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Victor J. Mizel
Carnegie Mellon University

Kondagunta, -1931 Sundaresan

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ADDITIVE FUNCTIONALS ON SPACES WITH
NON-ABSOLUTELY-CONTINUOUS NORM

V. J. Mizel

and

K. Sundaresan

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by

V. J. Mizel and K. Sundaresan

Let \((T, \Sigma, \mu)\) be a complete finite positive measure space, and let \((X, || \cdot ||)\) be a seminormed vector space of real valued measurable functions defined on \(T\). We suppose that (a) The space \((X, || \cdot ||)\) obtained by identifying functions of \(X\) which are equal a.e. is a Banach space. (b) If \(x \in X\) and \(E\) is a measurable set then \(x I_E \in X\) where \(I_E\) is the characteristic function of \(E\), (c) If \(x \in X\) and \(y\) is a real valued measurable function satisfying \(|y| (t) \leq |x| (t)\) a.e. then \(y \in X\), (d) \(L^\infty (\mu) \subset X\).

The problem of characterizing nonlinear functionals \(F\) on \(X\) which admit integral representations of the form \(F(x) = \int_T \phi(x(t), t) \, d\mu(t)\)

where \(\phi\) is a Caratheodory function has been discussed in Drewnowski and Orlicz [1] and Mizel [3]. In [3] the case \(X = L^p(\mu), 1 \leq p \leq \infty\), is discussed while in [1] \(X\) is assumed to be of absolutely continuous norm. The purpose of the present note is to extend these results to the case when \(X\) is not necessarily of absolutely continuous norm.

We recall a few definitions and notations before presenting the main results of the paper.

In the sequel the set of all measurable real-valued functions on the complete finite positive measure space \((T, \Sigma, \mu)\) is denoted by \(m\), \(L^\infty (\mu) \subset m\) is the subspace of essentially bounded functions, and \(X \subset m\) denotes a fixed subspace satisfying (a), (b), (c) and (d) above. We denote the real line by \(R\). A functional \(F: X \to R\) is
said to be **additive** (orthogonally additive) if \( F(x + y) = F(x) + F(y) \) whenever \( x, y \) are of disjoint support i.e. \( \mu(\{t \mid x(t)y(t) \neq 0\}) = 0 \).

A function \( \varphi : \mathbb{R} \times T \to \mathbb{R} \) is a **Caratheodory function** if for each \( \gamma \in \mathbb{R} \), \( \varphi(\gamma, \cdot) \) is a measurable function, and for \( t \) a.e., \( \varphi(\cdot, t) \) is a continuous function on \( \mathbb{R} \). If \( x \) is a real valued function then \( \varphi \circ x(t) = \varphi(x(t), t) \). If \( x \) is a constant function taking the value \( h \), \( \varphi \circ x \) will be denoted by \( \varphi_h \).

We proceed to the representation theorem of the paper. We present separately the necessary and sufficient conditions guaranteeing the representation. We recall a lemma given in [3] (see also Krasnoselskii [2]).

**Lemma 1.** If \( \varphi \) is a Caratheodory function and \( \eta \) is a positive real number then there exists a measurable set \( S_\eta \) such that (1) \( \mu(T - S_\eta) < \eta \) and (2) \( \varphi(\cdot, t) \) is uniformly continuous on each bounded interval \( J \subseteq \mathbb{R} \), uniformly for \( t \in S_\eta \).

**Theorem 1.** Let \( F \) be a real valued function on \( X \) such that (i) \( F \) is additive, (ii) \( F \) is uniformly continuous on every ball in \( (C^\infty(\mu), \| \cdot \|_\infty) \), (iii) \( F \) is continuous with respect to dominated a.e. convergence, i.e. whenever \( \{x_n\} \), \( x, y \in X \) satisfy \( x_n \to x \) a.e. and \( |x_n(t)| \leq |y(t)| \) a.e. for all \( n \), then \( F(x_n) \to F(x) \).

