y)$. *Suppose in addition that* f *has diminishing orbital diameters and that for some* $x \in X$, *the sequence of iterates* $\{f^n(x)\}$ *has a convergent subsequence, then* f *has a fixed point in* X.

PROOF. Denote $\varphi(x) = \sup_{i,j} \rho(f^i(x), f^j(x)) = \delta(\mathcal{O}(x))$ which we know from Corollary 5 that it is lower semi-continuous. To show that $\varphi(x)$ is also upper semi-continuous, we need to verify that for each $b > 0$, $\bigcap_{i,j} \{x : \rho(f^i(x), f^j(x)) < b\} = S_b$ is open. Let $x_o \in S_b$ and choose $\delta > 0$ so that $\varphi(x_o) + 2C\delta < b$. Thus, for all $x \in X$, $\rho(x_o, x) < \delta$, we have

$$\begin{aligned}\rho(f^i(x), f^j(x)) &\leq \rho(f^i(x), f^i(x_o)) + \rho(f^i(x_o), f^j(x_o)) + \rho(f^j(x_o), f^j(x)) \\ &\leq C\rho(x, x_o) + \varphi(x_o) + C\rho(x_o, x) \\ &= 2C\delta + \varphi(x_o) < b.\end{aligned}$$

It is also clear that φ is invariant under f and also regular with respect to f. That f have diminishing orbital diameter implies that φ is weakly contractive on f, and the conclusion follows from Theorem 2.

Corollary 10. (Edelstein [8]) *Let X be a complete metric space and f be contractive on X. Suppose that there exists $x \in X$ whose sequence of iterates $\{f^j(x)\}$ has a convergent subsequence, then f has a fixed point in X.*

Corollary 11. *Let X be a complete metric space and f be a non-expansive mapping of X into itself. Suppose that f is also weakly contractive on X and there exists $x \in X$ such that its sequence of iterates $\{f^j(x)\}$ has a convergent subsequence; then f has a fixed point in X.*

Corollary 10 follows from Corollary 11 trivially. To see how Corollary 11 follows from Theorem 2, one need only to define $\varphi(x) = \rho(x, f(x))$ and observe that f non-expansive implies that φ is regular with respect f. We also remark that Corollary 9 remains valid if the hypothesis that f has diminishing orbital diameters is replaced by the weaker assumption that f has shrinking orbits. Similarly, we may formulate results similar to Corollaries 6 and 7 using the notions of diminishing orbital radii and weakly contractive orbital radii respectively. Since the procedure is clear, we omit the details.

At this point, we would also like to mention the following generalization of the Contraction Mapping Principle:

Corollary 12, (Boyd and Wong [5]) Let X be ja complete metric space and f be a, nonlinear contraction on X. Then f has ja fixed point in X.

Let <p(x) = p(x,f(x)) it is easy to see that <p satisfies all the required hypothesis in Theorem 2. The existence of some element *xeX* whose iterates (f^(x)} contains convergent subsequence follows from definition of nonlinear contraction, but the details are a little complicated; hence, we refer the reader to [5]. In this connection, Corollary 12 may also be considered as a consequence of Corollary 10. (Cf also [17].)

4. We shall now consider continuous mappings on complete metric spaces which shrinks distances locally in the sense that the various distance-diminishing properties holds between points in X whose distances are small, e.g. properties (9) and (10). There exists a close relationship between the existence of periodic points and locally contractive mappings as reported in Edelstein [8] and Bailey [3]. In this section, we show how results on period points may similarly be obtained following the same lines as in the previous two sections.

THEOREM 3. <u>Let</u> X <u>be a compact metric space and</u> f <u>a continuous mapping from</u> X <u>into itself. Suppose that there exists a non-negative lower semi-continuous function</u> φ <u>which is invariant under</u> f^k <u>for some positive integer and</u> ϵ-<u>weakly contractive on</u> f <u>for some</u> $\epsilon > 0$. <u>If there exists</u> $z \epsilon X$ <u>such that</u> $\varphi(x) < \epsilon$, <u>then</u> f^k <u>has a fixed point in</u> X.