Then there exists a Caratheodory function \( \varphi \) on \( \mathbb{R} \times T \) such that for all \( x \in X \)

\[
(*) \quad F(x) = \int_T \varphi \circ x \, d\mu.
\]
Proof. Since the restriction of $F$ to $L^\infty(\mu)$ satisfies the hypotheses in theorem 1 of [3], it follows that there exists a Carathéodory function $\varphi$ such that for all $x \in L^\infty(\mu)$ the representation (*) is valid. Let now $x \in X$. It follows from conditions (ii) and (iii) that the set function $F(x_1^A)$ on $\Sigma$ is absolutely continuous with respect to $\mu$. Thus it has a Radon-Nikodym derivative $g \in L^1(\mu)$, unique up to a null set, such that (a) $F(x_1^A) = \int_A g \, d\mu$, for $A \in \Sigma$. We proceed to verify that $g = \varphi \circ x$ a.e. Let for each real number $C \geq 0$,

$$A_C = \{ t \mid |x(t)| \leq C \}. \text{ Given a fixed sequence of real numbers } \{\eta_m\} \text{ converging to } 0 \text{ let us denote the corresponding measurable sets } S_m \text{ whose existence is assured by lemma 1, by } S_m. \text{ It is easily verified that the sequences of measurable sets } \{A_n\} \text{ and } \{S_m\} \text{ both converge to } T \text{ as } n, m \to \infty. \text{ Let } B_m, n = S_m \cap A_n. \text{ We proceed to show that } g = \varphi \circ x \text{ a.e. on } B_m, n. \text{ Since } |x(t)| \leq n \text{ on } B_m, n \text{ there exists a sequence } \{y_k\} \text{ of simple functions such that } y_k \to x \text{ uniformly on } B_m, n \text{ and } |y_k| \leq |x|. \text{ By lemma 1, it follows that } \varphi \circ y_k \to \varphi \circ x \text{ uniformly on } B_m, n \text{ hence if } E \text{ is any measurable subset of } B_m, n \text{ it follows that }$

$$\lim \int_E \varphi \circ y_k \, d\mu = \int_E \varphi \circ x \, d\mu.$$

Further since $y_k I_E \to x I_E$ a.e. and $|y_k| \leq |x|$ it follows from property (b) of $X$ and condition (iii) that $F(y_k I_E) \to F(x I_E)$ for each $E \in \Sigma$. Thus $\int_E g \, d\mu = \int_E \varphi \circ x \, d\mu$ for all $E \subseteq B_m, n$. Hence $g = \varphi \circ x$ a.e. on $B_m, n$. Since $T - \bigcup_{m,n} B_m, n$ is a null set, this completes the proof of the theorem.

We proceed next to the converse of Theorem 1.
Theorem 2. If \( \varphi \) is a Caratheodory function on \( \mathbb{R} \times T \) such that 
\[ \varphi(o, t) = 0 \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and} \quad \varphi \circ x \in L^1(\mu) \quad \text{for each} \quad x \in X, \]
then the functional \( F(x) = \int_T \varphi \circ x \, d\mu \) satisfies conditions (i), (ii) and (iii) of Theorem 1.

Before proceeding to the proof we note that by the converse assertion in Theorem 1 of [3] \( F \) certainly satisfies conditions (i) and (ii) above. We verify that \( F \) also satisfies the condition (iii) after establishing the following lemmas.

Lemma 2. If \( \varphi \) is a Caratheodory function and \( x \in L^\infty(\mu) \) then the (almost everywhere finite) function \( \alpha_x \) defined below is measurable.

\[
\alpha_x(\xi) = \sup \{ |\varphi_h(\xi)| \mid |h| \leq |x(\xi)| \}
\]

Further for each \( \epsilon > 0 \) there is a function \( y \in L^\infty(\mu) \) such that

1. \( |y| \leq |x| \)
2. \( |(\varphi \circ y)(\xi)| - \alpha_x(\xi) | < \epsilon \quad \text{a.e.} \)

Proof. First let \( x \) be a simple function of the form \( \sum x_1 1_{A_i} \).

Let \( \alpha'_x(\xi) = \sup_{x_1} |\varphi_h(\xi)|, h \text{ rational}. \) Clearly \( \alpha'_x \) is a measurable function since on each of the sets \( A_i \) it is the supremum of a countable family of measurable functions. Moreover since \( \varphi(\cdot, x) \) is continuous for \( x \) a.e. it is verified that \( \alpha_x = \alpha'_x \quad \text{a.e.} \). Thus \( \alpha_x \) is also measurable. Now each \( x \in L^\infty(\mu) \) is the pointwise limit of a sequence of simple functions \( \{ x_n \} \) satisfying \( |x_n| \uparrow |x| \quad \text{a.e.} \). Therefore \( \alpha^+_x = \alpha_x \quad \text{a.e.} \) and the measurability result holds for such \( x \) as well. Notice that by a similar argument the functions \( \alpha^\pm_x \) defined by
\[
\alpha_+^\times (\xi) = \sup_{0 < h \leq |x(\xi)|} |\varphi_h(\xi)|, \quad \alpha_-^\times (\xi) = \sup_{-|x(\xi)| \leq h < 0} |\varphi_h(\xi)|
\]
are both measurable.

Now let \( \{\eta_m\} \) be a sequence of positive reals such that \( \eta_m \to 0 \) and let the corresponding sets \( \{S_{\eta_m}\} \) whose existence is assured by lemma 1 be denoted by \( \{S_m\} \). For each \( \eta_m \) there exists by lemma 1 a \( \delta^m \) such that \( |\varphi(h, \xi) - \varphi(h', \xi)| < \epsilon \) for all \( \xi \in S_m \) whenever \( |h - h'| \leq \delta^m \) and \( |h|, |h'| \leq ||x||_\infty \). Let
\[
A^+=\{\xi| \alpha^\times_\xi(\xi) = \sup_{0 < h \leq |x(\xi)|} |\varphi(h, \xi)| \} = \{\xi| \alpha^+_\xi(\xi) = \alpha^\times_\xi(\xi)\}. \]
By the results above \( A^+ \) is a measurable set. Now define sets \( E^\pm_{mj} \) as follows.
\[
E^+_{m0} = \{\xi \in S_m| |\alpha^\times_\xi(\xi)| < \epsilon\}, \quad E^+_{m1} = \{\xi \in S_m \cap A^+| |\varphi(\delta^m, \xi)| - \alpha^\times_\xi(\xi)| < \epsilon\} \sim E_{m0}
\]
\[
E^-_{ml} = \{\xi \in S_m \sim A^+| |\varphi(-\delta^m, \xi)| - \alpha^\times_\xi(\xi)| < \epsilon\} \sim E_{m0}, \text{ and more generally}
\]
\[
E^+_{mj} = \{\xi \in S_m \cap A^+| |\varphi(j\delta^m, \xi)| - \alpha^\times_\xi(\xi)| < \epsilon\} \sim \bigcup_{i \leq j-1} (E^+_{mi} \cup E^-_{mi}).
\]
\[
E^-_{mj} = \{\xi \in S_m \sim A^+| |\varphi(-j\delta^m, \xi)| - \alpha^\times_\xi(\xi)| < \epsilon\} \sim \bigcup_{i \leq j-1} (E^+_{mi} \cup E^-_{mi}).
\]
Clearly each set \( E^\pm_{mj} \) is measurable and if \( y^\epsilon_m = \sum_{l=1}^{m} (j \delta^m I_{E^+_{mj}} + -j \delta^m I_{E^-_{mj}}) \)
then \( |\varphi \circ y^\epsilon_m(\xi)| - \alpha^\times_\xi(\xi)| < \epsilon \) for a.e. \( \xi \in S_{\eta_m} \). Now let
\[
y^\epsilon = \sum_{m \geq 1} y^\epsilon_m I_{S_m-S_{m-1}}. \quad \text{From the construction of the } \{y^\epsilon_m\} \text{ it is verified that } |y^\epsilon| \leq |x| \text{ and satisfies } |\varphi \circ y^\epsilon| - |\alpha^\times_\xi| < \epsilon \text{ except on the null set } T \sim \bigcup_{m} S_{\eta_m}. \quad \text{By construction each } y^\epsilon_m \text{ is a } k\delta^m \text{-valued function such that } \varphi \circ y^\epsilon_m \text{ approximates } \alpha^\times_\xi \text{ within } \epsilon \text{ on } S_{\eta_m}.\]
Lemma 3. If \( x \in \mathcal{L}^\infty (\mu) \) then there exists a function \( y_x \in \mathcal{L}^\infty (\mu) \) such that (1) \( |y_x| \leq |x| \) a.e. and (2) \( \phi \circ y_x = \alpha_x \) a.e.

Proof. Let \( \{ \epsilon_n \} \) be a sequence of positive real numbers such that (1) \( \epsilon_n \to 0 \). Then the functions \( y^\epsilon_n \) (following the notation in lemma 2) are all dominated by \( x \) and by construction converge pointwise on each set \( S \). In fact, denoting \( U_S = S \), one has

\[
(*) \lim_{n \to \infty} y^\epsilon_n(\xi) = \begin{cases} 
\text{minimum } c \geq 0 \text{ s.t. } |\phi(c, \xi)| = \alpha_x(\xi) \xi \in S \cap A^+ \\
\text{maximum } c \leq 0 \text{ s.t. } |\phi(c, \xi)| = \alpha_x(\xi) \xi \in S \sim A^+
\end{cases}
\]

Let \( y_x(\xi) = \lim_{n \to \infty} y^\epsilon_n(\xi) \) for \( \xi \in S \) and = 0 otherwise. Clearly \( y_x \) is measurable and \( |y_x| \leq |x| \) since \( |y^\epsilon_n| \leq |x| \) for all \( n \geq 1 \).