PROOF. Define $X_k = \{x : \varphi(x) \leq \varphi(z)\}$ which is non-empty by assumption. Since φ is lower semi-continuous, X_k is also closed and hence compact. Let $x_o \epsilon X_k$ be the point at which φ attains its infimum. Since φ is ϵ-weakly contractive on f, we have $\varphi(f^{n(x_o)} x_o) < \varphi(x_o)$ and hence $\varphi(x_o) = 0$ or x_o is a fixed point under f^k.

<u>Corollary 13</u>. (Bailey [3]) <u>Let</u> X <u>be compact metric and</u> f <u>is continuous which is</u> ϵ-<u>weakly contractive on</u> X. <u>Then</u> f <u>has at least one periodic point</u>.

PROOF. Using the compactness of X, we can deduce the existence of the smallest integer k so that there exists at least one zeX satisfying $p(z^{\wedge}T^{k}z) < e$. Define $<p(x) = p(x,T^{k}x)$. It is readily verified that all hypothesis in Theorem 3 are satisfied hence the existence of periodic point follows.

THEOREM 4. Let X be a. complete metric space and f be a. continuous mapping from X into itself. Suppose that there exists ja non-negative continuous function <p which is invariant under f^ for some positive integer k, e-regular with respect to f, and e-weakly contractive on f. JTE there exists xeX such that its sequence of iterates $(f^{i}(x)\}$ contains a. convergent subsequence with limit zeX satisfying $<p(x) < e$, then f^ has a. fixed point in X.

PROOF. Since $<p$ is e-weakly contractive on f, there exists a positive integer N(z) such that $<p(f^{N}(z)) < (p(z) < e$. The proof that $(p(z) = 0$ is identical with that given in Theorem 2 and will be omitted.

Corollary 14. (Edelstein [8]) Let X be a complete metric space and f JLE e-contractive on X. Ij! there exists xeX such that its sequence of iterates $\{f^{x}(x)\}$ contains ja convergent subsequence with limit zeX, then f has a^ periodic point in X.

PROOF. Let $\{f^{n_i}(x)\}$ be the subsequence of $(f^{i}(x)\}$ which has z as its limit. Choose N_o so that $p(f^{n_i}(x), z) < \frac{1}{2}e$ for

$i \geq N_o$. After $n_{i+1} - n$^ iterations, we obtain

$$\rho(z, f^{n_{i+1}-n_i}(z)) \leq \rho(z, f^{n_{i+1}}(x)) + \rho(f^{n_{i+1}}(x), f^{n_{i+1}-n_i}(z))$$

$$< f + p(f^{n_i}(x), z) < f + f = €.$$

Denote $k = n_{i+1} - n_i$ for any $i \geq N_Q$ and define $<p(x) = p(x, f^k(x))$ which is clearly invariant under f. We have just seen that $<p(z) < e$. Since f is €-contractive it follows that $<p(f^n(x))$ is decreasing in n, thus showing that $<p$ is $e_{\sim}$regular with respect to f. It follows from Theorem 4 that f has a fixed point in X.

By considering various choices of the function $<p$ and imposing appropriate hypothesis on f, one can obtain similar results as Corollaries 13 and 14. We leave the details to the interested reader. We remark also that local contractive conditions together with certain chainable condition on X may also yield fixed points instead of periodic points, see for example Edelstein [9] and Boyd and Wong [5]. These chainable conditions on the metric space X are usually satisfied in metric linear spaces, in particular, normed linear spaces. In this regard, the results in this section may also be formulated as fixed point theorems in these slightly specialized spaces.

5. This section is devoted to a discussion of miscellaneous questions concerning non-expansive and contractive mappings on compact metric space. The following propositions are simple observations from earlier results:

PROPOSITION 1. *Let* X *be compact metric space and* f *be non-expansive on* X. *Then* f *is an isometry on* $Y = \bigcap_{n=1}^{\infty} f^n(x)$.