From (*) it follows that \( |\phi \circ y_x| = \alpha_x \) a.e.

Lemma 4. For each \( x \in X \) there exists a function \( y_x \in X \) such that (1) \( |y_x| \leq |x| \) a.e. and (2) \( \phi \circ y_x = \alpha_x \) a.e.

Proof. Define recursively a sequence of pairwise disjoint measurable sets \( \{ E_n \} \) as follows.

\( E_1 = \{ \xi \mid |x(\xi)| \leq 1 \}, E_2 = \{ \xi \mid |x(\xi)| \leq 2 \} \sim E_1 \)

and in general

\( E_n = \{ \xi \mid |x(\xi)| \leq n \} \sim \bigcup_{j=1}^{n-1} E_j \). It follows from the preceding lemma that for each integer \( n \geq 1 \) there is a function \( y_n \) in \( \mathcal{L}^\infty (\mu) \) such that \( |\phi \circ y_n(\xi)| = \alpha_x(\xi) \) for all \( \xi \in E_n \) and \( y_n(\xi) = 0 \) otherwise. Let \( y_x = \sum y_n I_{E_n} \). Then on each set \( E_n \) \( |y_x| \leq |x| \) and \( |\phi \circ y_x| = \alpha_x \). Since \( \mu(T \sim \bigcup E_n) = 0 \) it follows that \( |y_x| \leq |x| \) a.e., so that \( y_x \in X \), and \( |\phi \circ y_x| = \alpha_x \) a.e.
We complete the proof of Theorem 2 by proving the following corollary to lemma 4.

**Corollary.** If \( \varphi \) is a Caratheodary function and if for each \( x \in X \), \( \varphi \circ x \in L^1(\mu) \) then the functional \( F(x) = \int \varphi \circ x \, d\mu \) has the property (iii) of Theorem 1.

**Proof.** Let \( z \in X \) and \( \{x_n\} \) be a sequence in \( X \) such that \( |x_n| \leq |z| \). Then by the construction of \( y_z \) it follows that \( |\varphi \circ x_n| \leq |\varphi \circ y_z| \) and \( \varphi \circ y_z \in L^1(\mu) \). If further the sequence \( \{x_n\} \) converges to some function \( x \) a.e. where \( x \in X \) then by the continuity of \( \varphi(\cdot, \xi) \) it follows that \( \varphi \circ x_n \to \varphi \circ x \) a.e. Since \( |\varphi \circ x_n| \leq |\varphi \circ y_z| \) we have by the dominated convergence theorem

\[
F(x_n) \to F(x).
\]

In conclusion we mention that Theorems 1 and 2 can be extended to the case when \((T, \Sigma, \mu)\) is \( \sigma \)-finite measure space. As this generalization is straightforward and the proof is very similar to that of Theorem 2 in [3] once the results for the finite case are obtained, we content ourselves by stating the theorem without proof.

In the next theorem \((T, \Sigma, \mu)\) is a complete \( \sigma \)-finite measure space.

**Theorem 3.** Let \((X, \|\cdot\|)\) be as in the introduction except that \( X \) satisfies instead of condition (c) the following condition (c') \( f \in L^\infty(\mu), E \in \Sigma \) s.t. \( \mu(E) < \infty \) implies \( f I_E \in X \).

Suppose the function \( F: X \to \mathbb{R} \) satisfies conditions (i) and (iii) of theorem 1 as well as the condition,

(ii') \( F \) is uniformly continuous on each set of the form \((X \cap Y, \|\cdot\|_\infty)\)

where \( Y \) is a bounded subset of \( L^\infty(\mu) \) supported by a set of finite measure.
Then there exists a Caratheodory function \( \varphi \), satisfying \( \varphi \circ x \in L^1(\mu) \) for each \( x \in X \), for which the following representation holds

\[
(*) \quad F(x) = \int_{\mathbb{T}} \varphi \circ x \, d\mu.
\]

Conversely, each Caratheodory function \( \varphi \) which satisfies

1. \( \varphi \circ x \in L^1(\mu) \) for each \( x \in X \), and
2. \( \varphi(0, \xi) = 0 \) a.e., determines by means of (*) a function \( F \) which satisfies (i), (iii) and (ii').

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References