PROOF. Clearly $f(Y) = Y$ and Y is compact. Since f is onto Y, it follows from the result of Freudenthal and Hurewicz [12] that f is an isometry.

Corollary 15. (Edelstein [10]) *Let* X *be a compact metric space and* f *be non-expansive on* X. *Then for each* $x \in \bigcap_{n=1}^{\infty} f^n(X)$, $(f^n(x)\}$ *forms an isometric sequence,* i.e. $p(f^m(x), f^n(x)) = p(f^{m+k}(x), f^{n+k}(x))$ *for all* $k,m,n, = 1,2,3,\ldots$.

PROPOSITION 2. *Let* X *be compact metric space and* f *be* €-*non-expansive on* X. *Then* f *is an* e-*isometry on* Y, i.e. *for all* $x,y \in Y$ *with* $p(x,y) < €$, *we have* $p(f(x),f(y)) = P(x,y)$.

PROOF. Consider $Y = \bigcap_{n=1}^{\infty} f^n(X)$. Since $f(Y) = Y$, and f is onto Y, hence it follows from a result of Edrei [13] that f is an €-isometry on Y.

A similar corollary as Corollary 15 holds for e-nonexpansive mappings, in this case, every point $x \in Y$ generates an €-isometric sequence, in the sense that $p(f^m(x), f^n(x)) = p(f^{m+k}(x), f^{n+k}(x))$ for all $m,n,k = 1,2,\ldots$ whenever $p(f^m(x), f^n(x)) < e$. The main result of this section is the following:

THEOREM 5. *Let* X *be compact metric space and* f *be contractive on* X. *Then for each* λ, $0 < \lambda < 1$, *there exists an equivalent metric* ρ_λ *with respect to which* f *is a contraction, namely satisfying* (1).

We note that a contractive mapping f on a compact metric space is not necessarily a contraction. Take for example $X = [0,1]$ given with the Euclidean metric and define $f(x) = \frac{1}{2}x + \frac{1}{4}x^2$. Clearly $f(X) \subseteq X$, and f is contractive on X, since

$$\left|\tfrac{1}{2}x + \tfrac{1}{4}x^2 - \tfrac{1}{2}y - \tfrac{1}{4}y^2\right| \leq \tfrac{1}{2}|x - y| + \tfrac{1}{4}|x + y||x - y|$$
$$< \tfrac{1}{2}|x - y| + \tfrac{1}{2}|x - y| = |x - y|.$$

On the other hand, $\sup\limits_{x,y\in X} \dfrac{|f(x) - f(y)|}{|x - y|} = 1$. (Take any two sequence of numbers $\{x_n\},\{y_n\}$ with $x_n > y_n$ for all n and $x_n \to 1$, $y_n \to 1$ as $n \to \infty$.) In view of this example, Theorem 5 becomes an interesting observation.

PROOF OF THEOREM 5. Since f is contractive on X, it follows from Theorem 3 that $\bigcap\limits_{n=1}^{\infty} f^n(X) = \{\omega\}$ a singleton set. Now, a result of Janos [13] implies that for each $\lambda \in (0,1)$, there exists an equivalent metric ρ_λ relative to which f is a contraction, namely $\rho_\lambda(f(x),f(y)) \leq \lambda\rho_\lambda(x,y)$ for all $x,y \in X$.

We remark that the conclusion $\bigcap\limits_{n=1}^{\infty} f^n(X) = \{\omega\}$ is stronger than the convergence of successive approximations, i.e. $\rho(f^j(x),x_0) \to 0$

as $j \to \infty$ for all $x \in X$. Consider the example given in [16]: $X = \{z : z = e^{i\theta}\}$ with the ordinary Euclidean metric and $f(z) = \frac{1}{2}z$. Clearly X is compact and $f^n(z) \to 0$ as $n \to \infty$ for all $z \in X$, but $\bigcap_{n=1}^{\infty} f^n(X) = X$. Note also that f is not contractive there.

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